Notes on Domain Theory

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1 Introduction

Domain theory is by now a rather large mathematical theory with applications in many areas such as denotational semantics, type theory and recursive mathematics. For a short course on the mathematical theory of domains it is therefore necessary to be selective and to choose a particular perspective. We have chosen to see domain theory as a theory of approximation and a theory of computability via approximations and have selected our topics accordingly.

In Section 2 we develop the basic theory of algebraic and continuous domains. By analysing a notion of approximation we arrive naturally at the axioms for domains. In Section 3 we develop the theory of domain representability. We show how a large class of structures can be given natural representations using domains. Together with an effective theory of domains this provides a uniform method to study computability on such structures, which is the topic of Section 5. The general theory of effective domains is described in Section 4. In Section 6 we discuss bifinite domains and show that the category of effective bifinite domains is cartesian closed. The various power domain constructions are briefly considered in Section 7. In the final Section 8 we barely touch on the problem of representing relations and non-continuous functions using domains.

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2 Basic theory

2.1 Approximation structures and domains

We take as our starting point the following problem. Suppose we want to compute on a possibly uncountable structure such as the field of real numbers $\mathbb{R}$. The elements of $\mathbb{R}$ are in general truly infinite objects (Cauchy sequences or Dedekind cuts) with no finite description. However, real omputations, that is computations that can in principle be performed by a digital computer or Turing machine must operate on 'concrete' objects. By an element being concrete we mean (at least) that it is finitely describable or, equivalently, coded by a natural number. In particular, the structure on which the computations are performed must be countable. Therefore it is not possible to compute directly on $\mathbb{R}$, we can at best compute on concrete approximations of elements in $\mathbb{R}$. If the approximations are such that each real number is a limit of its approximations then we can extend a computation to $\mathbb{R}$ by interpreting a computation on a real number as the 'limit' of the computations on its approximations, where such a limit exists. This is the technique used in recursive analysis, see e.g. Pour-El and Richards [35].

Is there a general method or a general class of structures which captures computations via approximations as in the example of the reals $\mathbb{R}$? We shall show that a simple analysis of the notion of approximation leads naturally to the class of structures called *algebraic domains* which has the generalisation, also natural from the approximation point of view, to *continuous domains.*

Let us consider the problem of approximation abstractly. Suppose that X is a set (or more generally a structure). To say that a set P is an approximation for X should mean that elements of P are approximations for elements of X. That is, there is a relation $\prec$, the *approximation relation*, from P to X with the intended meaning for $p \in P$ and $x \in X$,

$$p \prec x \iff \text{“}p \text{ approximates } x\text{”}.$$

We illustrate this with a few relevant examples.

Example 2.1. Let $P = \{[a, b] \colon a \leq b, a, b \in \mathbb{Q}\}$ and $X = \mathbb{R}$. Define

$$[a, b] \prec x \iff x \in [a, b].$$

Note that P consists of concrete elements in the sense that an interval $[a, b]$ is finitely describable from finite descriptions of the rational numbers a and b and the symbols "[", "]" and ",".

Example 2.2. Let X be a topological space with a topological base $\mathcal{B}$. For $B \in \mathcal{B}$ and $x \in X$ define

$$B \prec x \iff x \in B.$$

For second countable spaces the set of approximations $\mathcal{B}$ can be chosen (by definition!) to be countable.

Example 2.3. Let $P = \mathbb{Q}$ and $X = \mathbb{R}$. For $a \in \mathbb{Q}$ and $x \in \mathbb{R}$ define

$$a \prec x \iff a < x$$

where $<$ is the usual order on $\mathbb{R}$.

Note that Example 2.1 provides a better approximation of $\mathbb{R}$ than Example 2.3 in that $[a, b] \prec x$ gives (roughly) more information than $a \prec x$.

Let P and X be sets and let $\prec$ be a relation from P to X. Then $\prec$ induces a relation $\sqsubseteq$ on P, called the *refinement (pre-)order* obtained from or induced by $\prec$, in a natural way: for $p, q \in P$ let

$$p \sqsubseteq q \iff (\forall x \in X)(q \prec x \Longrightarrow p \prec x).$$

Thus $p \sqsubseteq q$ expresses that q *is a better approximation than* p, or q *refines* p, in the sense that q approximates fewer elements in X than does p. Note that the induced refinement order indeed is a *preorder*, i.e. it is reflexive and transitive.

We now put some reasonable requirements on P and $\prec$ in order to obtain an *approximation structure* for X. We require that

- each element $x \in X$ is uniquely determined by its approximations, and
- each element $x \in X$ is the 'limit' of its approximations.

In addition it is useful to require P to have a trivial approximation, i.e., an approximation which approximates all elements of X (and hence contains no information about elements of X). This leads us to

Definition 2.4. Let P and X be sets, $\prec$ a relation from P to X and $\sqsubseteq$ the refinement preorder obtained from $\prec$. Then $(P; \sqsubseteq)$ is an *approximation structure* for X w.r.t. $\prec$ if

(i) $(\forall x, y \in X)(\{p \in P \colon p \prec x\} = \{p \in P \colon p \prec y\} \iff x = y)$;

(ii) $p \prec x$ and $q \prec x \Longrightarrow (\exists r \prec x)(p \sqsubseteq r$ and $q \sqsubseteq r)$;

(iii) $(\exists p \in P)(\forall x \in X)(p \prec x)$.

Examples 2.1 and 2.3 are approximation structures when we add a trivial approximation. Example 2.2 gives an approximation structure precisely when the space X is T_0. In this sense (i) in Definition 2.4 is a T_0 property.

Let $(P; \sqsubseteq)$ be an approximation structure for X w.r.t. $\prec$. Then each $x \in X$ is uniquely identified with the set $\{p \in P \colon p \prec x\}$. Note that if $p \sqsubseteq q \prec x$ then $p \prec x$. Together with (ii) and (iii) in Definition 2.4 we see that $\{p \in P \colon p \prec x\}$ is an *ideal* over $(P; \sqsubseteq)$.

For a preorder $P = (P; \sqsubseteq)$, a set $A \subseteq P$ is said to be *directed* if A is non-empty and if $p, q \in A$ then there is an $r \in A$ such that $p, q \sqsubseteq r$, i.e., every finite subset of A has an upper bound in A. A subset $I \subseteq P$ is an *ideal* over P if I is directed and downwards closed. We often use the notation $\downarrow p = \{q \in P \colon q \sqsubseteq p\}$ and $\uparrow p = \{q \in P \colon p \sqsubseteq q\}$. Note that $\downarrow p$ is an ideal, the *principal ideal* generated by p. We denote by $\mathrm{Idl}(P; \sqsubseteq)$, or just $\mathrm{Idl}(P)$, the set of all ideals over $(P; \sqsubseteq)$.

Thus given an approximation structure $(P; \sqsubseteq)$ of X w.r.t. $\prec$ we obtain an injection of X into $\mathrm{Idl}(P)$, i.e. X "lives" in $\mathrm{Idl}(P)$. (However, this may not be the best injection, as we shall see later.) In addition $\mathrm{Idl}(P)$ contains the approximation structure P that we started with by means of the principal ideals $\downarrow p$. So $\mathrm{Idl}(P)$ contains both the original space and its approximations.

$\mathrm{Idl}(P)$ is naturally ordered by inclusion $\subseteq$. For if ideals $I \subseteq J$ then J contains more approximations and hence more information about the elements approximated than does I. We therefore consider $\mathrm{Idl}(P)$ as a structure ordered by inclusion.

Definition 2.5. Let $P = (P; \sqsubseteq)$ be a preorder. The *ideal completion* of P is the structure $\bar{P} = (\mathrm{Idl}(P); \subseteq)$.

We also denote the set $\mathrm{Idl}(P)$ by $\bar{P}$. Here are the relevant properties of the ideal completion.

Theorem 2.6. *Let $P = (P; \sqsubseteq)$ be a preorder and let $\bar{P} = (\bar{P}; \subseteq)$ be the ideal completion of P ordered by inclusion.*

(i) *$\subseteq$ is a partial order.*

(ii) *If $\mathcal{F} \subseteq \bar{P}$ is a directed family of ideals then $\bigcup \mathcal{F}$ is an ideal and $\bigcup \mathcal{F} =$ supremum of $\mathcal{F}$ w.r.t. $\subseteq$.*

(iii) *If $p \in P$, $\mathcal{F} \subseteq \bar{P}$ is directed and $\downarrow p \subseteq \bigcup \mathcal{F}$ then there is $I \in \mathcal{F}$ such that $\downarrow p \subseteq I$.*

(iv) *Let $I \in \bar{P}$. Suppose for each directed $\mathcal{F} \subseteq \bar{P}$ such that $I \subseteq \bigcup \mathcal{F}$ there is $J \in \mathcal{F}$ such that $I \subseteq J$. Then $I = \downarrow p$ for some $p \in P$.*

(v) *Let $I \in \bar{P}$. Then the set $\{\downarrow p : \downarrow p \subseteq I\} \subseteq \bar{P}$ is directed and $I = \bigcup \{\downarrow p : \downarrow p \subseteq I\}$.*

Proof. Straightforward. (For (iv) let $\mathcal{F} = \{\downarrow p : p \in I\}$.) $\square$

The interesting point for us is that the theorem, with notions motivated from simple considerations about approximations, gives the axioms for an *algebraic cpo* simply by replacing $(\bar{P}; \subseteq)$ by a partial order $(D; \sqsubseteq)$.

Note that if P contains a 'least' element $\perp$ as in an approximation structure then $\downarrow\perp$ is least in $\bar{P}$. It is often convenient to have a least element, e.g., for the existence of fixed points and for the function space construction.

