Holomorphic Curve Theories in Symplectic Geometry
Lecture VI

Georgios Dimitroglou Rizell

Uppsala University
Goal of lecture

Today:

- Definition of Floer homology in the exact case.
- Application: Obstructs Hamiltonian displaceability.
- Moduli space of discs with boundary punctures (Associahedra).
**Take-home Message**

If a closed Lagrangian can be displaced by a Hamiltonian isotopy, then it admits a non-constant pseudoholomorphic disc.

**Figure:** The blue Lagrangian $L$ is displaceable by a Hamiltonian isotopy and bounds a holomorphic disc; the green Lagrangian which is homologically essential is not Hamiltonian displaceable.
Plan

1. Goal of lecture
2. Diplaceability
3. The maximum principle
4. The Floer complex
5. Associahedra
6. References
Displaceability

**Definition**

We say that a compact subset $C \subset (X, \omega)$ is *Hamiltonian displaceable* if there exists a Hamiltonian $H : X \times \mathbb{R} \to \mathbb{R}$ for which $C \cap \phi_H^1(C) = \emptyset$.

- Gromov showed in [Gro85] that a closed Lagrangian $L$ which is Hamiltonian displaceable must admit a $J$-holomorphic disc with boundary on $L$ for all tame $J$. Also see Chekanov’s refinement [Che98].
- Floer later refined this to a chain complex in [Flo88], whose homology is a lower bound for

$$|L \cap \phi_H^1(L)| \geq 0$$

This complex is typically impossible, or at least difficult, to define in the presence of pseudoholomorphic discs with boundary on $L$. 

Georgios Dimitroglou Rizell (Uppsala University)
Holomorphic Curve Theories in Symplectic Geometric
Displaceability

Example

- Any compact subset $C \subset (\mathbb{C}^n, \omega_0)$ is Hamiltonian displaceable; e.g. the translation $x_i \mapsto x_i + t$ is generated by the Hamiltonian $H = -y_i$.

- For curves in surfaces, the question of Hamiltonian displaceability can be solved completely by *area considerations*.

Remark

In order to deduce the existence of a $J$-holomorphic disc with boundary on a closed displaceable Lagrangian inside a non-compact symplectic manifold, one needs to control the behaviour of $J$ outside of a compact subset.
Displaceability

Why we need control at infinity:

- There are plenty of Hamiltonian isotopies, and they act on the space of tame almost complex structures:

  \[ (\phi^1_H)_* J := (D\phi^1_H)^{-1} \circ J \circ D\phi^1_H. \]

- We may use this action repeatedly to push some part of the interior of a \( J_0 \)-holomorphic curve in \( \mathbb{C}^n, n > 1 \), with boundary on a fixed Lagrangian out to \( \infty \) (make sure it leaves every compact subset).
Conditions on $J$

To gain control at infinity of a noncompact $(X, \omega)$ we will here assume that $J \in \mathcal{J}^{tame}(X, \omega)$ satisfies the following property: there exists a smooth proper function $f : X \to [-N, +\infty)$ such that:

Non-negativity of the “Levi two-form” on $J$-complex lines, i.e.

$$-dd^c f(v, Jv) \geq 0, \quad v \in T_pX,$$

holds for all $p \in f^{-1}[0, +\infty) \subset X$.

(This is a condition only on $J$ outside of some compact subset.) We will now derive a maximum-principle under these assumptions, that makes it impossible for $J$-holomorphic curves to partly escape to infinity.
Conditions on $J$
The maximum principle

Conditions on $J$

Proposition

Let $u : (\Sigma, j) \to (X, J)$ be pseudoholomorphic, and $\Sigma$ a connected (possibly open) Riemann surface whose boundary is contained inside $f^{-1}[-N, 0]$. If $f \circ u : \Sigma \to \mathbb{R}$ assumes a positive value, then $f \circ u$ is constant.

Proof (1/2).

It suffices to show that $f \circ u$ is sub-harmonic in local coordinates near a point which maps to a positive value under $f \circ u$. 
The maximum principle

Conditions on $J$

**Proposition**

Let $u: (\Sigma, j) \rightarrow (X, J)$ be pseudoholomorphic, and $\Sigma$ a connected (possibly open) Riemann surface whose boundary is contained inside $f^{-1}[-N, 0]$. If $f \circ u: \Sigma \rightarrow \mathbb{R}$ assumes a positive value, then $f \circ u$ is constant.

