L. XII

$\mathcal{Z} = (C, \underline{a}, M, z)$ pt'd matched circle

$\rightsquigarrow \mathcal{A}(\mathcal{Z})$ dg-alg. $\rightsquigarrow I(S) \in \mathcal{A}(\mathcal{Z})$

$S \subseteq \{1, \dots, 2k\}$
$\shortparallel$
set of α-arcs
in a bordered diagram

collection of horizontal strands over pts $a_i \in \underline{a}$ s.t. $M(a_i) \in S$

$I(S)^2 = I(S)$

we call $\mathbb{I} \subseteq \mathcal{A}(\mathcal{Z})$

the subspace gen by $I(S)$

for $S \subseteq \{1, \dots, 2k\}$, $|S| = k$

This is a subalgebra: $I(S) \cdot I(T) = 0$ if $S \neq T$.

"The idempotent ring of the algebra"

$\underline{\rho} = \{\rho_1, \dots, \rho_n\}$ "Reeb chords" of $\mathcal{Z}$ =

orientation preserving arcs in C w. endpoints in $\underline{a}$

not passing through z.

$\rightsquigarrow a(\rho) \in \mathcal{A}(\mathcal{Z})$ sum of

all ways to complete ρ to

an element of $\mathcal{A}(\mathcal{Z})$ by adding matched horiz. lines

$(\bar{\Sigma}, \underline{\alpha}^c \cup \underline{\alpha}^a, \underline{\beta}, z)$ bordered Heegaard diagram

$g(\bar{\Sigma}) = g$

$|\underline{\alpha}^a| = 2k$

Bordered Floer generator:

$x = (x_1, \dots, x_g)$ s.t.

- each β-curve contains exactly one point
- each α-circle contains exactly one point.
- each α-arc contains at most one point.

exactly k α-arcs contain a point (occupied arcs)

exactly k α-arcs contain no point (unoccupied arcs)

$x \rightsquigarrow I_A(x), I_D(x) \in \mathbb{I} \subseteq \mathcal{A}$

$I_A(x)$ idempotent associated to the set of occupied arcs

$I_D(x)$ —— " —— unoccupied arcs.

in $\mathcal{A}(-Z)$

$= \mathcal{A}(Z)^{op}$.

$$\mathbb{I}(s)\,x = \begin{cases} x & \text{if } s = I_D(x) \\ 0 & \text{o.w.} \end{cases} \qquad (L)$$

$$x\,\mathbb{I}(s) = \begin{cases} x & \text{if } s = I_A(x) \\ 0 & \text{o.w.} \end{cases} \qquad (R)$$

Fix $\mathcal{H} = (\Sigma, \underline{\alpha}^c \cup \underline{\alpha}^a, \underline{\beta}, z)$ bordered diagram

$\widetilde{CFD}(\mathcal{H})$ vector space over $\mathbb{F}_2$ generated by the bordered Floer generators defined by (L)

$\widehat{CFD}(\mathcal{H}) = \mathcal{A}(-\mathcal{Z}) \otimes_{\mathbb{I}} \widetilde{CFD}(\mathcal{H})$ This inherits an $\mathcal{A}(-\mathcal{Z})$-module structure

<u>Define</u>: ∂ on $\widehat{CFD}(\mathcal{H})$ by:

$$\partial_D(x) = \sum_{y, \rho_1, \dots, \rho_n} \#\mathcal{M}(x, y, \rho_1, \dots, \rho_n) \cdot a(-\rho_1) \cdot \dots \cdot a(-\rho_n) \cdot y$$

where $\mathcal{M}(x, y, \rho_1, \dots, \rho_n)$ is the moduli space of embedded index 1 curves.

J-holomorphic curves $u : \dot{S} \to \mathbb{R} \times [0,1] \times \Sigma$

S is a Riemann surface with ∂, $\dot{S} = S \setminus 2g+n$ punct. on ∂S.

condition on punctures:

g + - punctures

g - - punctures

n e - punctures

- every component of ∂S has at least one +/- punct.
- +/- punct alternates on ∂S (so at least one + & one -)

If q_i^+ is a + puncture:

$$\begin{tikzcd} & \mathbb{R}\times[0,1]\times\Sigma \arrow[r, "\pi_\Sigma"] \arrow[dl, "\pi_{\mathbb{R}}"'] \arrow[d, "\pi_D"] & \Sigma \\ \mathbb{R} & \mathbb{R}\times[0,1] & \end{tikzcd}$$

$$\lim_{w\to q_i^+} \pi_\Sigma \circ u(w) = y_i$$

pos punct:

