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Holomorphic Curve Theories in Symplectic Geometry

Lecture V

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Goal of lecture

Today:

- Gromov's Compactness for pseudoholomorphic discs.
- "Uniruledness" proofs for certain Lagrangians in $(\mathbb{C}P^n, \omega_{\mathsf{FS}}, J)$.
- The superpotential: a Hamiltonian isotopy invariant for monotone Lagrangians.

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Plan

Gromov's compactness for discs

- To control global properties of the moduli-space of pseudoholomorphic discs we need Gromov's compactness theorem.
- The energy $E(u) = \int_u \omega$ of a pseudoholomorphic disc is defined in the same way as in the closed case, and it is positive whenever u is non-constant.
- The condition for a nodal pseudoholomorphic disc to be stable seems complicated at first sight (but is natural from the point of view of doubling).

Nodal pseudoholomorphic discs

Definition

A *nodal pseudoholomorphic disc* is a continuous map $u_\infty \colon (D^2, \partial D^2) \to (X, L)$ which is J-holomorphic for some almost complex structure j_∞ defined on $D^2 \setminus \Gamma$, such that

- The double $(\mathbb{C}P^1, \Gamma^{dbl}, j_\infty^{dbl})$ of (D^2, Γ, j_∞) is a nodal pseudoholomorphic sphere.

Collapsed boundary

The case $\partial D^2 \subset \Gamma$ happens exactly when $u_\infty(\partial D^2) \subset L$ is a single point. In this case, the component of $D^2 \setminus \Gamma$ adjacent to the boundary is necessarily a *punctured sphere* (and not a disc!)

- $(B^2, j_0) \not\cong (\mathbb{C}P^1 \setminus \{\infty\} = \mathbb{C}, j_0)$
- Carleman's similarity principle implies that a holomorphic disc with constant boundary is itself constant.
- When the nodal disc has no component with a boundary, the boundary corresponds to a node. We say that the bounadary has "collapsed to a node".

Stable nodal pseudoholomorphic discs

Definition

A nodal pseudoholomorphic disc is *stable* if the doubled nodal sphere is stable.

In particular this means that a constant component of $(D^2 \setminus \Gamma, j_\infty)$ has

- at least three nodes if is a (punctured) sphere; and
- at least
  - three boundary nodes,
  - one boundary node and one internal node, or
  - two internal nodes,

  if it is a (punctured) disc.

Stable nodal pseudoholomorphic discs

Remark

- In a stable nodal disc, each constant component has only finitely many automorphisms that fix the nodes. (If the nodes are supposed to be fixed point-wise, then there are even no non-trivial automorphisms.)

Stable nodal pseudoholomorphic discs

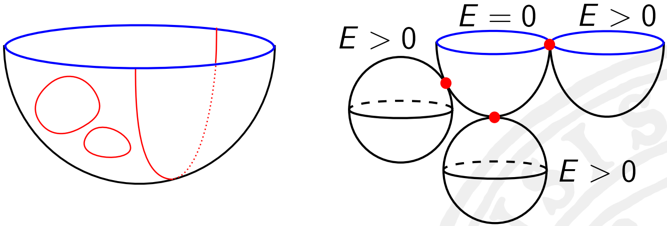
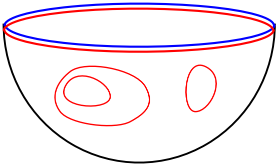


Figure: Example of a stable nodal disc: there are two components which are punctured holomorphic *discs*, while two components are punctured holomorphic spheres. OBS: $(B^2, j_0) \ncong (\mathbb{C}P^1 \setminus \{\infty\} = \mathbb{C}, j_0)$.

Stable nodal pseudoholomorphic discs



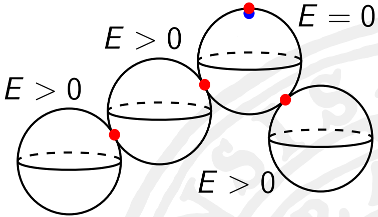


Figure: Example of a stable nodal disc: all components are punctured *spheres*, i.e. the boundary of the disc has collapsed to a node (of the double). OBS: $(B^2, j_0) \ncong (\mathbb{C}P^1 \setminus \{\infty\} = \mathbb{C}, j_0)$.