**Proof (2/2).**

For a loc. def. $J$-holomorphic map $u: (B^2_\epsilon, j_0) \rightarrow (X, J)$ we have

$$4 \Delta (f \circ u) dx \wedge dy = 2i \partial \overline{\partial} (f \circ u) = -dd^c (f \circ u)$$

(see [Lecture I]). The assumption on the Levi form gives

$$4 \Delta (f \circ u) dx \wedge dy (\partial_x, j_0 \partial_y) \geq 0$$
Conditions on $J$

- The above is clearly satisfied for $\mathbb{C}^n$: for $J = J_0$ and $f = \|z\|^2/4$ we even have
  $$-dd^c f = \omega_0.$$ 

- More generally: when $(X, \omega)$ is symplectomorphic to a “half symplectisation”
  $$((0, +\infty)_t \times Y, d(e^t \alpha))$$
  outside of a compact subset (where $(Y, \alpha)$ is a closed contact manifold) one can take $f = e^t$ and $J$ to be cylindrical in the same subset. (See next slide.)

Example

The latter condition is satisfied for $(T^*M, d\theta_M)$ for $M$ closed; take e.g. the complement of any fibre-wise convex smooth domain.
Cylindrical almost complex structures

**Definition**

An almost complex structure $J \in J^{\text{comp}}$ on the symplectisation $(\mathbb{R}_t \times Y, d(e^t\alpha))$ is said to be **cylindrical** if

- $J$ is $t$-invariant,
- $J(\ker \alpha \cap TY) = \ker \alpha \cap TY$ (i.e. $J$ preserves the contact planes $\ker \alpha \subset TY$)
- $J\partial_t \in TY$ and satisfies

$$d\alpha(J\partial_t, \cdot) = 0 \text{ and } \alpha(J\partial_t) = 1$$

(i.e. $J\partial_t$ is the Reeb vector field on $Y$ defined by $\alpha$).
Cylindrical almost complex structures

An almost complex structure $J \in J^{comp}$ on the symplectisation

$$(\mathbb{R}_t \times Y, d(e^t \alpha))$$

which is cylindrical satisfies

$$-dd^c e^t = d(e^t \alpha) = \omega.$$ 

In other words, if $(X, \omega)$ can be equipped with an almost complex structure which is “cylindrical outside of a compact subset,” then the aforementioned maximum principle is satisfied.
The Floer complex: The vector space

For two closed Lagrangians $L_0, L_1 \subset (X, \omega)$ that intersects transversely $L_0 \pitchfork L_1$ (the intersection is a finite number of points) we define:

$$CF(L_0, L_1) = \bigoplus_{x \in L_0 \cap L_1} \mathbb{F} \cdot x$$

- The above canonical basis is graded under the presence of suitable additional data. (To be dealt with later.)
- The coefficients are taken to be in a suitable field $\mathbb{F}$; in general one needs the “Novikov field”, i.e. power series of the form

$$\Lambda^R := \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i}, \ a_i \in \mathbb{R}, \lambda_i \in \mathbb{R}, \lim_{i \to +\infty} \lambda_i = +\infty \right\}.$$
The Floer complex: The vector space

For two closed Lagrangians $L_0, L_1 \subset (X, \omega)$ that intersects transversely $L_0 \pitchfork L_1$ (the intersection is a finite number of points) we define:

$$CF(L_0, L_1) = \bigoplus_{x \in L_0 \cap L_1} F \cdot x$$

- We can always take $F = \mathbb{Z}_2$ when both $L_i$ are exact Lagrangians.
- If, in addition, $L_i$ are equipped with spin structures we can take $F = \mathbb{Q}$ (or even $\mathbb{Z}$).
The Floer complex: The vector space

From now on we will assume that $L_i$ are closed, connected, exact Lagrangians and will take $\mathbb{F} = \mathbb{Z}_2$. In particular $(X, \omega) = (X, d\lambda)$ and $\lambda|_{T L_i} = df_i$ for primitives $f_i : L_i \to \mathbb{R}$.