Definition 2.7.

(i) Let $D = (D; \sqsubseteq, \perp)$ be a partially ordered set with least element $\perp$. Then D is a *complete partial order* (abbreviated *cpo*) if whenever $A \subseteq D$ is directed then $\bigsqcup A$ (the least upper bound or supremum of A) exists in D.

(ii) Let D be a cpo. An element $a \in D$ is said to be *compact* or *finite* if whenever $A \subseteq D$ is a directed set and $a \sqsubseteq \bigsqcup A$ then there is $x \in A$ such that $a \sqsubseteq x$. The set of compact elements in D is denoted by D_c.

(iii) A cpo D is an *algebraic cpo* if for each $x \in D$, the set

$$\mathrm{approx}(x) = \{a \in D_c\colon a \sqsubseteq x\}$$

is directed and $x = \bigsqcup \mathrm{approx(x)}$.

With this terminology, Theorem 2.6 states that $\bar{P} = (\bar{P}; \subseteq, \downarrow\perp)$ is an algebraic cpo.

Here is a representation theorem for algebraic cpos. For its simple proof see Stoltenberg-Hansen et al. [39].

Theorem 2.8. *Let $D = (D; \sqsubseteq, \perp)$ be an algebraic cpo and let $\overline{D_c}$ be the ideal completion of $D_c = (D_c; \sqsubseteq)$. Then $D \simeq \overline{D_c}$.*

Note that if D is an algebraic cpo then $(D_c; \sqsubseteq)$ is an approximation structure for D w.r.t. $\prec$, where for $a \in D_c$ and $x \in D$,

$$a \prec x \Longleftrightarrow a \sqsubseteq x.$$

We have shown that algebraic cpos are precisely ideal completions of approximation structures.

It is well-known that the class of algebraic cpos is not closed under the function space construction. The usual added requirement is the following.

Definition 2.9. Let D be a cpo. Then D is said to be *consistently complete* if whenever $x, y \in D$ are consistent in D (i.e. have a common upper bound) then their supremum $x \sqcup y$ exists in D.

A consistently complete algebraic cpo will be called an *algebraic domain*. Other terms in the literature are Scott domain or Scott-Ershov domain after the originators of our theory.

For an approximation structure $P = (P; \sqsubseteq)$ we often want the ideal completion $\bar{P}$ to be an algebraic domain. It is an easy exercise to see that a necessary and sufficient condition (when $\sqsubseteq$ is a partial order) is that P is a *conditional upper semilattice* or *cusl*, i.e., if $p, q \in P$ are consistent in P then $p \sqcup q$ exists in P. The approximation structures in Examples 2.1 and 2.3 are cusls and the topological base $\mathcal{B}$ in Example 2.2 can be chosen such that it is a cusl.

Assume P is an approximation structure for X via $\prec$. Then we obtain an induced approximation from X to X by

$$x \leq y \Longleftrightarrow (\forall p \in P)(p \prec x \Longrightarrow p \prec y).$$

Clearly, $(X; \leq)$ is an approximation structure for X w.r.t. $\leq$.

In Example 2.1 the induced relation $\leq$ is discrete, i.e., $\leq$ is $=$. The induced relation in Example 2.2 is known as the *specialisation order*. It is discrete if, and only if, X is a T_1 space. The induced relation in Example 2.3 coincides with the usual ordering on $\mathbb{R}$ and hence is far from discrete.

In applications we are mainly interested in approximations P of X such that the induced relation $\leq$ on X is discrete. The reason is the following. Suppose $I \in \bar{P}$ determines or represents a unique element $x \in X$ and suppose $J \supseteq I$. Then J contains as much information as I and hence should also represent or determine x. That is, it is desirable that the representations in $\bar{P}$ are *upwards closed*.

Define a function rep: $\bar{P} \to \wp(X)$ by

$$\operatorname{rep}(I) = \{x \in X \colon (\forall p \in I)(p \prec x)\}.$$

We say that $I \in \bar{P}$ is *convergent* if $\operatorname{rep}(I)$ is a singleton and then we denote $\operatorname{rep}(I) = \{x\}$ by $I \to x$. Let $\bar{P}^R = \{I \in \bar{P} : I \text{ convergent}\}$ and define $\nu \colon \bar{P}^R \to X$ by

$$\nu(I) = x \Longleftrightarrow I \to x.$$

If the representation is upwards closed then ν is a surjection.

It is not necessarily the case that each $x \in X$ has a *unique* representation in $\bar{P}^R$. In Example 2.1 each irrational has a unique representation while each rational has exactly four representations. (Can you describe all these representations?)

From an approximation P for X we have constructed an algebraic cpo (often an algebraic domain) $\bar{P}$ which includes both the approximations for elements of X and also (representations of) the elements of X. Now we can use the general theory of domains to study the structure X via the mapping ν, including

- fixed point theorems (generalises Banach's fixed point theorem, used for example in studying iterated function systems, see Edalat [11, 12] and Blanck [6]);
- ability to build higher type objects (e.g. streams and stream transformers, see [7]);
- computability, inherited from the computability of P.

Our claim is that the use of domains (of various kinds) provides a general, uniform and useful way to study computability via approximations on a large class of structures. It is a hope that the use of domain representability will help to give insights into ways of doing feasable exact or secure computations on certain topological algebras such as $\mathbb{R}$. For some interesting work in this direction see Potts [34].

2.2 Scott topology

There is a natural topology on cpos given by Scott.

Definition 2.10. Let $D = (D; \sqsubseteq, \perp)$ be a cpo. Then $F \subseteq D$ is *closed* if

(i) whenever $A \subseteq F$ is directed then $\bigsqcup A \in F$; and

(ii) if $x \in F$ and $y \sqsubseteq x$ then $y \in F$.

A directed set should be thought of as a generalised sequence. In a cpo all such “sequences” converge. Thus condition (i) has a clear motivation.

Examples of closed sets are $\downarrow x = \{y \in D\colon y \sqsubseteq x\}$. In fact, the topological closure $\overline{\{x\}}$ of $\{x\}$ is $\downarrow x$.

Open sets are complements of closed sets. Thus $U \subseteq D$ is *open* if

(i) whenever $A \subseteq D$ is directed and $\bigsqcup A \in U$ then $A \cap U \neq \emptyset$; and

(ii) if $x \in U$ and $x \sqsubseteq y$ then $y \in U$.

Part (ii) has been motivated as follows (see Smyth [38]). The ordering $\sqsubseteq$ on D is considered as an information ordering, i.e., y contains more information than x if $x \sqsubseteq y$. Open sets are thought of as “observable properties”. If x has enough information for the property U to hold, i.e., $x \in U$, and $x \sqsubseteq y$ then y also contains enough information for the property U to hold. Thus we assume information to be consistent, and hence to satisfy a property is definite when the information is sufficient.

Remark 2.11. If D is an algebraic cpo then the family $\{\uparrow a\colon a \in D_c\}$ is a topological base for the Scott topology.

Exercise 2.12. Let (X, τ) be a topological space. The specialisation order $\leq$ on X is defined by

$$x \leq y \Longleftrightarrow \forall U^{open}(x \in U \Longrightarrow y \in U).$$

Show that for a cpo $D = (D; \sqsubseteq, \bot)$ equipped with the Scott topology the order $\sqsubseteq$ is the specialisation order. Conclude that D is a T_0 space and that D is a T_1 space only if $D = \{\bot\}$.

The Scott topology is the correct one for the natural order theoretic version of continuity.

Proposition 2.13. *Let D and E be cpos. Then a function $f\colon D \to E$ is continuous with respect to the Scott topology if, and only if, f is monotone and for each directed set $A \subseteq D$,*

$$f(\bigsqcup A) = \bigsqcup f[A].$$

The proof is a straightforward exercise.

Consider the unit interval $[0, 1]$. It is a cpo under the usual ordering $\leq$. However, the Scott topology is radically different from the usual Euclidean topology on $[0,1]$. For example, the step function $f\colon[0,1]\to[0,1]$ defined by

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \le 1 \end{cases}$$

is Scott continuous but not continuous with respect to the usual topology. A non-monotone function is not Scott continuous.

For cpos D and E we define the *function space* $[D \to E]$ of D and E by

$$[D \to E] = \{f\colon D \to E \mid f \text{ continuous}\}.$$

We order $[D \to E]$ by

$$f \sqsubseteq g \iff (\forall x \in D)(f(x) \sqsubseteq g(x)).$$

It is easy to see that $[D \to E]$ is a cpo where for a directed set $\mathcal{F} \subseteq [D \to E]$ and $x \in D$,

$$(\bigsqcup \mathcal{F})(x) = \bigsqcup\{f(x)\colon f \in \mathcal{F}\}.$$

We form a category $\mathbf{CPO}_\perp$whose objects are cpos and whose morphisms are continuous functions between cpos. It is well-known and easy to prove that $\mathbf{CPO}_\perp$ is a cartesian closed category, where the product of cpos D and E is given by

$$D \times E = \{(x,y)\colon x \in D, y \in E\}$$

and ordered by

$$(x,y) \sqsubseteq (z,w) \iff x \sqsubseteq_D z \text{ and } y \sqsubseteq_E w.$$

The exponent of D and E is the function space $[D \to E]$. In particular this means that the corresponding projection functions for the product are continuous. Furthermore eval: $[D \to E] \times D \to E$ defined by

$$\mathrm{eval}(f,x) = f(x)$$

is continuous and curry: $[D \times E \to F] \to [D \to [E \to F]]$ defined by

$$\mathrm{curry}(f)(x)(y) = f(x,y)$$

is continuous.

Finally we have the existence of least fixed points.