Gromov's Compactness Theorem

Theorem (Gromov [Gro85] and Frauenfelder [Fra08])

Assume that $0 < E(u_i) \leq C$ *is uniformly bounded. After passing to a subsequence, we may assume that there exists either:*

1. *A sequence* $\phi_i \in \mathrm{Aut}(D^2)$ *of reparametrisations that makes* $\|d(u_i \circ \phi_i)\|$ *uniformly bounded, and the subsequence* $\{u_i \circ \phi_i\}$ *is* C^∞*-convergent to a* J*-holomorphic disc* u_∞*.*
2. *A stable nodal pseudoholomorphic disc* u_∞ *whose double has* <u>*at least two*</u> *non-constant components, and reparametrisations* ϕ_i*, such that:*
   - $(\phi_i)^* j$ *is a sequence of complex structures on* D^2 *which* C^∞_{loc}*-converges to the complex structure* j_∞ *on the nodal disc;*
   - $u_i \circ \phi_i$ *converges uniformly to* u_∞ *and* C^∞_{loc}*-converges on* $\mathbb{C}P^1 \setminus \Gamma$ *to* u_∞*.*

Example of Gromov's compactness

Consider the sequence of biholomorphisms in $\mathrm{Aut}(\mathbb{C}P^1)$ of the form

$$z \mapsto \frac{z}{tz+1} = \frac{1}{t+z^{-1}}, \;\; t \in [0,+\infty),$$

determined uniquely by the property

$$0 \mapsto 0, \;\; 1 \mapsto \frac{1}{1+t}, \;\; \infty \mapsto 1/t.$$

Hence these automorphism fix $\mathbb{R}P^1$ and restrict to automorphisms of either disc (hemisphere) $\{\pm\mathfrak{Im} z \geq 0\} \subset \mathbb{C}P^1$.

Example of Gromov's compactness

The elements

$$z \mapsto \frac{z}{tz+1} = \frac{1}{t+z^{-1}}, \ \ t \in [0,+\infty),$$

in $\mathrm{Aut}(\mathbb{C}P^1)$ will be used to act on the holomorhic disc

$$(\{\mathfrak{Im} z \geq 0\}, \mathbb{R}P^1) \cong (D^2, \partial D^2) \to (\mathbb{C}P^1, \mathbb{R}P^1),$$
$$z \mapsto z^2.$$

Remark

The map is degree zero when restricted to the boundary, since it only covers the positive real part

$$\{\mathfrak{Re} z \geq 0\} \subset \mathbb{R}P^1.$$

Example of Gromov's compactness

The composition of the disc $z \mapsto z^2$ with the automorphism $\frac{z}{tz+1}$ yields a sequence of pseudoholomorphic discs

$$(\{\mathfrak{Im} z \geq 0\}, \mathbb{R}P^1) \cong (D^2, \partial D^2) \to (\mathbb{C}P^1, \mathbb{R}P^1),$$
$$z \mapsto \frac{z^2}{tz^2+1} = \frac{1}{t+z^{-2}}, \;\; t \in [0, +\infty)$$

which

- takes the boundary $\mathbb{R}P^1 \to \mathbb{R}P^1 \cap \{\mathfrak{Re} z \in [0, 1/t]\}$, and
- and which takes $0 \mapsto 0$, $i\sqrt{t} \mapsto \infty, \infty \mapsto 1/t$.

Example of Gromov's compactness

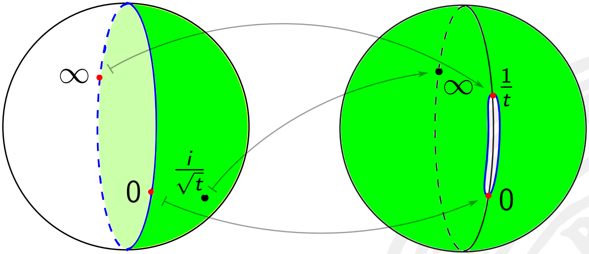


Figure: As $t \to +\infty$ the discs converge to a nodal disc which consists of a single component which is a sphere, i.e. the boundary collapses to a node (of the double).

Example of Gromov's compactness

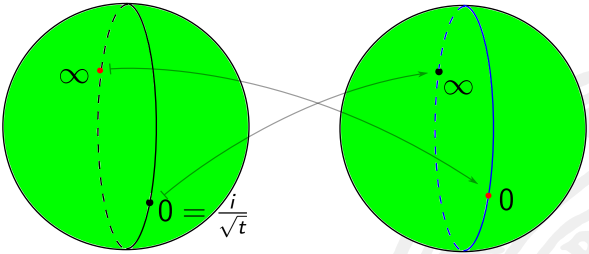


Figure: As $t \to +\infty$ the discs converge to a nodal disc which consists of a single component which is a nodal sphere of one component. I.e. the boundary collapses to a node (of the double), which is mapped to the point $0 \in \mathbb{R}P^1$.

Example of Gromov's compactness

Gromov's compactness theorem for discs implies that the moduli space of the above discs, modulo reparametrisation, is a two-dimensional manifold with boundary.

- Indeed, the Maslov index is twice the Maslov index of the hemisphere, i.e.

$$\mu = 2c_1^{T\mathbb{C}P^1}[\mathbb{C}P^1] = 4$$

  and thus $v\dim = 1 - 3 + 4 = 2$.
- The sphere with a node (collapsed boundary) which maps to $\mathbb{R}P^1$ is a part of the boundary stratum.

Exercise

Give a description of the entire moduli space and its boundary.