**Definition**

The action of an intersection $x \in L_0 \cap L_1$ is

$$a(x) := f_0(x) - f_1(x) \in \mathbb{R}$$

- The primitives $f_i$ of $d\lambda|_{T L_i}$ are only determined by the embedding $L_i \subset (X, d\lambda)$ up to an unspecified constant.
- Likewise, only difference of action between two generators is uniquely determined by $L_i \subset X$. 
The Floer complex: The differential

The differential
\[ d : CF(L_0, L_1) \to CF(L_0, L_1) \]
is defined as follows.

- The differential \( d \) depends only on the choice \( J_t \) of a generic one-parameter family of tame almost complex structures on \((X, \omega)\).
- For \( x \in L_0 \cap L_1 \) a basis element of \( CF(L_0, L_1) \) we define

\[
d(x) = \sum_{y \in L_0 \cap L_1} \sum_{\substack{M \in \pi_0(M_{J_t}(x, y)) \\text{index } M = 1}} y
\]

where we proceed to describe the moduli space \( M_{J_t}(x, y) \).
The Floer complex: The moduli space

The “moduli space of Floer strips from $x$ to $y$”

$$\mathcal{M}_{J_t}(x, y)$$

consists of those smooth maps

$$u: ([s + it; t \in [0, 1]], \{ t = 0 \}, \{ t = 1 \}) \to (X, L_0, L_1)$$

which

- are pseudoholomorphic for the domain-dependent complex structure $J_t$ on $X$, i.e.
  $$du(\partial_t) = du(j_0 \cdot \partial_s) = J_t \cdot du(\partial_s)$$
  is satisfied.
- have finite energy $0 \leq \int_u \omega < \infty$.
- $t \mapsto u_s(t) = u(s + it)$ converge uniformly to the constant map $t \mapsto x \in L_0 \cap L_1$ (resp. $t \mapsto y$) as $s \to +\infty$ (resp. $s \to -\infty$).
The Floer complex: The moduli space

Figure: A Floer strip used in the differential. The input is $x \in L_0 \cap L_1$ and the output is $y \in L_0 \cap L_1$. 
The Floer complex: The moduli space

**Figure:** A strip whose symplectic area is infinite; it is given as the universal cover of a holomorphic annulus with boundary on two disjoint Lagrangians.
The Floer complex: The moduli space

- The reason why we need a $t$-dependence is to achieve transversality, so that the moduli spaces of Floer strips becomes a smooth manifold of the expected dimension i.e. the Fredholm index.

- In [EES07] Ekholm–Etnyre–Sullivan managed to get rid of this assumption in the exact case (for a carefully chosen almost complex structure).
The Floer complex: The moduli space

- The requirement of finite symplectic area (energy) together with holomorphy gives an a priori uniform convergence of the functions $t \mapsto u_s(t) = u(s + it)$ to constants as $s \to \pm \infty$.
- One can get rid of the domain-dependence of the Cauchy–Riemann equation by considering instead $J$-holomorphic sections over

$$X \times \{s + it; \ t \in [0, 1]\} \to \{s + it; \ t \in [0, 1]\}$$

with $J_{(pt, s, t)} = J_t(pt) \oplus j_0$. 
The Floer complex: The moduli space

By Stokes’ theorem, any \( u \in \mathcal{M}_{J_t}(x, y) \) satisfies

\[
\int_u \omega = \int_u d\lambda = a(x) - a(y)
\]

(Exactness is of course crucial here!) On the other hand, recall that pseudoholomorphic maps satisfy

\[
\int_u \omega \geq 0
\]

with equality if and only if \( u \) is constant.
The Floer complex: The moduli space

- The moduli space of Floer strips has a linearisation which is elliptic, and hence there is a well-defined Fredholm index.
- The index of a constant strip in $\mathcal{M}_{J_t}(x, x)$ which maps into a double point $x \in L_0 \cap L_1$ is equal to zero.
- In the exact case, only the constant strip lives in the moduli space $\mathcal{M}_{J_t}(x, x)$; the formula for its symplectic area in terms of the asymptotics yields $a(x) - a(x) = 0$.
- The moduli space has a natural $\mathbb{R}$-action by reparametrisation $s \rightarrow s + s_0$ which is free unless the strip is constant (by its asymptotic properties).
The Floer complex: The differential