Proposition 2.14. *Let $D = (D; \sqsubseteq, \perp)$ be a cpo.*

(i) *For each $f \in [D \to D]$ there is a least $x \in D$ such that $f(x) = x$, i.e., f has a least fixed point.*

(ii) *The function* fix : $[D \to D] \to D$ *defined by*

$$\mathrm{fix}(f) = \text{least fixed point of } f$$

is continuous.

Proof. The least fixed point of f is $\bigsqcup_n f^n(\perp)$ where

$$\begin{cases} f^0(\perp) & = & \perp \\ f^{n+1}(\perp) & = & f(f^n(\perp)). \end{cases}$$

We leave the continuity of fix as an exercise. □

2.3 Continuous and algebraic domains

We will briefly discuss continuous domains, which is a broader class of structures than algebraic domains. It turns out that for the purpose of using domains to study computability on topological algebras it suffices, and is in our view sometimes preferable, to consider the simpler structures of algebraic domains. However, when dealing with topological algebras which can only have weak computability properties then continuous domains may seem more natural.

2.3.1 Continuous cpos

In this section we discuss the appropriate approximation relation on cpos. Then we isolate those cpos which are well-behaved with respect to this approximation relation. These are the continuous cpos.

Definition 2.15. Let $D = (D; \sqsubseteq, \perp)$ be a cpo. Then for $x, y \in D$ we say x is *way below* y, denoted $x \ll y$, if for each directed set $A \subseteq D$,

$$y \sqsubseteq \bigsqcup A \Longrightarrow (\exists z \in A)(x \sqsubseteq z).$$

The way below relation is sometimes called the *approximation relation*.

Note that $x \in D$ is compact if, and only if, $x \ll x$. Of course, $\perp \ll x$ always holds.

Exercise 2.16. Show that the way below relation on an algebraic cpo D is characterised by

$$x \ll y \Longleftrightarrow (\exists a \in D_c)(x \sqsubseteq a \sqsubseteq y).$$

Lemma 2.17. *Let $D = (D; \sqsubseteq, \perp)$ be a cpo and $x, y, z, w \in D$. Then the following hold.*

(i) $x \ll y \implies x \sqsubseteq y$.

(ii) $z \sqsubseteq x \ll y \sqsubseteq w \implies z \ll w$.

In particular we see that $\ll$ is transitive and antisymmetric. But it need not be reflexive.

Example 2.18. Let $\mathbb{CI} = ([0, 1]; \leq, 0)$ where $\leq$ is the usual order. Then

$$x \ll y \Longleftrightarrow x < y.$$

Example 2.19. Let $\mathbb{CIR} = \{[x, y] : x, y \in \mathbb{R}, x \leq y\} \cup \{\mathbb{R}\}$ ordered by reverse inclusion. Then

$$[x, y] \ll [z, w] \Longleftrightarrow x < z \leq w < y,$$

i.e., $[z, w] \subseteq (x, y)$, the interior of $[x, y]$.

Exercise 2.20. Determine the compact elements in the above examples.

Definition 2.21. A cpo $D = (D; \sqsubseteq, \perp)$ is *continuous* if for each $x \in D$,

(i) the set $\{y \in D\colon y \ll x\}$ is directed (w.r.t. $\sqsubseteq$); and

(ii) $x = \bigsqcup\{y \in D\colon y \ll x\}$.

Useful notation for sets determined by the way below relation are, for x an element of a cpo D,

$$\twoheaddownarrow x = \{y \in D\colon y \ll x\} \qquad \text{and} \qquad \twoheaduparrow x = \{y \in D\colon x \ll y\}.$$

As observed above, the way below relation $\ll$ is not necessarily reflexive. However, for continuous cpos it satisfies the following crucial *interpolation property*.

Lemma 2.22. *Let D be a continuous cpo. Let $M \subseteq D$ be a finite set and suppose $M \ll y$, i.e., $(\forall z \in M)(z \ll y)$. Then there is $x \in D$ such that $M \ll x \ll y$.*

Proof. (Sketch) Suppose $M \ll y$. Let $A = \{x \in D\colon \exists x'(x \ll x' \ll y)\}$. Using the continuity of D one shows that A is directed and $\bigsqcup A = y$. Say $M = \{z_1, \ldots, z_n\}$. Then we know from the above that there are $a_i \in A$ such that $z_i \sqsubseteq a_i$ and hence there is $a \in A$ such that $M \sqsubseteq a$. But then, by the definition of A, there is x such that $M \ll x \ll y$. $\square$

It follows that if D is a continuous cpo then $\twoheaddownarrow y = \{x \in D\colon x \ll y\}$ is directed with respect to $\ll$ for each $y \in D$.

For an algebraic cpo D we have seen that all the information is contained in the behaviour of the compact elements D_c of D. We say that D_c is a *base* for D. This is essential when using domains in order to study computability. We now introduce the analogous notion for continuous cpos.

Definition 2.23. Let $D = (D; \sqsubseteq, \perp)$ be a cpo. A subset $B \subseteq D$ is a *base* for D if for each $x \in D$,

$$\text{approx}_B(x) = \{y \in B\colon y \ll x\}$$

is directed and $\bigsqcup \text{approx}_B(x) = x$.

Actually it suffices to require that for each $x \in D$ there is a directed subset $A \subseteq \mathrm{approx}_B(x)$ such that $x = \bigsqcup A$. Note that if B is a base for a cpo D then $D_c \subseteq B$.

If D is a continuous cpo then trivially D is a base for D. As already mentioned, D_c is a base for an algebraic cpo D. In Example 2.18, $[0,1] \cap \mathbb{Q}$ is a base and in Example 2.19, the intervals $[a,b]$ with $a, b \in \mathbb{Q}$ (along with bottom) make up a base.

The following proposition is an easy exercise.

Proposition 2.24. *A cpo is continuous if, and only if, it has a base.*

Exercise 2.25. Show that a cpo D is algebraic if, and only if, D_c is a base for D. Also show that a continuous cpo D has a least base if, and only if, D is algebraic.

Just as for an algebraic cpo the behaviour of a continuous cpo is determined by a base. For example, the interpolation property (Lemma 2.22) is such that the witness always can be chosen from a base of D.

Theorem 2.26. *Let D be a cpo with a base B. Then a topological base for the Scott topology on D is given by the family $\{\twoheaduparrow x : x \in B\}$.*

Proof. First we note that $\twoheaduparrow x$ is open for any $x \in D$. It is clearly upwards closed. Suppose $x \ll \bigsqcup A$ where A is directed. Then, by interpolation, there is y such that $x \ll y \ll \bigsqcup A$. Hence there is $a \in A$ such that $y \sqsubseteq a$, so $x \ll a$.

Suppose $w \in \twoheaduparrow x \cap \twoheaduparrow y$, i.e., $x, y \ll w$. By interpolation we choose $b \in B$ such that $x, y \ll b \ll w$. Thus $w \in \twoheaduparrow b \subseteq \twoheaduparrow x \cap \twoheaduparrow y$. Similarly one shows that for each open $U \subseteq D$,

$$U = \bigcup_{x \in U \cap B} \twoheaduparrow x.$$

$\square$

The continuous functions between continuous cpos are characterised by their behaviour on bases of the cpos. The following Proposition will often be tacitly used. We leave its easy proof as an exercise.

Proposition 2.27. *Let D and E be continuous cpos with bases B_D and B_E respectively.*

(i) *A function $f : D \to E$ is continuous if, and only if, f is monotone and for each $x \in D$,*

$$(\forall b \in \mathrm{approx}_{B_E}(f(x)))(\exists a \in \mathrm{approx}_{B_D}(x))(b \ll f(a)).$$

(ii) *For each monotone function* $f\colon B_D \to E$ *There is a largest continuous function* $\bar{f}\colon D \to E$ *given by*

$$\bar{f}(x) = \bigsqcup\{f(a)\colon a \in \mathrm{approx}_{B_D}(x)\}$$

such that for each $a \in B_D$, $\bar{f}(a) \sqsubseteq f(a)$.

Recall that if D and E are algebraic then the way below relation $\ll$ in the above proposition is just $\sqsubseteq$.

We conclude this section by providing a connection between algebraic cpos and continuous cpos. Of course, every algebraic cpo is continuous.

Let D and E be cpos. Then a pair of functions $e\colon D \to E$ and $p\colon E \to D$ is a *projection pair* from D to E if they are continuous and

$$p \circ e = \mathrm{id}_D \qquad \text{and} \qquad e \circ p \sqsubseteq \mathrm{id}_E$$

where id is the identity function.

Let D be a continuous cpo with a base B and let $E = \mathrm{Idl}(B; \sqsubseteq)$, the ideal completion of the partial order $(B; \sqsubseteq)$. It follows by Theorem 2.6 that E is an algebraic cpo. Define $e\colon D \to E$ and $p\colon E \to D$ by

$$e(x) = \mathrm{approx}_B(x) = \{y \in B\colon y \ll x\} \qquad \text{and} \qquad p(I) = \bigsqcup{}_D I.$$

Proposition 2.28. *The pair* (e, p) *is a projection pair from* D *to* E.

The proof is straightforward, noting that the continuity of e depends on the interpolation property.

2.3.2 The function space

In this section we discuss the function space construction for *continuous domains* and *algebraic domains*, i.e., continuous and algebraic cpos which are consistently complete.

It is well-known that the categories of continuous cpos and algebraic cpos are not cartesian closed. The categories of continuous domains and algebraic domains *are* cartesian closed and these are the ones most often considered in semantics. However, there are important larger cartesian closed subcategories of the continuous and algebraic cpos. The category of bifinite domains is particularly important. The latter category is essential in connection with the Plotkin power domain construction. Bifinite domains (or a continuous analogue) are also important from the point of view of domain representability. For example, they are used to study the computability of iterated function systems (see Blanck [6] and Edalat [11, 12]). Bifinite domains and different power domain constructions will be discussed in Sections 6 and 7.