The case of $\mathbb{R}P^n \subset \mathbb{C}P^n$

A real line

$$\mathbb{C}P^1 \to \mathbb{C}P^n,$$
$$[x_1 : x_2] \mapsto x_1 \cdot P_1 + x_2 \cdot P_2, \quad P_1, P_2 \in \mathbb{R}P^n$$

splits into two holomorphic discs with boundary on $\mathbb{R}P^n$.

The case of $\mathbb{R}P^n \subset \mathbb{C}P^n$

A real line

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splits into two holomorphic discs with boundary on $\mathbb{R}P^n$. Last time we computed:

- The Maslov index of the disc is equal to the first chern class of the double (i.e. the complex line)
$$\mu = c_1^{T\mathbb{C}P^n}(L) = n + 1.$$
- The discs thus live in a moduli space of
$$v \dim = n - 3 + n + 1 = 2n - 2$$
after taking the quotient by the three-dimensional group $\mathrm{Aut}(D^2)$.

Real lines

There is a unique real J_0-holomorphic line $\mathbb{C}P^1 \to \mathbb{C}P^n$ through every pair of points on $\mathbb{R}P^n$. Each real line splits into *two* J_0-holomorphic discs with coinciding boundaries, equal to a *real* line $\mathbb{R}P^1 \subset \mathbb{R}P^n$.

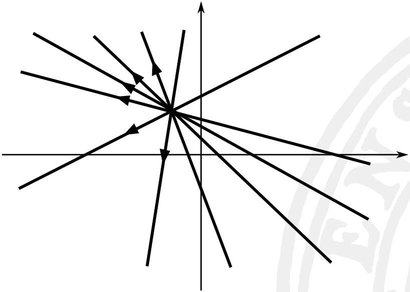


Figure: Real lines $\mathbb{R}P^1 \subset \mathbb{R}P^2 \subset \mathbb{C}P^2$. Each *oriented* line is the boundary of a unique holomorphic disc of Maslov class $\mu = 3$.

The case of $\mathbb{R}P^n \subset \mathbb{C}P^n$

Consider the long exact sequence

$$\begin{array}{ccccccccc} H_2(\mathbb{R}P^n) & \overset{0}{\to} & H_2(\mathbb{C}P^n) & \to & H_2(\mathbb{C}P^n, \mathbb{R}P^n) & \to & H_1(\mathbb{R}P^1) & \to & H_1(\mathbb{C}P^1) = 0 \\ \| & & \| & & & & \| & & \| \\ \mathbb{Z}_2 & & \mathbb{Z} \cdot L & & & & \mathbb{Z}_2 & & 0 \end{array}$$

of homology groups. Hence

$$\beta \in H_2(\mathbb{C}P^n, \mathbb{R}P^n) = \mathbb{Z} \cdot \beta, \;\; 2\beta = L$$

The case of $\mathbb{R}P^n \subset \mathbb{C}P^n$

The relative second homology is generated by "half a line":

$$\beta \in H_2(\mathbb{C}P^n, \mathbb{R}P^n) = \mathbb{Z} \cdot \beta, \ \ 2\beta = L.$$

Since the space of holomorphic lines (holomorphic spheres in class L) can be classified by positivity of intersection [see Lecture III]. Since hol. discs in class β can be doubled to a hol. sphere in class L by a standard application of a *Schwarz reflection* we obtain:

Proposition

All holomorphic discs in class β are "half complex lines".

The case of $\mathbb{R}P^n \subset \mathbb{C}P^n$

- The discs are in bijection with *oriented* real lines in the class
- The space of oriented real lines is equal to $\mathcal{M}_{J_0}(\beta) = Gr_2^+(\mathbb{R}^{n+1})$ (Grassmannian of *oriented* two-planes).
- The space of oriented real lines through a point in $\mathbb{R}P^n$ is equal to $\mathcal{M}_{J_0}(\beta; \mathrm{pt}) = S^{n-1}$.
- The bundle of boundary points of the discs that pass through pt is equal to

$$S^1 \to \tilde{\mathcal{M}}_{J_0}(\beta; \mathrm{pt}) \to \mathcal{M}_{J_0}(\beta)$$

  and is of dimension n. (In fact $\tilde{\mathcal{M}}_{J_0}(\beta; \mathrm{pt}) = S^1 \times \mathcal{M}_{J_0}(\beta)$ is trivial in this case.)

The case of $\mathbb{R}P^n \subset \mathbb{C}P^n$

Similarly to the case of the uniruledness of $\mathbb{C}P^1$, the existence of a cobordism between moduli spaces with evaluation maps shows that

Theorem

For an arbitrary generic tame almost complex structure J on $\mathbb{C}P^n$, there exists a J-holomorphic disc in class β that passes through two any pair $\mathrm{pt}, \mathrm{pt}' \in \mathbb{R}P^n$ of distinct points.

