UPPSALA UNIVERSITET
Matematiska institutionen
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Exercises 2
Logik II
2008

Deadline:	2008-11-04 (at 10.00)
Hjälpmedel:	Course material or any book.
Maximal poäng:	None
Instruktioner:	See below.

Submit at least neatly handwritten (and individually prepared) solutions of the following problems. You may use any book as help, but state and refer clearly to which reference (and what part of that reference you use) if you do so. All numbered references below are to chapter I of the course literature unless otherwise stated. It is a part of all problems to decide how detailed solution you should hand in. (A guideline is to think that the solution should be clear to all participants of the course.) Ask me if you have any questions!

Hand in the problems to me personally or in my mailbox before deadline. No solutions handed in after the deadline will be considered unless you have an agreement with me.

1. Let $\mathcal{A}$ and $\mathcal{B}$ be L-structures such that $\mathcal{A} \equiv_L \mathcal{B}$. Suppose that for each $a \in \mathcal{A}$ there is a closed term t in L such that $t^{\mathcal{A}} = a$. Show that $\mathcal{A}$ can be embedded in $\mathcal{B}$. Suppose that we also have that for each element $b \in \mathcal{B}$ there is a closed term s in L such that $s^{\mathcal{B}} = b$. Then show that $\mathcal{A} \cong \mathcal{B}$.

2. Let $\mathcal{A}$ and $\mathcal{B}$ be L-structures, where L has at least one constant symbol. Let $\mathcal{A}_0$ and $\mathcal{B}_0$ be their least substructures. (**Pause**: Why do $\mathcal{A}_0$ and $\mathcal{B}_0$ exist?)
   (a) Show that if $\mathcal{A}_0 \cong \mathcal{B}_0$ if and only if $\mathcal{A}_0 \equiv_L \mathcal{B}_0$.
   (b) Show that if $\mathcal{A} \equiv_L \mathcal{B}$ then $\mathcal{A}_0 \cong \mathcal{B}_0$.

3. Let $\mathcal{A}$ be an L-structure and let $X \subseteq |\mathcal{A}|$. What was the minimal requirement on X to be a substructure of $\mathcal{A}$?

4. Show that a theory T is complete if and only if all models of T are elementarily equivalent.

5. Let DLO denote (a set of non-logical axioms for) the theory of *dense linear orderings without endpoints* over the language $L = \{<\}$. Write down a set of axioms for DLO. (Hint: Five axioms are sufficient.) That is, $\mathcal{A} \models DLO$ if and only if $\mathcal{A}$ is a dense linear ordering without endpoints.
   (a) Show that all countable models of DLO are isomorphic.
   (b) Show that DLO is complete.
   (c) Show that if $\mathcal{A} \models DLO$ and $\mathcal{B} \models DLO$ then $\mathcal{A} \equiv_L \mathcal{B}$.
   (d) Give examples of non-isomorphic models $\mathcal{A}$ and $\mathcal{B}$ of same cardinality.

6. Let $L' = \{<, \bar{a_i} : i \in \mathbb{N}\}$ and let DLO' be the theory (over L') consisting of the axioms DLO together with axioms $\{\bar{a_i} < \bar{a_{i+1}} : i \in \mathbb{N}\}$. Correspondingly, let $L_n = \{<, \bar{a_i} : i \leq n\}$ and DLO(n) be the theory (over L_n) consisting of the axioms DLO together with axioms $\{\bar{a_i} < \bar{a_{i+1}} : i < n\}$.
   (a) Show that DLO' is consistent.
   (b) Show that DLO' has two non-isomorphic countable models.
   (c) Show that DLO(n) is complete.
   (d) Show that DLO' is complete.

7. As in problem 5, suppose that you have written down a set of axioms for LO(least), the theory of linear orderings with a least element. Is LO(least) an existential theory?

(var god vänd!)

8. (This is probably a difficult one.) Let ACF(p) denote (a set of non-logical axioms for) the theory of *algebraically closed fields of characteristic* p over the language $L = \{+, *, 0, 1\}$. **Fact**: We know that ACF(p) is a complete and decidable theory! We have the following lemma (corresponding to the one in the proof of the theorem for fields by Robinson which we did in some lecture):

   Lemma: Let φ be a closed formula in L. Let $\mathbb{C}$ be the (algebraically closed) field of complex numbers. Then the following are equivalent:

   (a) $\mathbb{C} \models \varphi$.

   (b) $ACF(0) \models \varphi$.

   (c) $ACF(p) \models \varphi$ for all $p \geq p_0$, some $p_0 \in \mathbb{N}$.

   (d) $ACF(p) \models \varphi$ for infinitely many $p \in \mathbb{N}$.

   Suppose that $f : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial function. Use the lemma to show that if f is injective then f is surjective.