The differential

\[ d : CF(L_0, L_1) \rightarrow CF(L_0, L_1) \]

is defined on a basis element \( x \in L_0 \cap L_1 \) by

\[ d(x) = \sum_{y \in L_0 \cap L_1} \sum_{\substack{M \in \pi_0(M^*J_t(x,y)) \\ \text{index}M = 1}} y \]

where the area formula (Stoke’s theorem) implies that \( d(x) \) is a sum of generators action strictly lower than \( a(x) \).
The Floer complex

Theorem

Floer [Flo88] When $L_i \subset (X, d\lambda)$, $i = 0, 1$, are closed exact Lagrangian submanifolds and $J_t$ is cylindrical outside of a compact subset then

1. $d$ is well-defined;
2. $d^2(x) = 0$;
3. A compactly supported Hamiltonian isotopy $\phi^t_H$ of $(X, d\lambda)$, and choice of two-parameter family of almost complex structures $J_{s,t}$, induces a chain map

$$\Phi_{H,J_{s,t}} : CF(L_0, L_1; J_{-1,t}) \to CF(L_0, \phi^1_H(L_1); J_{1,t})$$

which induces isomorphism in homology; and
The Floer complex

Theorem

Floer [Flo88] When \( L_i \subset (X, d\lambda) \), \( i = 0, 1 \), are closed exact Lagrangian submanifolds and \( J_t \) is cylindrical outside of a compact subset then

When \( L_1 \) is obtained by perturbing \( L_0 \) to the graph of \( dg \in \Omega^1(L) \) inside a Weinstein neighbourhood \( U \subset T^*L_0 \) of \( L_0 \), then suitable choices yields an identification

\[
(C\!F(L_0, L_1), d) = (C^{Morse}(-g), \partial^{Morse}),
\]

of complexes with action filtration (R.H.S. is the Morse homology complex of \( -g : L_0 \rightarrow \mathbb{R} \) generated by \( \text{crit}(g) \)).
The Floer complex: An example

Figure: The Floer homology complex $CF(L_0, L_1)$ for $L_0 = 0_M$ the zero section in $(T^*S^1, dp \wedge d\theta)$ and $L_1 = dg$ the exterior derivative of a Morse function $g : S^1 \to \mathbb{R}$ with precisely two critical points. The two holomorphic strips contribute $d(x) = y - y = 0$. 
The Floer complex

Corollary

When the homology $HF(L_0, L_1)$ is nonzero, then $L_0$ intersects any image of $L_1$ under any compactly supported Hamiltonian isotopy.

Since the Morse homology always is non-zero, it follows that a closed exact Lagrangian is not Hamiltonian displaceable.

Instead of proving isomorphism with Morse homology, the next lecture we will mimic the proof of the fact that “Morse homology is nonvanishing” to give a condition for when the Floer homology is nonvanishing.
Proof that $d$ is well-def.

This follows from a version of Gromov’s compactness theorem that we will formulate later:

- Since $L_i$ are exact, the components of the moduli spaces $\mathcal{M}_{J_t}(x, y)$ which consist of solutions of index $= 1$ become compact zero-dimensional manifold after taking quotients by automorphisms (translations).

- For compactness, the fact that the energy of solutions in $\mathcal{M}_{J_t}(x, y)$ are automatically bounded, is crucial. (Gromov’s compactness needs an assumption of energy bound!)
The Floer complex

Proof that $d^2 = 0$.

This follows from a compactness argument together with a gluing argument, that we will postpone until next time.

Roughly:

- Two strips $u, v$ can be glued to a new solution $u \# v$ if their asymptotics match;
- The Fredholm index is additive under this operation i.e. $\text{index}(u \# v) = \text{index}(u) + \text{index}(v)$;
- After taking a quotient by reparam. we obtain a compact $1 + 1 - 1 = 1$-dimensional manifold; A compact one-dimensional manifold has an even number of boundary points!
The Floer complex

Proof of invariance (1/3).

Today we define the chain map

$$\Phi_{H,J,s,t} : CF(L_0, L_1; J_{-1}, t) \to CF(L_0, \phi_H^t(L_1); J_1, t).$$

The chain-map property, as well as the property of being invertible in homology, will be postponed until next time.