To know the structure of a base for the function space, rather than just its existence, is essential in order to determine the effectivity of the function space.

In what follows we will describe the construction for continuous domains. This includes the construction for algebraic domains, where the cusl of compact elements plays the role of a base, recalling the characterisation of the way below relation for the algebraic case.

Given cpos D and E with bases B_D and B_E we want to construct a base for the function space $[D \to E]$. It turns out that such a base, under appropriate conditions, can be taken as finite suprema of step functions determined from B_D and B_E. Here is the definition of a step function.

Definition 2.29. Let $D = (D; \sqsubseteq, \perp)$ and $E = (E; \sqsubseteq, \perp)$ be cpos. For $a \in D$ and $b \in E$ define $\langle a; b\rangle \colon D \to E$ by

$$\langle a; b\rangle(x) = \begin{cases} b & \text{if } a \ll x \\ \perp & \text{otherwise.} \end{cases}$$

Note that each step function $\langle a; b\rangle$ is continuous since $\mathord{\uparrow\!\!\!\uparrow} a$ is open for each $a \in D$.

Proposition 2.30. *Let D and E be cpos and let $a \in D$ and $b \in E$.*

(i) *Suppose $f \colon D \to E$ is continuous. Then*

$$b \ll f(a) \Longrightarrow \langle a; b\rangle \ll f.$$

(ii) *If D and E are continuous cpos with bases B_D and B_E and $f \colon D \to E$ is continuous then*

$$f = \bigsqcup\{\langle a; b\rangle \colon a \in B_D, \quad b \in B_E, \ \langle a; b\rangle \ll f\}.$$

Proof.

(i) Suppose $f \sqsubseteq \bigsqcup \mathcal{F}$ where $\mathcal{F} \subseteq [D \to E]$ is directed and $b \ll f(a)$. Then $f(a) \sqsubseteq \bigsqcup\{g(a) : g \in \mathcal{F}\}$ and hence $b \sqsubseteq g(a)$ for some $g \in \mathcal{F}$, i.e., $\langle a; b\rangle \sqsubseteq g$.

(ii) Suppose $\{\langle a; b\rangle : a \in B_D, \ b \in B_E, \ \langle a; b\rangle \ll f\}$ has g as an upper bound and let $x \in D$. Then $x = \bigsqcup\{a \in B_D : a \ll x\}$ and hence $f(x) = \bigsqcup\{f(a) : a \ll x\}$. For $b \ll f(x)$ we obtain (by interpolation) $a \ll x$ such that $b \ll f(a)$ and hence, by (i), $\langle a; b\rangle \ll f$. Thus $\langle a; b\rangle \sqsubseteq g$. In particular, $b = \langle a; b\rangle(x) \sqsubseteq g(x)$ so $f(x) \sqsubseteq g(x)$.

$\square$

It is clearly not the case that the set in (ii) is directed in general. In order to obtain a directed set we expand it by including suprema of finite subsets of step functions way below f. Consistent completeness of E suffices to obtain such suprema. (In Section 6 we consider a weaker sufficient condition.)

The following characterisation is important when considering the effectivity of the functions space construction.

Proposition 2.31. *Let D be a continuous cpo, E a consistently complete cpo, and let $a_1, \dots, a_n \in D$ and $b_1, \dots, b_n \in E$. Then*

$$\{\langle a_1; b_1\rangle, \dots, \langle a_n; b_n\rangle\} \quad \textit{is consistent in} \quad [D \to E]$$

if, and only if,

$$\forall I \subseteq \{1, \dots, n\}(\bigcap_{i \in I} \Uparrow a_i \neq \emptyset \Longrightarrow \{b_i : i \in I\} \quad \textit{consistent}).$$

Proof. For the non-trivial direction define $h: D \to E$ by

$$h(x) = \bigsqcup \{b_i : a_i \ll x\}.$$

Then h is well-defined by consistent completeness and h is monotone. Suppose $A \subseteq D$ is directed and $a_i \ll \bigsqcup A$. By the continuity of D there is $d_i \in A$ such that $a_i \ll d_i$ and hence $b_i \sqsubseteq h(d_i)$. Thus

$$h(\bigsqcup A) = \bigsqcup \{b_i : a_i \ll \bigsqcup A\} \sqsubseteq \bigsqcup h[A].$$

$\square$

Note that if $\{\langle a_1; b_1\rangle, \dots, \langle a_n; b_n\rangle\}$ is consistent then the function h in the proof *is* $\bigsqcup_{i=1}^n \langle a_i; b_i\rangle$.

Exercise 2.32. Let D be a continuous cpo. Show that if $a, b \ll x$ and $a \sqcup b$ exists then $a \sqcup b \ll x$. Also show that if a base B_D of D is consistently complete then D is consistently complete.

Using Proposition 2.30 (ii) it is now straightforward to prove that the categories of continuous domains and algebraic domains are cartesian closed.

Theorem 2.33. *Let D and E be continuous cpos with bases B_D and B_E. If E is consistently complete then $[D \to E]$ is continuous and consistently complete, i.e., a continuous domain. A base for $[D \to E]$ is*

$$B_{[D \to E]} = \{\bigsqcup_{i=1}^{n} \langle a_i; b_i\rangle : a_i \in B_D,\ b_i \in B_E,\ \{\langle a_1; b_1\rangle, \dots, \langle a_n; b_n\rangle\} \text{ consistent}\}.$$

For algebraic domains we let the bases be cusls of compact elements D_c and E_c. For $a \in D_c$ and $b \in E_c$ the step function $\langle a; b\rangle$ is compact. It follows that $B_{[D\to E]}$ is a base for $[D \to E]$ consisting only of compact elements. This shows that $[D \to E]$ is an algebraic domain.

In conclusion we emphasise again that a continuous or algebraic domain is completely determined by a base (and hence by the compact elements in the algebraic case). In addition, a continuous function from a domain into a cpo is completely determined by its values on a base.

3 Domain representability

In this section we describe how large classes of topological algebras can be given natural domain representations which later will be shown to satisfactorily model concrete computations on the algebras. To a topological algebra A is associated a domain (of some kind) D_A from which a subset D_A^R is selected to make a representation of A via a surjective quotient map $\nu\colon D_A^R \to A$. Also the operations on A must be tracked by continuous operations on D, i.e., D must be a Σ-domain when representing a topological Σ-algebra.

The notion of effective domain representability for topological algebras was, as far as we know, first made explicit in Stoltenberg-Hansen and Tucker [40] where it was used to study the effective content of the completion of a computable Noetherian local ring. It was further extended to ultrametric spaces and locally compact regular spaces in [41, 42, 43] and to metric spaces in the thesis [3].

However, it was clear from the beginning of the development of domain theory that domain theory is a theory of approximation and computation, and that computability often implies continuity. This was exploited in [15] where Ershov gave a domain representation of the Kleene-Kreisel continuous functionals. An effective and adequate domain model of Martin-Löf partial type theory is given in Palmgren and Stoltenberg-Hansen [32] which has been extended in Waagbø [45] to provide a domain representation of Martin-Löf total type theory (see also Berger [9] and Normann [30]). Also related to domain representability is Weihrauch and Schreiber [46] where embeddings of metric spaces into complete partial orders equipped with weight and distance is considered.

The work on domain representability referred to above use algebraic domains. Related work on domain representability using continuous (most often non-algebraic) domains has been pursued by Edalat and his group; for a survey of their work see Edalat [13].

3.1 Basic definitions

Let X and Y be topological spaces. A function $\nu: X \to Y$ is a *quotient mapping* if $U \subseteq Y$ is open if, and only if, $\nu^{-1}[U]$ is open in X. In case ν is surjective we then have that $X/\sim$ and Y are homeomorphic spaces when the former is given the quotient topology and where $\sim$ is the equivalence relation induced on X by ν, i.e., $x \sim y \iff \nu(x) = \nu(y)$.

The concepts below are valid both for continuous domains and algebraic domains. Therefore in this section we simply use the term *domain* to mean continuous domain or, if the reader so desires, algebraic domain. For D a domain we use the notation B_D for a base of D. In case D is algebraic the reader should read D_c for B_D, recalling that D_c is a base of D if, and only if, D is algebraic.

Definition 3.1. Let X be a topological space, let D be a domain and D^R a subset of D. Then (D, D^R, ν) is a *domain representation* of X in case $\nu: D^R \to X$ is a surjective quotient map when D^R is given the (relativised) Scott topology.

In the discussion in Section 2.1 we created a domain representation via an approximation structure for a *set* X. In that case the projection ν induced a topology on X from the Scott topology of the domain. In the present situation we are given a topology on X. Thus the domain representation must be such that the induced topology from the Scott topology of the representing domain coincides with the given topology. In the examples already given the approximation structure for the topological space X was chosen to respect the topology on X. It is interesting, at least initially, that all topological spaces have a domain representation and hence a topology induced by the Scott topology via a quotient map.

As already remarked, a domain representation (D, D^R, ν) of X contains both concrete and proper approximations of elements of X, the elements in a base B_D, and "total" elements in D^R containing sufficient information to represent elements of X exactly via ν. Since the function ν in the definition above is a quotient map we have

$$D^R/\sim \;\cong\; X.$$

Here are some useful and natural properties desirable for domain representations.