The case of $\mathbb{R}P^n \subset \mathbb{C}P^n$

To produce a compact cobordism obtained by considering a one-parameter family of almost complex structures that interpolates between the standard J_0 and the arbitrary tame J, we need *Gromov's compactness theorem*. (Non-compact cobordisms with boundary are not so useful...)
Gromov's compactness applies since:

The symplectic area of the discs

$$\int_\beta \omega_{\mathsf{FS}} = \frac{1}{2}\int_L \omega_{\mathsf{FS}} = \pi/2$$

is *minimal* among discs of positive symplectic area.(And there are no spheres in this class).

The case of $\mathbb{R}P^n \subset \mathbb{C}P^n$

However, at one point the cobordism argument needs to be modified:

- The evaluation map ev $\mathcal{M}_J(\beta; \mathrm{pt}) \to \mathbb{R}P^n$ takes values in a non-orientable manifold if n is even. In addition the degree is $0 \in \mathbb{Z}_2$ when $n \geq 2$.

This can be amended lifting the evaluation map to the universal two-fold cover $S^n \to \mathbb{R}P^n$; the *lift* has degree $1 \in \mathbb{Z}_2$.

Coherent orientations

- **Recall:** In the case of the moduli space of pseudoholomorphic spheres, the moduli space has itself an almost complex structure, and is thus naturally an *oriented* manifold (whenever transversality is achieved)!
- This is obviously *not* the case for the space of discs with a Lagrangian boundary condition.

Example

Take the moduli space of discs of vanishing symplectic area (i.e. constant discs) with boundary on $L = \mathbb{R}P^{2k} \subset \mathbb{C}P^{2k}$:

$$\left\{ u\colon (D^2, \partial D^2) \to (X, L); \int_u \omega = 0 \right\} \cong L.$$

Coherent orientations

Remark

In the above case:

- the Maslov class vanishes, and hence index $= n + 0 = \dim L$,
- transversality holds (since the linearisation of $\overline{\partial}_J$ is easy to compute at a constant u – it is the Cauchy–Riemann operator on a trivial complex bundle – transversality is easy to check),
- since $\mathrm{Aut}(D^2)$ does not act freely: taking the quotient by reparam. does not make sense in this case. ☡.

Coherent orientations

Answer is given by Fukaya–Ohta–Ono–Oh [FOOO09, Chapter 8].

Theorem

A spin structure on L, i.e.

- *the choice of a trivialisation of TL (when $\dim L \leq 2$: replace TL by $TL \oplus \mathbb{R}^2$, i.e. stable trivalisation) along the one-skeleton of L (in particular, this orients L); which moreover*
- *extends over the two-skeleton of L,*

induces an orientation of the moduli space of psh. discs with boundary on L. Moreover:

- *Reversing the orientation of L reverses the orientation of the moduli spaces.*
- *Changing the spin-structure along $u(\partial D^2) \subset L$ reverses the orientation of the moduli space at the point u.*

The case of $\mathbb{C}P^1$

- **Recall:** In the previous lecture we saw that the tori $\mu^{-1}(a) \subset \mathbb{C}P^1$ bounds precisely two holomorphic discs of Maslov index two.
- Next we generalise these disc counts to certain torus fibres of the momentum maps $\boldsymbol{\mu}$ on $(\mathbb{C}P^1)^n$ and $\mathbb{C}P^n$.

A basis of holomorphic discs

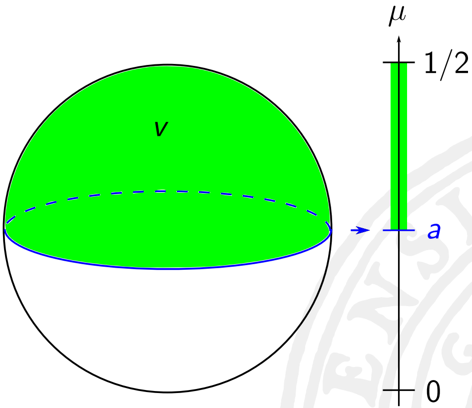


Figure: A pseudoholomorphic disc v inside $\mathbb{C}P^1$ with Maslov index two with boundary on a fiber of the momentum map μ.

A basis of holomorphic discs

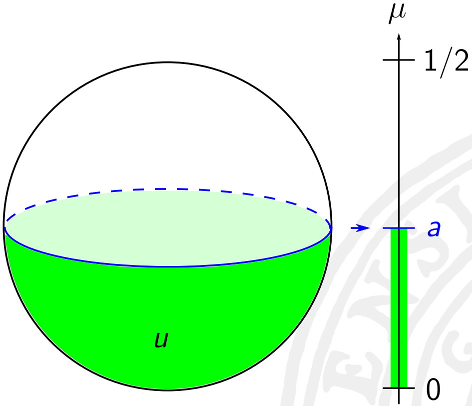


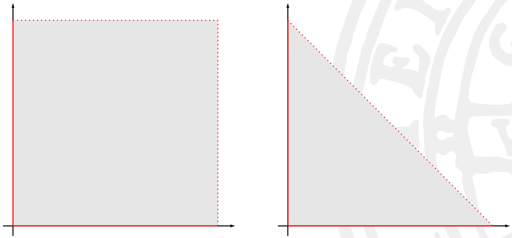
Figure: A pseudoholomorphic disc u inside $\mathbb{C}P^1$ with Maslov index two with boundary on a fiber of the momentum map μ.