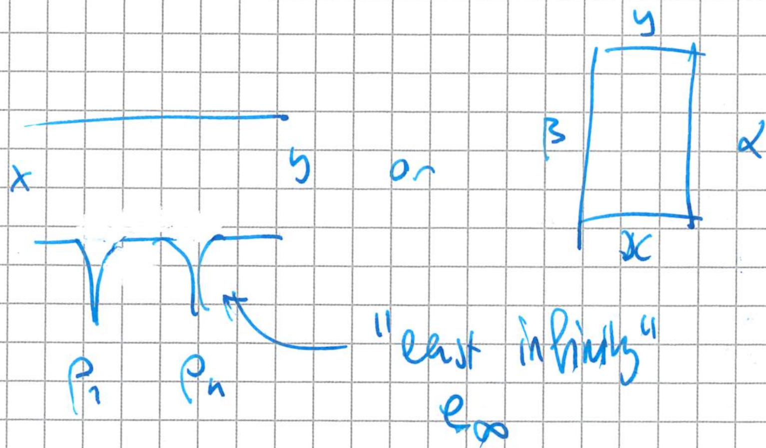
$$\lim_{w\to q_i^+} \pi_{\mathbb{R}} \circ u(w) = +\infty$$

if q_i^- is a - punct, then

$$\lim_{w\to q_i^-} \pi_\Sigma \circ u(w) = x_i \ , \quad \lim_{w\to q_i^-} \pi_{\mathbb{R}} \circ u(w) = -\infty$$

If q_i^e is a e-punct, then

$$\lim_{w \to q_i^e} \pi_\Sigma \circ u(w) = e_i, \quad \lim_{w \to q_i^e} \pi_D \circ u(w) = (s_i, 0)$$



$\partial_D(a \otimes x) = \partial a \otimes x + a\,\partial_D x$ Leibniz rule.

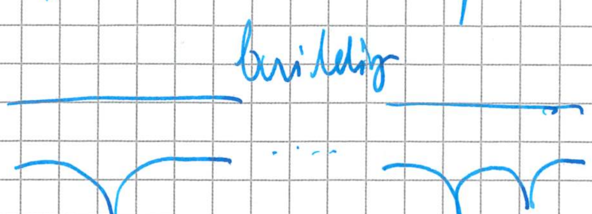
$$\partial_D(x) = \sum_{y, e_1, \dots, e_n} \#\mathcal{M}(x, y, e_1, \dots, e_n)\, a(e_1) \cdots a(e_n)\, y$$

where $\mathcal{M}(x, y, e_1, \dots, e_n)$ is the moduli space of embedded, index 1

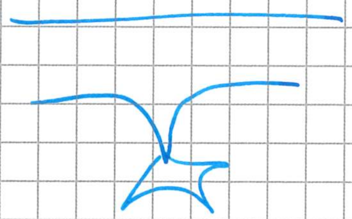
Theorem $\partial_D^2 = 0$

"Proof": analyse boundary of 1-dim moduli space

- two floor buildings

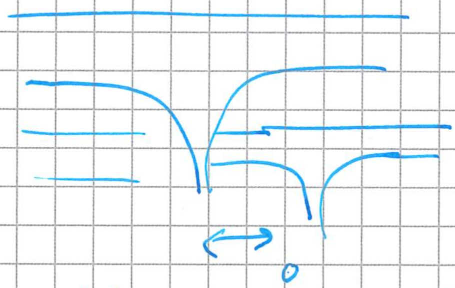


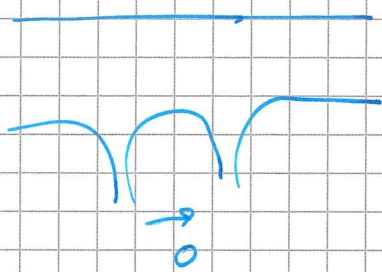
• "holomorphic combs" where curves split off at e_∞



comb

• collapse of levels: when s_i becomes equivalent to s_{i+1}





violates condition of different heights for e-punctures.