Definition 3.2. A domain representation (D, D^R, ν) of a space X is

(i) *upwards closed* if whenever $x \in D^R$ and $x \sqsubseteq y$ then $y \in D^R$ and $\nu(x) = \nu(y)$;

(ii) *dense* if for each $a \in B_D$, $\Uparrow a \cap D^R \neq \emptyset$;

(iii) *local* if $(\forall x, y \in D^R)(\nu(x) = \nu(y) \implies x$ and y are consistent$)$.

Upwards closed domain representations (D, D^R, ν) are natural when regarding the ordering $\sqsubseteq$ on D as an information ordering. If $x \in D^R$ completely determines $\nu(x) \in X$ and $x \sqsubseteq y$ then y contains all the information of x and hence also completely determines $\nu(x)$. Note that if the represented space X is T_1 then the condition $\nu(x) = \nu(y)$ is redundant.

We also distinguish between domain representations by putting requirements on the quotient map ν. For a thorough study of various kinds of domain representations we refer to Blanck [5]. Here we will just consider the following.

Definition 3.3. Let (D, D^R, ν) be a domain representation on X.

(i) (D, D^R, ν) is a *retract representation* of X if ν is a *retract*, i.e., if there is a continuous function $e\colon X \to D$ such that $\nu \circ e = \mathrm{id}_X$.

(ii) (D, D^R, ν) is an *open representation* of X if ν is an open mapping.

(iii) (D, D^R, ν) is a *homeomorphic representation* of X if ν is a homeomorphism.

Domain representations are uniformly closed under most of the usual constructions, such as retracts, quotients, disjoint sums and direct limits. However, perhaps surprisingly, domain representations are not uniformly closed under products. That is, if (D, D^R, ν) and (E, E^R, μ) are domain representations of X and Y respectively, then $\nu \times \mu\colon D^R \times E^R \to X \times Y$ need not be a quotient map (see [5]). Retract representations are uniformly closed under (arbitrary) products.

The next step is to represent continuous functions between topological spaces.

Definition 3.4. Let (D, D^R, ν) and (E, E^R, μ) be domain representations of X and Y respectively. A function $f\colon X \to Y$ is *represented* by (or *lifts* to) a continuous function $\bar{f}\colon D \to E$ if $\bar{f}[D^R] \subseteq E^R$ and $\mu(\bar{f}(x)) = f(\nu(x))$, for all $x \in D^R$.

Let (D, D^R, ν) and (E, E^R, μ) be domain representations of X and Y respectively. Suppose $\bar{f}\colon D \to E$ is such that $\bar{f}[D^R] \subseteq E^R$ and such that $\nu(x) = \nu(y) \implies \mu(\bar{f}(x)) = \mu(\bar{f}(y))$. Then $\bar{f}$ induces a unique function $f\colon X \to Y$ defined by $f(\nu(x)) = \mu(\bar{f}(x))$. In the terminology above, f is represented by $\bar{f}$.

Observe that if (D, D^R, ν) is local and upwards closed and (E, E^R, μ) is upwards closed then every continuous function $\bar{f}\colon D \to E$ such that $\bar{f}[D^R] \subseteq E^R$ induces a unique function $f\colon X \to Y$.

Proposition 3.5. *If $f\colon X \to Y$ is represented by a continuous $\bar{f}\colon D \to E$ then f is continuous.*

The proof is simple and depends on the fact that ν is assumed to be a quotient (only continuity of μ is required). It is a topological result and has nothing to do with the special case of domains.

Thus every continuous function $\bar{f}: D \to E$, such that $\bar{f}[D^R] \subseteq E^R$ and $\nu(x) = \nu(y) \implies \mu(\bar{f}(x)) = \mu(\bar{f}(y))$ for $x, y \in D^R$, trivially induces a *continuous* function $f: X \to Y$. The problematic part is the converse.

Question: When does a continuous function $f: X \to Y$ have a lifting $\bar{f}: D \to E$?

Let (D, D^R, ν) be a domain representation of a space X. Then ν induces a relation $\equiv$ on D by

$$x \equiv y \iff \nu(x) = \nu(y).$$

The relation $\equiv$ is symmetric and transitive on D, i.e., $\equiv$ is a *partial equivalence relation* (abbreviated *per*) on D with field D^R.

Let D be a domain and $\equiv$ a per on D. We say the the pair $(D, \equiv)$ is a *domain with per*. Given a domain with per $(D, \equiv)$ we extract the set

$$\bar{D} = \{x \in D : x \equiv x\} = \text{Field}(\equiv).$$

Then the pair $(D, \bar{D})$ is a *domain with totality*. We now take the quotient $\bar{D}/\equiv$ of $\bar{D}$ with $\equiv$, defined by

$$\bar{D}/\equiv \; = \{[x]: x \in \bar{D}\}$$

where $[x] = \{y \in \bar{D}: x \equiv y\}$ is the equivalence class of x, and give it the quotient topology. The latter is defined by

$$U \subseteq \bar{D}/\equiv \quad \text{open} \quad \iff \quad \bigcup U \subseteq \bar{D} \quad \text{open}$$

where $\bar{D}$ has the subspace topology from the Scott topology on D. Define $\nu: \bar{D} \to \bar{D}/\equiv$ by

$$\nu(x) = [x].$$

Then ν is a quotient map, so $(D, \bar{D}, \nu)$ is a domain representation of $\bar{D}/\equiv$.

Now let (D, D^R, ν) and (E, E^R, μ) be domain representations of X and Y respectively and let $\equiv_D$ and $\equiv_E$ be the induced pers. Suppose $f: X \to Y$ is represented by $\bar{f}: D \to E$. Then it is immediate from Definitions 3.1 and 3.4 that for each $x, y \in D$,

$$x \equiv_D y \implies \bar{f}(x) \equiv_E \bar{f}(y). \tag{1}$$

If $\bar{g}: D \to E$ is another lifting of f then

$$(\forall x, y \in D)(x \equiv_D y \implies \bar{f}(x) \equiv_E \bar{g}(y)). \tag{2}$$

Conversely, if $\bar{f}: D \to E$ is a continuous function satisfying (1) then $\bar{f}$ is the lifting of a continuous function $f: X \to Y$ by Proposition 3.5.

We have shown that an equivalent alternative to domain representability is the use of domains with pers.

Let $(D, \equiv_D)$ and $(E, \equiv_E)$ be domains with pers. Then, guided by (2), we define a relation $\sim$ on $[D \to E]$ by

$$f \sim g \iff (\forall x, y \in D)(x \equiv_D y \Longrightarrow f(x) \equiv_E g(y)). \tag{3}$$

It is immediate that $\sim$ is a per on $[D \to E]$.

The domains with pers make up a category **PER(Dom)** in the following way. Let **Dom** be a category of domains (continuous or algebraic) with continuous functions as morphisms. An object in **PER(Dom)** is a domain with a per $(D, \equiv_D)$. A morphism from $(D, \equiv_D)$ to $(E, \equiv_E)$ is obtained as follows. Let $\sim$ be the per on $[D \to E]$ defined by (3). For $f \in [D \to E]$ such that $f \sim f$ let $[f]$ be the equivalence class of f w.r.t. $\sim$. Then we say that $[f]: (D, \equiv_D) \to (E, \equiv_E)$ is a morphism. Furthermore, the operation on morphisms is defined by $[f] \circ [g] = [f \circ g]$ and $\mathrm{id}_{(D, \equiv_D)} = [\mathrm{id}_D]$. It is an easy exercise to show, using the equivalence given by (3), that

Proposition 3.6. PER(Dom) *is a cartesian closed category.*

This means that we can build type structures in a natural way over a domain representable space X. We return to this topic in Section 3.4.

We have defined what is meant by a topological space being domain representable and by a continuous function to be representable. We now extend this in the natural way to topological algebras.

Let Σ be a (finite) signature. Recall that a *topological* Σ*-algebra* A is a structure $A = (A; \sigma_1, \ldots, \sigma_k)$ where A is a topological space and the operations $\sigma_i: A^{n_i} \to A$ are continuous. The signature Σ gives k and the arity of each σ_i. (A constant is regarded as a 0-ary operation.)

Definition 3.7. A structure $D = (D; \sqsubseteq, \bot; \psi_1, \ldots, \psi_k)$ is a *structured domain* or Σ*-domain* for a signature Σ if

(i) $(D; \sqsubseteq, \bot)$ is a domain, and

(ii) each ψ_j is a continuous n_j-ary operation on D, that is $\psi_j : D^{n_j} \to D$ is continuous, where D^{n_j} is given the product topology, and k and the arities n_j are given by Σ.

By D_A being a Σ-substructure of a Σ-domain D we mean that D_A is a substructure of D with respect to the operations named by Σ.

Definition 3.8. A topological Σ-algebra $A = (A; \sigma_1, \ldots, \sigma_k)$ is *domain representable by* a Σ-domain $D = (D; \sqsubseteq, \bot; \hat{\sigma}_1, \ldots, \hat{\sigma}_k)$ if there is a Σ-substructure $D_A = (D_A; \hat{\sigma}_1, \ldots, \hat{\sigma}_k)$ of D and a Σ-epimorphism

$$\nu_A : D_A \to A$$

which is a quotient map with respect to the subspace topology of D_A. The triple (D, D_A, ν_A) is a *domain representation* of A.

We say that a topological Σ-algebra A is *domain representable* if A is representable by some Σ-domain D.

3.2 Ultrametric algebras

In this section we consider domain representations of ultrametric algebras. It is a known fact that both domains and (ultra-)metric spaces behave well and in similar ways when used in denotational semantics, despite the fact that domains and metric spaces have drastically different topological properties. A perhaps naive explanation for this is that (ultra-)metric spaces have natural domain representations. In particular, by Proposition 3.21 we see that the Banach fixed point construction over a complete ultrametric space is precisely the least fixed point construction over the representing domain. This is also the case for a complete metric space.