In the affine part

Observe that both

$$(\mathbb{C}P^1)^n \setminus \bigcup_{i=1}^{n} \{z_i = \infty\} \text{ and } \mathbb{C}P^n \setminus \mathbb{C}P^n_\infty$$

are biholomorphic to the affine plane $\mathbb{C}^n$. However, the symplectic forms are very different, which is exhibited by the images of the momentum maps:



In the affine part

Nevertheless, in the either affine plane

$$(\mathbb{C}P^1)^n \setminus \bigcup_{i=1}^{n} \{z_i = \infty\} \text{ or } \mathbb{C}P^n \setminus \mathbb{C}P^{n-1}_{\infty}$$

the Lagrangian fibres of $\boldsymbol{\mu}$ are orbits of the standard $U(1)^n$-action in the affine space, and hence of the form

$$\{\|z_1\|^2 = A_1, \ldots, \|z_n\|^2 = A_n\}$$

for *both* symplectic structures on $\mathbb{C}^n$. **Recall:**

$$\mu_i = \frac{1}{2}\frac{\|z_i\|^2}{1+\|z_i\|^2}$$

in the case of $(\mathbb{C}P^1)^n$, while in the case of $\mathbb{C}P^n$:

$$\mu_i = \frac{1}{2}\frac{\|z_i\|^2}{1+\|\mathbf{z}\|^2}$$

In $(\mathbb{C}P^1)^n$

Recall the basis

$$H_2((\mathbb{C}P^1)^n, \boldsymbol{\mu}^{-1}(\mathbf{a})) = \bigoplus_{i=1}^{n} \mathbb{Z}[u_i] \oplus \mathbb{Z}[v_i]$$

which can be represented by holomorphic discs u_i, v_i of Maslov index $\mu = 2$.

Here u_i parametrises the disc

$$\{\sqrt{A_1}\} \times \ldots \times \{\sqrt{A_{i-1}}\} \times D^2_{\sqrt{A_i}} \times \{\sqrt{A_{i+1}}\} \times \ldots \times \{\sqrt{A_n}\}$$

while v_i parametrises the disc

$$\{\sqrt{A_1}\} \times \ldots \times \{\sqrt{A_{i-1}}\} \times (\mathbb{C}P^1 \setminus B^2_{\sqrt{A_i}}) \times \{\sqrt{A_{i+1}}\} \times \ldots \times \{\sqrt{A_n}\},$$

i.e. the reflection of u_i.

In $(\mathbb{C}P^1)^n$

Lemma

The above discs u_i and v_i both have Maslov index two.

Proof.

The calculation follows from $c_1^{T\mathbb{C}P^1}[\mathbb{C}P^1] = 2$ together with the fact that the normal bundle of

$$A = \{\sqrt{A_1}\} \times \ldots \times \{\sqrt{A_{i-1}}\} \times \mathbb{C}P^1 \times \{\sqrt{A_{i+1}}\} \times \ldots \times \{\sqrt{A_n}\},$$

is a trivial complex bundle, and hence

$$c_1^{TX}(A) = c_1^{T\mathbb{C}P^1}[\mathbb{C}P^1] + 0 = 2$$

by the splitting principle for the first Chern class. □

In $(\mathbb{C}P^1)^n$

A dual basis of $[u_i]$, $[v_i]$ is represented by the divisors

$$\{z_i = 0\} \text{ and } \{z_i = \infty\}.$$

More precisely:

$$[u_i] \bullet \{z_j = 0\} = \begin{cases} 1, & i = j, \\ 0, & \text{o.w.} \end{cases}, \text{ and } [u_i] \bullet \{z_j = \infty\} = 0,$$

$$[v_i] \bullet \{z_j = \infty\} = \begin{cases} 1, & i = j, \\ 0, & \text{o.w.} \end{cases}, \text{ and } [v_i] \bullet \{z_j = 0\} = 0.$$

In $(\mathbb{C}P^1)^n$

Proposition

The Maslov class of any continuous disc in $(\mathbb{C}P^1)^n$ with boundary on $\boldsymbol{\mu}^{-1}(\mathbf{a})$ is equal to twice the intersection number with the divisor

$$\bigcup_{i=1}^{n}\{z_i = 0 \text{ or } \infty\}.$$

In particular, positivity of intersection between curves and divisors implies that

Corollary

Any holomorphic disc of Maslov index two intersects the above divisor transversely in precisely one point.