- In the case of a degeneration at e_∞, we have the building of a split/join curve

- In the case of a collapse of levels $i, i+1$ we can have: • occupied arcs of p_i are disjoint from the occupied arcs of p_{i+1}, or

• initial pt of p_i is the same as the end pt of p_{i+1}.

write $\partial_D x = \sum a_{xy}\, y \qquad a_{xy} \in \mathcal{A}(-\xi)$

$$a_{xy} = \sum_{\rho_1, \dots, \rho_n} \# \mathcal{M}(x, y, \rho_1, \dots, \rho_n)\, a(-\rho_1) \cdots a(-\rho_n)$$

$\partial_D^2 = 0$ equivalent to $\partial a_{x_- x_+} + \sum_{x_0} a_{x_- x_0} a_{x_0 x_+}$

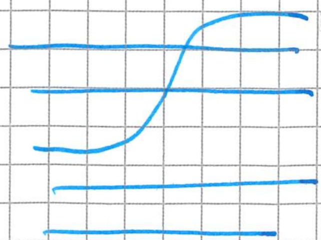
for all x_-, x_+.

$\sum_{x_0} a_{x_- x_0} a_{x_0 x_+}$ corresponds to two floor buildings

The term $\partial a_{x_+ x_-}$ corr. to split curve degenerations at e_∞.

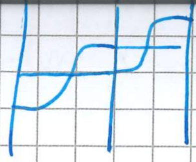
$$a_{x_- x_+} = \sum a(-\rho_1) \cdots a(-\rho_n)$$

$$\partial a_{x_- x_+} = \sum a(-\rho_1) \cdots \partial(a(-\rho_i)) \cdots a(-\rho_n)$$

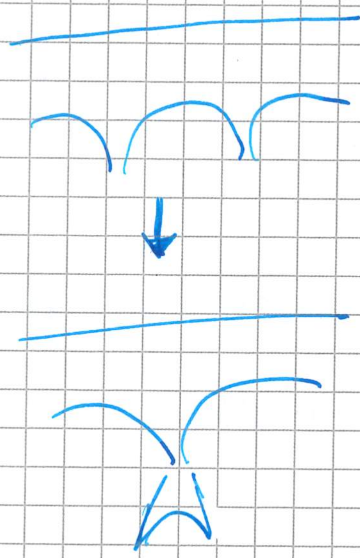


<u>Missing terms:</u>

1. join curve degenerations
2. crossing of levels
   (a) initial pt of ρ_i same as endpoint of ρ_{i+1}
   (b) ρ_i and ρ_{i+1} are disjoint or nested 

(c) interleaved  $= 0$ because of double crossing

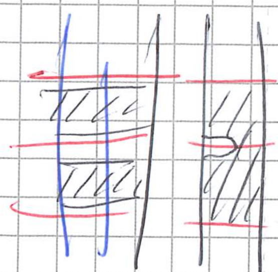
(1) cancels w. (2a) because $a(-\rho_i)\, a(-\rho_{i+1}) = a(-\rho_i \cup \rho_{i+1})$

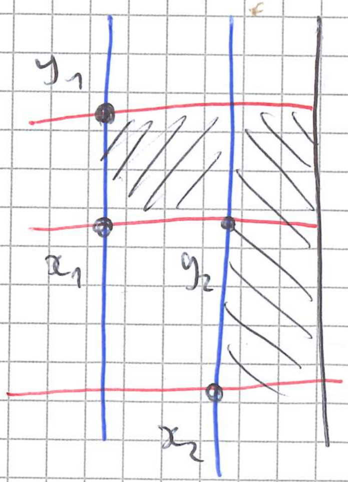


$a(-\rho_i)\, a(-\rho_{i+1})$ before crossing

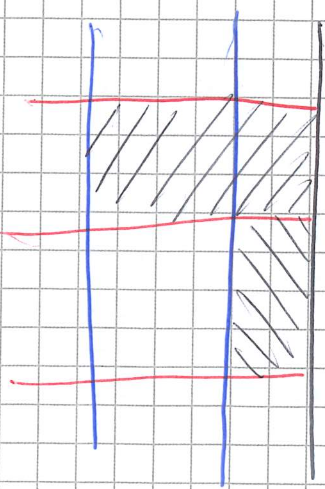
$\quad \|$

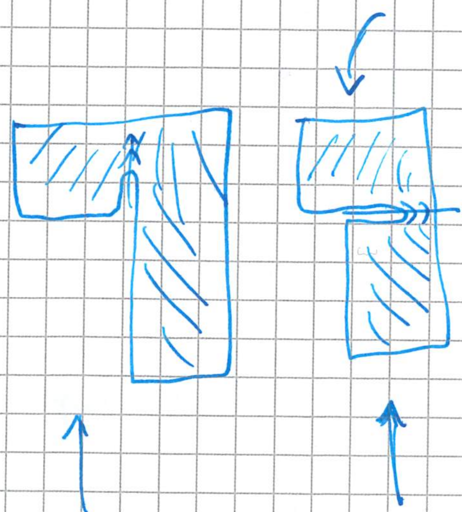
$a(-\rho_{i+1})\, a(-\rho_i)$ after crossing

conv. to. 