We will show how to represent completions of ultrametric algebras or, more generally, certain inverse limits of algebras by *algebraic* domains. A domain representation (D, D^R, ν) of the complete ultrametric space X will be such that $D^R = D_m$, the set of maximal elements of D, and $\nu \colon D^R \to X$ is a homeomorphism. This illustrates how the ideal completion corresponds to the ultrametric completion.

We consider certain inverse limits obtained from a family of *separating congruences* on an algebra A. Let $A = (A; \sigma_1, \ldots, \sigma_k)$ be a Σ-algebra. A binary relation $\equiv$ on A is said to be a *congruence relation* on A if it is an equivalence relation and if for each operation σ in A, say n-ary, if $x_i \equiv y_i$ for $i = 1, \ldots, n$, then $\sigma(x_1, \ldots, x_n) \equiv \sigma(y_1, \ldots, y_n)$.

Definition 3.9. Let $A = (A; \sigma_1, \ldots, \sigma_k)$ be a Σ-algebra and let $I = (I; \leq, 0)$ be a directed set with least element 0. Then $\{\equiv_i\}_{i \in I}$ is a *family of separating congruences* on A if

(i) each $\equiv_i$ is a congruence relation on A,

(ii) $j \geq i$ and $x \equiv_j y \Longrightarrow x \equiv_i y$, and

(iii) $\bigcap_{i \in I} \equiv_i \; = \{(x, x) : x \in A\}$.

For convenience we always assume $x \equiv_0 y$ for each $x, y \in A$. Of course, $\omega = (\omega; \leq, 0)$ is a directed set with least element, where we identify the set of natural numbers $\mathbb{N}$ with the ordinal ω. Furthermore, in order to obtain a consistently complete domain representation, one should assume that the index set I is closed under finite suprema and that the family of separating congruences $\{\equiv_i\}_{i \in I}$ on A is *upward consistent* , i.e.,

$$a \equiv_i b \;\&\; a \equiv_j b \Longrightarrow (\exists k \geq i, j)(a \equiv_k b).$$

This trivially holds for $I = \omega$.

Example 3.10. (i) On the natural numbers $\mathbb{N}$, let $\equiv_n$ be the equivalence relation corresponding to the partition

$$\{0\}, \{1\}, \{2\}, \ldots, \{n-1\}, \{n, n+1, n+2, \ldots\}.$$

Note that $\equiv_n$ is a congruence for addition and multiplication (and any monotone operation).

(ii) Let $T(\Sigma, X)$ be the term algebra over a signature Σ and a set of variables X. Then, for $t, t' \in T(\Sigma, X)$ let $t \equiv_n t'$ if t and t' are identical up to height $n-1$, for $n \in \omega$.

(iii) Let R be a local commutative ring whose unique maximal ideal is $\mathbf{m}$. Define for $x, y \in R$ and $n \in \omega, x \equiv_n y \Leftrightarrow x - y \in \mathbf{m}^n$. Then, by Krull's Theorem, $\{\equiv_n\}_{n\in\omega}$ is a family of separating congruences with respect to the ring operations (see Stoltenberg-Hansen and Tucker [40]).

(iv) Let $2^\omega = \{f \mid f : \omega \to \{0, 1\}\}$, the Cantor set. For $f, g \in 2^\omega, n \in \omega$ define

$$f \equiv_n g \Longleftrightarrow (\forall i < n)(f(i) = g(i)).$$

Then $\{\equiv_n\}_{n\in\omega}$ is a family of separating congruences on 2^ω.

(v) Suppose we are given an algebra A and a family of separating congruences $\{\equiv_i\}_{i\in I}$ on A. Let J be a non-empty possibly infinite set and let $A^J = \{f \mid f : J \to A\}$. We make A^J into a Σ-algebra in the usual way by saying, for $\sigma \in \Sigma$ n-ary,

$$\sigma(f_1, \ldots, f_n)(j) = \sigma_A(f_1(j), \ldots, f_n(j)).$$

We define a family of separating congruences on A^J as follows. Let

$$I^{(J)} = \{s \mid s : J \to I \text{ and } s(j) = 0 \text{ a.e.}\},$$

where a.e. means everywhere except on a finite set. We order $I^{(J)}$ by

$$s \leq t \Leftrightarrow (\forall j \in J)(s(j) \leq t(j)).$$

Clearly $I^{(J)}$ is a directed set with least element the constant zero function. For $a, b \in A^J$ and $s \in I^{(J)}$ define

$$a \equiv_s b \Leftrightarrow (\forall j \in J)(a(j) \equiv_{s(j)} b(j)).$$

It is easily shown that the family $\{\equiv_s\}_{s\in I^{(J)}}$ is a family of separating congruences on A^J over $I^{(J)}$.

An algebra A with a family $\{\equiv_i\}_{i\in I}$ of separating congruences has a natural topology with a base given by

$$B(a,i) = \{b \in A : a \equiv_i b\} \text{ for } a \in A \text{ and } i \in I.$$

Proposition 3.11. *The algebra A is a topological algebra.*

Proof. We must show that each operation in A is continuous. For an n-ary operation σ in A we have

$$\sigma^{-1}(B(a,i)) = \bigcup_{\sigma(x_1,\ldots,x_n)\equiv_i a} B(x_1,i) \times \cdots \times B(x_n,i).$$

□

Let (X,d) be a metric space. Then (X,d) is an *ultrametric space*, and d is an *ultrametric*, if d satisfies the following stronger form of the triangle inequality: for $x,y,z \in X$

$$d(x,y) \leq \max\{d(x,z), d(z,y)\}.$$

An ultrametric algebra (A,d) has *non-expansive* operations if each operation σ in A, say n-ary, is *non-expansive*, i.e., satisfies the following:

$$d(\sigma(x_1,\ldots,x_n), \sigma(y_1,\ldots,y_n)) \leq \ \max\{d(x_i,y_i) : 1 \leq i \leq n\}.$$

Given an ultrametric algebra (A,d) we define a family $\{\equiv_n\}_{n\in\omega}$ of separating equivalences on A by

$$x \equiv_n y \Longleftrightarrow d(x,y) \leq 2^{-n}.$$

If the operation σ is non-expansive then each $\equiv_n$ is a congruence. (The choice of 2^{-n} is not essential. Any strictly decreasing sequence approaching 0 will do.) Thus we see that to an ultrametric algebra A with non-expansive operations we naturally associate a family of separating congruences indexed by ω.

Conversely, suppose A is an algebra and $\{\equiv_n\}_{n\in\omega}$ is a family of separating congruences for A. Then define $d\colon A \times A \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 2^{-n} & \text{if } x \neq y, \text{ where } n \text{ is least s.t. } x \not\equiv_n y. \end{cases}$$

It easily follows from Definition 3.9 that d is an ultrametric and that each operation is non-expansive with respect to d.

Proposition 3.12. *Let (A,d) be an ultrametric algebra with non-expansive operations and let (A,d') be the ultrametric algebra obtained from (A,d) by composing the two operations above. Then $(A,d) \cong (A,d')$ as topological algebras.*

To be precise we mean that (A, d) and (A, d') are homeomorphic spaces with a witnessing homeomorphism which is also an isomorphism in the algebraic sense. Of course, we do not claim that (A, d) and (A, d') are isometric.

We now describe the construction of a domain $D(A)$ representing A. Let $A = (A; \sigma_1, \ldots, \sigma_k)$ be a Σ-algebra, let $\{\equiv_i\}_{i \in I}$ be a fixed family of separating congruences on A which is upward consistent, and assume that the index set I is closed under finite suprema. Let $A_i = A/{\equiv_i}$ be the set of equivalence classes of $\equiv_i$, and let

$$\mathcal{C} = \biguplus \{A_i : i \in I\},$$

the disjoint union of the A_i. We denote an element of $\mathcal{C}$ by $[a]_i$, where $a \in A$ and $i \in I$, thus letting the i indicate that $[a]_i$ is taken from A_i in the disjoint union. We define a partial order on $\mathcal{C}$ by

$$[a]_i \sqsubseteq [b]_j \iff i \leq j \text{ and } a \equiv_i b.$$

We claim that $\mathcal{C}$ with this ordering is a cusl. Clearly $[a]_0$ is the least element in $\mathcal{C}$ by our standing assumption on $\equiv_0$ that all elements are $\equiv_0$ equivalent, where 0 is the least element in I. Now suppose $[a]_i$ and $[b]_j$ are bounded in $\mathcal{C}$ by, say, $[c]_k$. Then $i, j \leq k$ and $a \equiv_i c$ and $b \equiv_j c$. Thus $[c]_{i \vee j}$ is an upper bound. To see that $[c]_{i \vee j}$ is the least upper bound let $[d]_m$ be another upper bound. This means that $i, j \leq m$ and $a \equiv_i d$ and $b \equiv_j d$. But then $c \equiv_i d$ and $c \equiv_j d$ and hence, by upward consistency, $c \equiv_{i \vee j} d$. It follows that $[c]_{i \vee j} \leq [d]_m$.

Having shown that $\mathcal{C}$ is a cusl we now define $D(A) = \overline{\mathcal{C}}$, the ideal completion of the cusl $\mathcal{C}$. Identifying the principal ideals in $D(A)$ with their generating elements in $\mathcal{C}$ we obtain $D(A)_c = \mathcal{C}$.

Note that if $I = \omega$ then $D(A)$ is a tree of height ω, where the maximal elements in the domain correspond to the infinite branches in the tree.