Discs under the momentum map

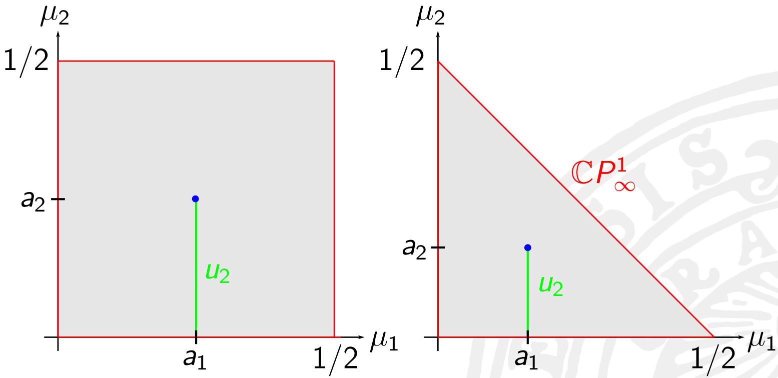
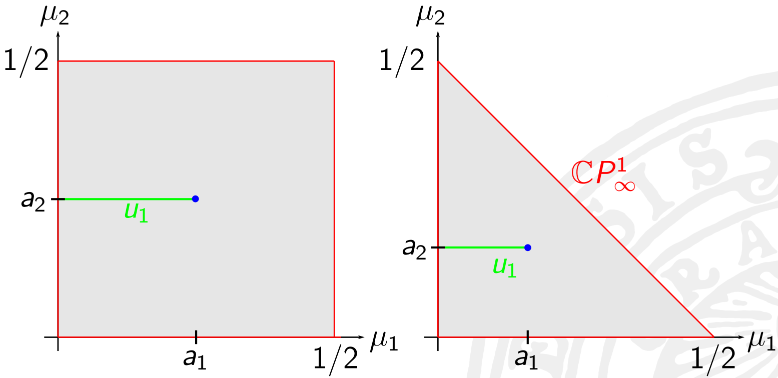


Figure: The image of the standard holomorphic disc under the momentum maps.

Discs under the momentum map



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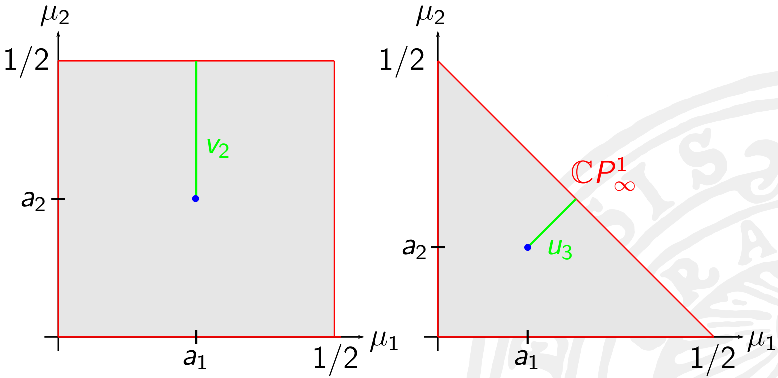

Figure: The image of the standard holomorphic disc under the momentum maps.

In $\mathbb{C}P^n$

In $\mathbb{C}P^n$ one can deduce a similar result:
Since $L_{\mathbf{a}} = \boldsymbol{\mu}^{-1}(\mathbf{a}) \subset \mathbb{C}^n$ is is null-homotopic, we get a long exact sequence

$$\begin{array}{ccccccccc} H_2(L_{\mathbf{a}}) & \overset{0}{\to} & H_2(\mathbb{C}P^n) & \to & H_2(\mathbb{C}P^n, L_{\mathbf{a}}) & \overset{\delta}{\to} & H_1(L_{\mathbf{a}}) & \to & H_1(\mathbb{C}P^1) = 0 \\ & & \| & & & & \| & & \| \\ & & \mathbb{Z} \cdot L & & & & \mathbb{Z}^n & & 0 \end{array}$$

where thus

$$H_2(\mathbb{C}P^n, L_{\mathbf{a}}) = \mathbb{Z} \cdot L \oplus \mathbb{Z}[u_1] \oplus \ldots \oplus \mathbb{Z}[u_n],$$

and u_i are holomorphic discs of Maslov index two which live in the affine part $\mathbb{C}^n$.

In $\mathbb{C}P^n$

- Recall that $\mathbb{C}P^n \setminus \mathbb{C}P^{n-1}_\infty$ is biholomorphic to the affine part of $(\mathbb{C}P^1)^n$, and that the torus $L_{\mathbf{a}} = \boldsymbol{\mu}^{-1}(\mathbf{a})$ is identified with a torus fibre under the biholomorphism. (In other words: The tori are simultaneously Lagrangian for both symplectic forms.)
- The discs $[u_i]$ with boundary on $L_{\mathbf{a}} \subset \mathbb{C}P^n$ can thus be identified with the analogous discs inside the affine part of $(\mathbb{C}P^1)^n$, and are of Maslov index two as well.