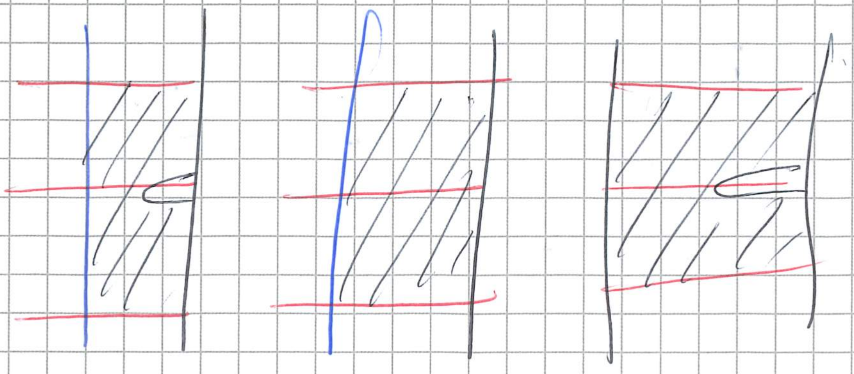
$\partial^2(\{x_1, x_2\})$





converges to a two-floor building

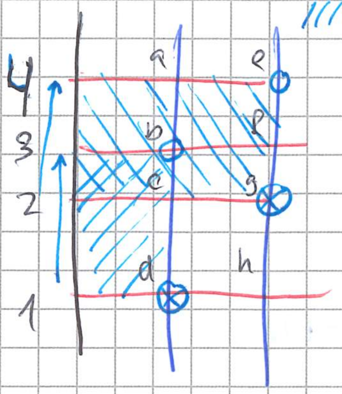
converges to a comb (w. join case)



$\backslash\backslash\backslash$ top

$///$ bottom

split curve

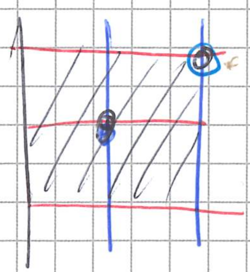


$\partial^2(\{b,e\})$ has a term:

$$\{b,e\} \xrightarrow{///} \rho_{13}\{d\,e\} \xrightarrow{\backslash\backslash\backslash} \begin{cases} /// & \text{bottom} \\ \backslash\backslash\backslash & \text{top} \end{cases}$$

$$\rho_{13}\rho_{24}\{d,g\}$$

on the other hand we have $///$ top $\backslash\backslash\backslash$ bottom



$\widehat{CFA}(\mathcal{H})$ as a vector space generated by bordered generators

right action of $\mathcal{I} \subseteq \mathcal{A}(\mathcal{Z})$ defined by (R)

<u>Maps</u>: $\mathcal{M}_k : \widehat{CFA}(\mathcal{H}) \otimes_{\mathcal{I}} \overbrace{\mathcal{A} \otimes_{\mathcal{I}} \cdots \otimes_{\mathcal{I}} \mathcal{A}}^{k-1 \text{ times}} \to$

$$\to \widehat{CFA}(\mathcal{H})$$

s.t. $$\sum_{i+j=k+1} m_i(m_j(x, a_1, \ldots, a_{j-1}), a_j, \ldots, a_k) +$$

$$\sum_{l=1}^{k-1} m_k(x, a_1, \ldots, \partial a_l, \ldots, a_{k-1}) +$$

$$\sum_{l=1}^{k-2} m_{k-1}(x, a_1, \ldots, a_l a_{l+1}, \ldots, a_{k-1}) = 0$$

$$m_k(x, a(\underline{\rho_1}), \ldots, a(\underline{\rho_{k-1}}))$$

$$\rho_i = \{ \rho_{i,1}, \ldots, \rho_{i,l_i} \}$$

$$\sum_y \# \mathcal{M}(x, \rho_1, \ldots, \rho_{k-1}) \, y$$

now the chords in ρ_i are all at the same level s_i, $s_1 < \ldots < s_n$