Given a Σ-algebra A with a family of separating congruences $\{\equiv_i\}_{i \in I}$ we form the Σ-algebra

$$\hat{A} = \lim_{\leftarrow} A/{\equiv_i}$$

the inverse limit of the $A/{\equiv_i}$ w.r.t. the homomorphisms $\phi_i^j : A_j \to A_i$ defined by $\phi_i^j([a]_j) = [a]_i$, for $i \leq j$. Without going into details of the inverse limit construction we only remark that $\lim_{\leftarrow} A/{\equiv_i}$ should be regarded as the *completion* of A.

The inverse limit $\hat{A} = \lim_{\leftarrow} A/{\equiv_i}$ has a natural topology obtained from the family of separating congruences $\{\equiv_i\}_{i \in I}$ as follows. Let $\hat{\phi}_i \colon \hat{A} \to A/{\equiv_i}$ be the homomorphisms obtained in the inverse limit. Then a base for the topology on $\hat{A}$ is given by

$$\hat{B}(a, i) = \{x \in \hat{A} : \hat{\phi}_i(x) = [a]_i\} \text{ for } a \in A \text{ and } i \in I.$$

This makes $\hat{A}$ into a topological Σ-algebra.

Example 3.13. Referring to Examples 3.10 we have that the completion of $\mathbb{N}$ as in (i) is the one-point compactification, where one element ∞ is added. The completion of $T(\Sigma, X)$ in (ii) is the set $T^{\infty}(\Sigma, X)$ of all finite and infinite terms. The completion of a local ring R as in (iii) is the standard construction of completion for local rings. Finally, the completion of the Cantor space in (iv) gives us the Cantor space back, since the Cantor space already is complete.

Proposition 3.14. *Let A be an ultrametric Σ-algebra and let $\{\equiv_n\}_{n\in\omega}$ be a family of separating congruences on A obtained from the metric as above. Let $\bar{A}$ be the metric completion of A. Then $\bar{A}$ is an ultrametric Σ-algebra and*

$$\bar{A} \cong \lim_{\leftarrow} A/\equiv_n$$

as topological algebras.

We now consider representations of operations on A.

Definition 3.15. Let A be a Σ-algebra with a family of separating congruences $\{\equiv_i\}_{i\in I}$ which is upward consistent and where the index set I is closed under finite suprema. Let $\lambda : I^n \to I$ be a function which is monotone in each argument. We say that a function $f : A^n \to A$ is λ-*congruent* if for each $i_1, \ldots, i_n \in I$,

$$a_1 \equiv_{i_1} b_1, \ldots, a_n \equiv_{i_n} b_n \Longrightarrow f(a_1, \ldots, a_n) \equiv_{\lambda(i_1,\ldots,i_n)} f(b_1, \ldots, b_n).$$

Let $f : A^n \to A$ be λ-congruent. Then define

$$\phi_f^\lambda([a_1]_{i_1}, \ldots, [a_n]_{i_n}) = [f(a_1, \ldots, a_n)]_{\lambda(i_1,\ldots,i_n)}.$$

Lemma 3.16. *Let $f : A^n \to A$ be λ-congruent. Then $\phi_f^\lambda : D(A)_c^n \to D(A)$ is well-defined and monotone.*

Proof. That ϕ_f^λ is well-defined follows from the λ-congruence of f. To prove that f is monotone it suffices to prove monotonicity in each argument. So we assume for notational simplicity that f is unary. Suppose $[a]_i \sqsubseteq [b]_j$, that is $i \leq j$ and $a \equiv_i b$. By the λ-congruence of f we have $f(a) \equiv_{\lambda(i)} f(b)$, and by the monotonicity of λ we have $\lambda(i) \leq \lambda(j)$. Thus

$$\phi_f^\lambda([a]_i) = [f(a)]_{\lambda(i)} = [f(b)]_{\lambda(i)} \sqsubseteq [f(b)]_{\lambda(j)} = \phi_f^\lambda([b]_j).$$

□

It follows that we can extend ϕ_f^λ uniquely to a continuous function $\phi_f^\lambda : D(A)^n \to D(A)$. This is the *representation* of f with respect to λ on $D(A)$.

The algebras A and $\hat{A}$ are topologically embedded into $D(A)$ as follows. $\Theta : A \to D(A)$ is defined by

$$\Theta(a) = \{[a]_i : i \in I\}$$

and $\Psi: \hat{A} \to D(A)$ is defined by

$$\Psi(x) = \{[\hat{\phi}_i(x)]_i : i \in I\}.$$

It is easy to see that these functions are well defined embeddings and in fact homeomorphisms between A and $\Theta[A]$ and $\hat{A}$ and $\Psi[\hat{A}]$, respectively.

Definition 3.17. Let $f : A^n \to A$ be λ-congruent. Then f is *continuous with respect to* λ if λ is unbounded, that is for each $i \in I$ there is $i_1, \ldots, i_n \in I$ such that $i \leq \lambda(i_1, \ldots, i_n)$.

Lemma 3.18. *Let $f : A^n \to A$ be continuous with respect to λ. Then ϕ_f^λ is a faithful representation of f, that is for each $a_1, \ldots, a_n \in A$,*

$$\Theta(f(a_1, \ldots, a_n)) = \phi_f^\lambda(\Theta(a_1), \ldots, \Theta(a_n)).$$

Proof. For notational simplicity we assume that $n = 1$. Note that for $a \in A$, $\Theta(a) = \bigsqcup\{[a]_i : i \in I\}$ and $\operatorname{approx}(\Theta(a)) = \{[a]_i : i \in I\}$. Thus

$$\begin{array}{rcl}
\phi_f^\lambda(\Theta(a)) & = & \phi_f^\lambda(\bigsqcup\{[a]_i : i \in I\}) \\
& = & \bigsqcup\{\phi_f^\lambda([a]_i) : i \in I\} \\
& = & \bigsqcup\{[f(a)]_{\lambda(i)} : i \in I\} \\
& \sqsubseteq & \bigsqcup\{[f(a)]_j : j \in I\} \\
& = & \Theta(f(a)).
\end{array}$$

The converse inequality follows from the continuity condition on λ. For given $j \in I$ choose $i \in I$ such that $\lambda(i) \geq j$. Then

$$\phi_f^\lambda(\Theta(a)) \sqsupseteq [f(a)]_{\lambda(i)} \sqsupseteq [f(a)]_j$$

so $\phi_f^\lambda(\Theta(a)) \sqsupseteq \Theta(f(a))$. □

We have shown that each function $f : A^n \to A$, continuous with respect to λ, extends continuously to the whole domain $D(A)$ by ϕ_f^λ. Suppose that the index set I is a lattice. Then, assuming σ_i is an n-ary operation in A, we define $\lambda_i : I^n \to I$ by

$$\lambda_i(i_1, \ldots, i_n) = i_1 \wedge \ldots \wedge i_n.$$

It follows that σ_i is λ_i-congruent and that λ_i is unbounded. Thus $\phi_{\sigma_i}^{\lambda_i}$ represents σ_i on $D(A)$.

We have arrived at the main theorem of this section. The remaining details to verify are left to the reader.

Theorem 3.19. *Let $A = (A; \sigma_1, \ldots, \sigma_k)$ be a Σ-algebra together with a family of separating congruences $\{\equiv_i\}_{i \in I}$ on A which is upward consistent and suppose the index set I is a lattice with a least element. Let $\hat{A} =$*

$\lim_{\leftarrow} A/{\equiv_i}$ *and let* $\Theta : A \to D(A)$ *and* $\Psi : \hat{A} \to D(A)$ *be the embeddings defined above. Then* A *and* $\hat{A}$ *are representable by the* Σ*-domain*

$$D(A) = (D(A); \sqsubseteq, \perp; \phi_{\sigma_1}^{\lambda_1}, \ldots, \phi_{\sigma_k}^{\lambda_k}),$$

where $(D(A), \Psi[\hat{A}], \Psi^{-1})$ *is a representation of* $\hat{A}$ *and* $(D(A), \Theta[A], \Theta^{-1})$ *is a representation of* A*.*

Using Proposition 3.14 we obtain the corresponding result for ultrametric algebras.

Theorem 3.20. *Suppose* $A = (A; \sigma_1, \ldots, \sigma_k)$ *is an ultrametric* Σ*-algebra with non-expansive operations and let* $\bar{A}$ *be the ultrametric completion of* A*. Then there is a* Σ*-domain* $D(A) = (D(A); \hat{\sigma}_1, \ldots, \hat{\sigma}_k)$ *such that*

(i) $D(A)_m$ *is a* Σ*-substructure of* $D(A)$*;*

(ii) $D(A)_m \cong \bar{A}$ *as topological algebras;*

(iii) $(D(A), D(A)_m, \Psi^{-1})$ *is a homeomorphic domain representation of* $\bar{A}$*; and*

(iv) $(D(A), \Theta[A], \Theta^{-1})$ *is a homeomorphic domain representation of* A*.*

We conclude this section by showing how the Banach fixed point theorem is obtained from the least fixed point theorem for domains.

Let (X, d) be a complete metric space. Then $f \colon X \to X$ is said to be a *contraction mapping* if there is a constant $c < 1$ such that for each $x, y \in X$,

$$d(f(x), f(y)) < c \cdot d(x, y).$$

Proposition 3.21. (Banach) *Let* (X, d) *be a complete metric space and suppose* $f \colon X \to X$ *is a contraction mapping. Then* f *has a unique fixed point, i.e., there is a unique* $x_0 \in X$ *such that* $f(x_0) = x_0$*.*

Proof. We prove the proposition for ultrametric spaces using our domain representation. A similar proof holds for arbitrary complete metric spaces using domain representations described in Section 3.3.