We assume that

- $J_{s,t}$ is constant inside $\{|s| \geq 1\}$;
- $\phi_H^s = \text{Id}_X$ for $s \leq 0$, and $\phi_H^s = \phi_H^1$ for $s \geq 1$. 
The Floer complex

Proof of invariance (2/3).

Consider a moduli space $\mathcal{M}_{J,s,t}(x, y)$ is defined similarly as before; It consists of smooth maps

$$u: (\{s + it; t \in [0, 1]\}, \{t = 0\}) \to (X, L_0)$$

which

- satisfy the boundary condition $u(s + i) \in \phi_{H}^{-s}(L_1)$
- satisfies the Cauchy–Riemann equation

$$du(\partial_t) = J_{-s,t}du(\partial_s)$$

- The asymptotic at $s = +\infty$ (resp. $s = -\infty$) is $x \in L_0 \cap L_1$ (resp. $y \in L_0 \cap \phi_{H}^{1}(L_1)$).
The Floer complex: The moduli space

**Figure:** A strip used in the continuation map, the input is \( x \in L_0 \cap L_1 \) while the output is \( y \in L_0 \cap \phi_H^1(L_1) \).
The Floer complex

Proof of invariance (3/3).

We finally define

\[ \Phi_{H,Js,t}(x) = \sum_{y \in L_0 \cap \phi^1_H(L_1)} \sum_{\substack{M \in \pi_0(\mathcal{M}_{J_s,t}(x,y)) \\text{index}M=0}} y \]

on any basis element \( x \in L_0 \cap L_1 \), where \( y \in L_0 \cap \phi^1_H(L_1) \).

Note that the components of the above moduli spaces that are counted all have expected dimension zero. (In the definition of the differential, the components had expected dimension one.)
The Floer complex

**Definition**

The map

\[ \Phi_{H,J_s,t} : CF(L_0, L_1; J_{-1}) \to CF(L_0, \phi_H^1(L_1); J_1) \]

between Floer complexes induced by the Hamiltonian isotopy \( \phi_H^t \) and path of almost complex structures \( J_{s,t} \) is called a *continuation map*.

**Exercise**

The continuation map induced by \( H \equiv 0 \) and \( J_{s,t} \equiv J_t \) is the *identity map*.
Associahedra

For more operations in Floer homology, we need to introduce the configurations space of boundary punctures on $D^2$.

**Recall:**

- There is a unique simply connected Riemann surface with boundary by the *uniformisation theorem*: $(D^2, j_0)$.
- The real Möbius transformations $\text{Aut}(D^2)$ act transitively on triples of distinct cyclically ordered points in $\partial D^2$. (Any element in $\text{Aut}(D^2)$ is determined uniquely by its image of such a triple.)
Associahedra

Set $p_0 = -1 \in \partial D^2$. The space of configurations of $d \geq 2$ additional distinct points

$$\iota : \{p_1, \ldots, p_d\} \hookrightarrow \partial D^2 \setminus \{p_0\},$$

called *boundary punctures*, which are required to

- respect the order on $(-\pi, \pi) = \partial D^2 \setminus \{p_0\}$, i.e.

$$\iota(p_1) < \ldots < \iota(p_d),$$

and where

- we identify two such configurations that differ by an element in $\text{Aut}(D^2)$ (which thus fixes $p_0$),

will be denoted by

$$\mathcal{R}_d = \text{Emb}^{ord}(\{p_1, \ldots, p_d\}, \partial D^2 \setminus \{p_0\}) / \sim$$
Associahedra

Since $\text{Aut}(D^2)$ acts transitively on three cyclically ordered distinct points, one deduces that

$$\mathcal{R}_d \cong \mathbb{R}^{d-2}, \quad d \geq 2.$$ 

We can e.g. pick the unique representatives which satisfy

$$p_1 \mapsto 1 \text{ and } p_2 \mapsto \sqrt{-1}.$$ 

BUT, there are of course many other choices: $\text{Aut}(D^2)$ is a non-compact group.

- Non-compactness of the space is a result of the fact that, in a sequence $\{r_i \in \mathcal{R}_n\}_i$, two or more points can collide.
Associahedra

Assume that \( \{r_i\} \) is a sequence of representatives of elements in \( \mathcal{R}_d \) which diverge.