In $\mathbb{C}P^n$

- L is a sphere, but can be considered as the class of a chain with a constant boundary on $L_{\mathbf{a}}$ of Maslov index

$$2c_1^{T\mathbb{C}P^n}(L) = 2(n+1).$$

Hence the class

$$[u_{n+1}] := L - ([u_1] + \ldots + [u_n])$$

is of Maslov index *two*.

- A dual basis for the basis $[u_1], \ldots, [u_n], [u_{n+1}]$ of $H_2(\mathbb{C}P^n, L_{\mathbf{a}})$ is moreover given by the divisors

$$\{z_1 = 0\}, \ldots, \{z_n = 0\}, \mathbb{C}P^{n-1}_\infty.$$

In $\mathbb{C}P^n$

Proposition

The Maslov class of any continuous disc in $\mathbb{C}P^n$ with boundary on $\boldsymbol{\mu}^{-1}(\mathbf{a})$ is equal to twice the intersection number with the divisor

$$\mathbb{C}P^{n-1}_{\infty} \cup \bigcup_{i=1}^{n} \{z_i = 0\}.$$

In particular, positivity of intersection between curves and divisors implies that

Corollary

Any such holomorphic disc of Maslov index two intersects the divisor transversely in precisely one point.

The standard discs

Proposition

For the fibre $L_{\mathbf{a}}$ of the momentum map in either $X = (\mathbb{C}P^1)^n$ or $\mathbb{C}P^n$, there only classes in $H_2(X, L_{\mathbf{a}})$ that admit J_0-holomorphic discs of Maslov index two are the classes in the aforementioned bases. Moreover, in each such homology class, there is a unique J_0-holomorphic disc of Maslov index two that passes through a fixed point on $L_{\mathbf{a}}$.

The standard discs

Proof.

Since there is a unique intersection with the divisor by the above corollaries, after applying a suitable holomorphic symplectomorphism of (X, ω) induced by a reordering of the coordinates, we may assume that the disc intersects only the divisor $\{z_1 = 0\}$. (Here it is important to note that such a coordinate change takes the fibres of the momentum map $\boldsymbol{\mu}$ to fibres.)
In particular, the disc may be assumed to be contained inside the affine part $\mathbb{C}^n$.
Since the projection to the coordinate planes is holomorphic, and since the boundary of the disc is on a product of circles, it follows by the *maximum principle* that only the projection to the 1:st coordinate is a non-constant holomorphic disc. □

Discs under the momentum map

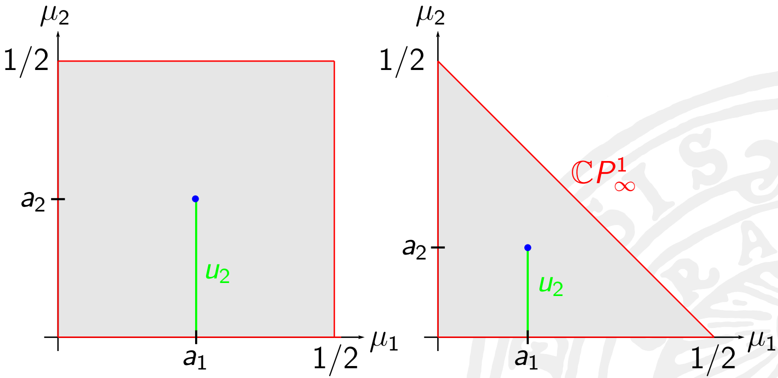

Figure: The image of the standard holomorphic disc under the momentum maps.

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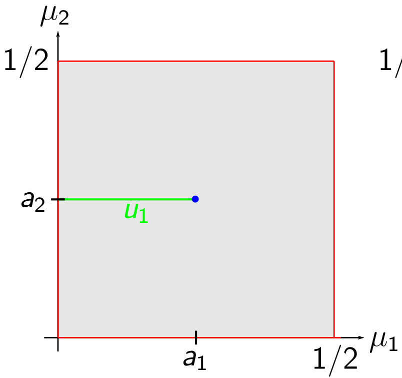

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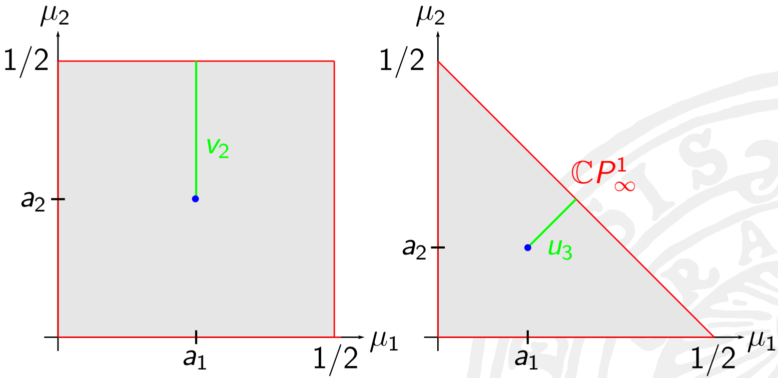

Figure: The image of the standard holomorphic disc under the momentum maps.

Monotonicity

To show that this type of uniruledness persists when changing the almost complex structure, we need to make an additional assumption to preclude non-trivial Gromov limits.

Definition

We say that a Lagrangian submanifold $L \subset (X, \omega)$ is *monotone* if there exists a constant $c \geq 0$ so that

$$\int_{[u]} \omega = c\mu^L[u]$$

is satisfied.

Monotonicity

Proposition

The so-called Clifford tori

$$L_{\mathbf{a}_0} \subset (\mathbb{C}P^1)^n, \;\; \mathbf{a}_0 = \left(\frac{1}{4}, \ldots, \frac{1}{4}\right),$$

$$L_{\mathbf{a}_0} \subset \mathbb{C}P^n, \;\; \mathbf{a}_0 = \left(\frac{1}{2(n+1)}, \ldots, \frac{1}{2(n+1)}\right),$$

are monotone with $c = \frac{\pi}{4}$ *and* $c = \frac{\pi}{2(n+1)}$*, respectively.*

Monotonicity

Exercise

- Use the Arnol'd–Liouville theorem from [Lecture IV] to compute the areas of $H_2(X, L_{\mathbf{a}_0})$ in the above cases, and deduce monotonicity for the Clifford tori $L_{\mathbf{a}_0}$.
- Show that pseudoholomorphic discs in the classes in $H_2(X, L_{\mathbf{a}_0})$ live in compact moduli spaces.

Uniruledness

Theorem

The moduli spaces of pseudoholomorphic discs with boundary on the monotone Clifford tori

$$L_{\mathbf{a}_0} \subset (\mathbb{C}P^1)^n, \ \ \mathbf{a}_0 = \left(\frac{1}{4}, \ldots, \frac{1}{4}\right),$$

$$L_{\mathbf{a}_0} \subset \mathbb{C}P^n, \ \ \mathbf{a}_0 = \left(\frac{1}{2(n+1)}, \ldots, \frac{1}{2(n+1)}\right),$$

have the property that, for any generic tame almost complex structure J, the maps $\mathrm{ev}\colon \tilde{\mathcal{M}}_J(\beta) \to L_{\mathbf{a}_0}$ is of degree σ_0 for the Lie group spin structure on $L_{\mathbf{a}_0}$ when $\beta \in H_2(X, L_{\mathbf{a}_0})$ is one of the standard basis elements above, while it is of degree 0 for any other class.

Uniruledness

Remark

Here we do not compute *which* of the two values $\sigma_0 \in \{\pm 1\}$ that the degree assumes, *however*...

- Since changing the orientation of L changes the orientation of both the domain (the moduli space) and the target space (i.e. the Lagrangian L) of the evaluation map, the sign σ_0 does not depend on the orientation of L.
- For the same reason, since there are holomorphic symplectomorphisms which act transitively on the elements of the above basis of $H_2(X, L_{\mathbf{a}})$, it follows that the sign of the evaluation map is the same for all basis elements. Here it is important to use the fact that the Lie group spin structure is preserved under linear automorphisms of the torus (even those which reverse orientation).

Uniruledness and the superpotential

In particular, we get an assignment

$$m\colon H_2(X, L_{\mathbf{a}}) \to \mathbb{Z},$$
$$\eta \mapsto \deg(\mathrm{ev}\colon \tilde{\mathcal{M}}_J(\beta) \to L_{\mathbf{a}})$$

which is defined to be zero whenever β is not a class of Maslov index two.

One can encode this invariant as the "Laurent polynomial"

$$\mathfrak{P}_{L_{\mathbf{a}}} := \sum_{\eta \in H_2(X, L_{\mathbf{a}})} m(\eta) \cdot [\delta\eta] \in \mathbb{Z}[H_1(L_{\mathbf{a}})]$$

called the "superpotential" of $L_{\mathbf{a}}$.

Uniruledness and the superpotential

Note that:

- The group ring of a free abelian group

$$\mathbb{Z}[H_1(\mathbb{T}^n)] \cong \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$$

  is the Laurent polynomial ring.
- Finiteness of the sum which defines $\mathfrak{P}_{L_a}$ is a consequence of Gromov's compactness theorem, where the assumption of monotonicity is crucial.

Uniruledness

The superpotential is a very powerful invariant that can be used for distinguishing Hamiltonian isotopy classes of monotone Lagrangian tori. Vianna showed in [Via16] that there are *infinitely* many Hamiltonian isotopy classes in these cases, which we distinguish by the above function.

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