After acting by \( \phi_i \in \text{Aut}(D^2) \) with \( \phi_i(-1) = -1 \), we obtain a possibly different divergent sequence.

For a suitable choice of sequence \( \phi_i \circ r_i \) of representatives, we may assume that a subsequence converges to an element in \( \mathcal{R}_{d'} \) for \( 2 \leq d' \leq d \). (Roughly speaking: the automorphisms \( \phi_i \) can be used to separate the limiting clusters of points, making sure that at least three clusters form.)
Associahedra

There are many different choices of reparametrisations which can be used to extract a limit configuration. Here is one example:

Exercise

For any \( j_0 \neq 0 \) (resp. \( j_0 = 0 \)), there is a sequence \( \{ \phi_i \} \), where

\[
\phi_i(-1) = -1,
\]

under which

- \( \phi_i \circ r_i(p_{j_0}) = 1 \) (resp. \( \phi_i \circ r_i(p_{j_0}) = -1 \)),
- no sequence \( \{ \phi_i \circ r_i(p_j) \} \) for \( j \neq j_0 \) has 1 (resp. \( -1 \)) as a limit point,
- there are at least three distinct limit points.

I.e. we can “zoom in” on the \( j_0 \):th boundary puncture in the limit, and extract an element in \( \mathcal{R}_{d'+1} \) in which \( p_{j_0} \) is not forming a cluster of colliding points.
Theorem (Devadoss [Dev99])

For a suitable metric on $\mathcal{R}_d$ there is a natural compactification

$$\overline{\mathcal{R}}_d \cong K_d$$

by adding “nodal configurations”, where $K_d$ denotes the $d-2$-dimensional associahedron (a.k.a. Stasheff polyhedron). Moreover, the boundary faces of the polyhedron $K_d = \overline{\mathcal{R}}_d$ of dimension $\dim \overline{\mathcal{R}}_d - 1 = d - 3$ is given by the products

$$K_{d'} \times K_{d''} = \overline{\mathcal{R}}_{d'} \times \overline{\mathcal{R}}_{d''}$$

with $d' + d'' = d + 1$, where these products naturally correspond to nodal configurations.
Associahedra

The metric on $K_d$ gives the same notion of convergence as in Gromov’s compactness theorem:

- There exists a nodal disc in the “Gromov sense”, whose every disc component has at least three boundary points which are either *nodes* or *boundary punctures*.
  - All nodes and boundary punctures are distinct.
- There exists a sequence of diffeomorphisms $\phi_i$ of $D^2$ which identifies $(D^2, j_0)$ with $(D^2, \Gamma, j_i)$, and where $(D^2, j_i)$ together and its boundary punctures converge in $C^\infty_{loc}$ to the nodal disc away from the curves $\Gamma$, and which respects the position of the boundary punctures.
The Floer complex: The moduli space

Figure: A nodal disc with boundary punctures which lives in $\mathcal{R}_3 \times \mathcal{R}_3 \times \mathcal{R}_2$. Note that each component has at least three boundary points which are either nodes or boundary punctures. In addition, nodes and boundary punctures are disjoint.
Associahedra

\[ K_2 : \quad 0 \rightarrow 1 \rightarrow 2 \]

\[ K_3 : \quad 0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \]

**Figure:** The assoceahedra \( K_2 = \overline{R}_2 = \{\star\} \) and \( K_3 = \overline{R}_3 = I \). The boundary vertices correspond to possible decompositions of the \( d \)-ary multiplication \( d \cdot (d-1) \cdot \ldots \cdot 1 \) into sequences of binary operations.
Figure: The associahedron $K_4 = \overline{R}_4$ is the pentagon. The boundary vertices corresponds to possible decompositions of the 4-ary multiplication $4 \cdot 3 \cdot 2 \cdot 1$ into sequences of binary operations.
Associahedra

Figure: The space $K_5$. Source: Wikipedia
Yu. V. Chekanov.
Lagrangian intersections, symplectic energy, and areas of holomorphic curves.

S. L. Devadoss.
Tessellations of moduli spaces and the mosaic operad.

T. Ekholm, John Etnyre, and Michael Sullivan.
Legendrian contact homology in $P \times \mathbb{R}$.

A. Floer.
Morse theory for Lagrangian intersections.

M. Gromov.
Pseudoholomorphic curves in symplectic manifolds.