We may without loss of generality assume X is bounded, say by 1. Uniqueness is proved in the usual way. Let $c < 1$ be the constant associated with the contraction mapping f. Define a family $\{\equiv_n\}_{n \in \omega}$ of separating equivalences on X by

$$x \equiv_n y \iff d(x, y) \leq c^n.$$

Let $\hat{X} = \lim_{\leftarrow} X/\equiv_n$, the completion of X w.r.t. $\{\equiv_n\}_{n \in \omega}$. Then $\hat{X}$ and X are homeomorphic since X is complete, so we identify $X, \hat{X}$ and $D(X)_m$.

Note that if $d(x, y) \leq c^n$ then

$$d(f(x), f(y)) \leq c \cdot d(x, y) \leq c^{n+1}$$

i.e. $x \equiv_n y \implies f(x) \equiv_{n+1} f(y)$. Thus f is s-congruent where s is the successor function $s(n) = n + 1$. Let $\rho: D(X) \to \omega \cup \{\omega\}$ be a rank function defined by

$$\rho(x) = \begin{cases} n & \text{if } x \in X/\equiv_n \\ \omega & \text{if } x \in D(A)_m. \end{cases}$$

Let $\phi_f^s: D(X) \to D(X)$ be the function representing f with respect to s. Thus for $x \in X$, $f(x) = \phi_f^s(x)$ by Lemma 3.18. Furthermore, from the definition of ϕ_f^s, $\rho(x) = n \implies \rho(\phi_f^s(x)) \geq n + 1$. It follows that

$$\rho(\text{fix}(\phi_f^s)) = \rho(\bigsqcup_n (\phi_f^s)^n(\perp)) = \omega$$

and hence $\text{fix}(\phi_f^s) \in X$. Now

$$f(\text{fix}(\phi_f^s)) = \phi_f^s(\text{fix}(\phi_f^s)) = \text{fix}(\phi_f^s),$$

i.e., $\text{fix}(\phi_f^s)$ is the fixed point of f. $\square$

3.3 Standard algebraic representations

In the previous section we showed that ultrametric spaces have algebraic homeomorphic domain representations where each representing element is maximal. Thus these representations are upwards closed, local and dense. It is the case that algebras important in logic and semantics often are ultrametric with non-expansive operations. However, in mathematical analysis this is no longer the case. For example, the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ are far from being ultrametric and the same is true for Banach spaces and Hilbert spaces usually studied. In this section we discuss domain representations for such spaces.

Exercise 3.22. Let D be an algebraic domain and let D_m denote the set of maximal elements in D. Show that the Scott topology on D_m has a clopen base (i.e., each element in the base is both open and closed).

It follows that D_m is *totally disconnected*. This means that for $x, y \in D_m$, if $x \neq y$ then there are disjoint open sets U and V such that $D_m = U \cup V$, $x \in U$ and $y \in V$.

Theorem 3.23. (Blanck [5]) *A compact topological space X has an algebraic homeomorphic representation by maximal elements if, and only if, X is totally disconnected.*

We conclude that if we want to represent spaces used in analysis, such as the reals $\mathbb{R}$, we have two alternatives. The first alternative is to use continuous domain representations. It is commonly the case that the represented space X can be topologically embedded into the maximal elements of the representing continuous domain. The second alternative, which we will pursue here, is to use algebraic domain representations (D, D^R, ν) where by necessity the quotient $D^R/\equiv_\nu$ is non-trivial.

Each approach has its advantage. A representation by maximal elements is appealing but continuous domains are generally more complicated objects than algebraic domains. Algebraic domains, on the other hand, are "concrete" and simple and arise in a natural way from approximations as described in Section 2.1. But an algebraic domain representation often necessitates a quotient structure. From a computational point of view the latter is not as problematic as it may seem since computations are performed on the representation and are only induced on the quotient structure.

From Proposition 2.28 we see that every continuous domain has an algebraic dense retract domain representation. It follows that every space X representable by a continuous domain is also representable by an algebraic domain. Thus the choice is mainly a matter of convenience.

Let X be a topological space. We shall (essentially) use the construction in Section 2.1 to obtain an algebraic retract representation (D, D^R, ν) of X.

We choose as a set of approximations a family of subsets of X satisfying the following with respect to the topology of X.

Definition 3.24. Let X be a topological space. Then a family P of non-empty subsets of X is a *neighbourhood system* if $X \in P$ and

(i) if $F, G \in P$ and $F \cap G \neq \emptyset$ then $F \cap G \in P$; and

(ii) if $x \in U$, where U is open, then $(\exists F \in P)(x \in F^\circ \subseteq \bar{F} \subseteq U)$.

Some remarks on notation and concepts. For $F \subseteq X$, F° denotes the *interior* of F, i.e., $F^\circ = \bigcup\{U : U \subseteq F \text{ and } U \text{ open}\}$. Similarly, $\bar{F}$ denotes the *closure* of F, i.e., $\bar{F} = \bigcap\{H : F \subseteq H \text{ and } H \text{ closed}\}$.

A Hausdorff space X is said to be *regular* if whenever $x \in U$, where U is open, there is V open such that $x \in V \subseteq \bar{V} \subseteq U$. Condition (ii) above implies that a Hausdorff space with a neighbourhood system is regular.

Here are some examples of neighbourhood systems for X.

Example 3.25.

(i) A topological base $\mathcal{B}$ for a regular space X closed under intersection.

(ii) The set P of all non-empty compact sets (and X) for a locally compact regular space X.

(iii) The set P of all non-empty closed subsets of a regular space.

Of course, there is no need generally to take *all* compact or closed sets in Example 3.25. It is enough to take sufficiently many. This is crucial from a computational point of view.

Example 3.26. For $\mathbb{R}$ we have the following natural neighbourhood systems.

(i) $P_1 = \{(a, b) \colon a < b, a, b \in \mathbb{Q}\} \cup \{\mathbb{R}\}$;

(ii) $P_2 = \{[a, b] \colon a \leq b, a, b \in \mathbb{Q}\} \cup \{\mathbb{R}\}$.

Note that P_1 is a neighbourhood system of type (i) in Example 3.25 whereas P_2 is of type (ii) and type (iii). Perhaps surprisingly, the neighbourhood system P_2 will turn out to be preferable to P_1.

Let P be a neighbourhood system for X. Then $P = (P; \supseteq, X)$ is a cusl and it is an approximation structure for X via the approximation

$$F \prec x \Longleftrightarrow x \in F.$$

Let $\bar{P}$ be the ideal completion of P. It is an algebraic domain.

Definition 3.27. An ideal $I \in \bar{P}$ *converges* to a point $x \in X$ if for every open set U containing x there is $F \in I$ such that $x \in F \subseteq U$. I converges to x is denoted by $I \to x$.

This definition differs slightly from the one given in Section 2.1. However, for neighbourhood systems as in Example 3.25 (ii) and (iii) (for complete metric spaces) they coincide. It is easy to see that if X is a Hausdorff space then a converging ideal converges to a unique point.

Let X be a Hausdorff space and let $\bar{P}^R = \{I \in \bar{P} \colon I \text{ convergent}\}$. Define $\nu \colon \bar{P}^R \to X$ by

$$\nu(I) = x \Longleftrightarrow I \to x.$$

We will show that $(\bar{P}, \bar{P}^R, \nu)$ is a retract representation of X. For $x \in X$ we define the important ideal

$$I_x = \{F \in P : x \in F^\circ\}.$$

Note that $I_x \to x$ and that $J \to x \Longleftrightarrow I_x \subseteq J$.

Theorem 3.28. *Let X be a Hausdorff space and P a neighbourhood system for X. Then $(\bar{P}, \bar{P}^R, \nu)$ is an algebraic retract representation of X which is upwards closed.*

Proof. Suppose $U \subseteq X$ is open and $\nu(I) = x \in U$. By Definition 3.24 (ii) there is $F \in P$ such that $x \in F^\circ \subseteq \bar{F} \subseteq U$. Thus $F \in I_x$ and hence $F \in I$. Suppose $J \in \bar{P}^R$ and $F \in J$. Then, clearly $\nu(J) \in \bar{F}$, i.e., $\uparrow F \subseteq U$. (As usual F is identified with its principal ideal $\downarrow F$.) Thus ν is continuous.

Define $\eta: X \to \bar{P}^R$ by $\eta(x) = I_x$. Then $\nu \circ \eta = \mathrm{id}_X$. Furthermore η is continuous since for $F \in P$,

$$\eta^{-1}(\uparrow F \cap \bar{P}^R) = \{x \in X: F \in I_x\} = \{x \in X: x \in F^\circ\} = F^\circ.$$

The representation is upwards closed since I_x is the smallest ideal converging to x. □

Exercise 3.29. Show that the domain representation of $\mathbb{R}$ obtained from P_1 in Example 3.26 is not local, but the one obtained from P_2 is local.

The exercise indicates why open neighbourhood systems such as P_1 may not be as useful as closed neighbourhood systems.

Suppose P is a neighbourhood system for X consisting of only closed sets. Then for $x \in X$,

$$I^x = \{F \in P: x \in F\}$$

is an ideal, $I^x \to x$, and I^x is the largest ideal converging to x.

Theorem 3.30. *Every regular Hausdorff space X has an algebraic retract representation which is upwards closed, dense and local.*

Proof. Choose the neighbourhood system P consisting of all non-empty closed subsets of X. Then the representation $(\bar{P}, \bar{P}^R, \nu)$ is dense since for $F \in P$ there is $x \in F$ and hence $F \in I^x$, i.e., $I^x \in \uparrow F$. The representation is local since I^x is the largest ideal converging to x. □

Example 3.31. The algebraic interval domain
We let $\mathcal{R}$ be the domain representation of $\mathbb{R}$ obtained from P_2 in Example 3.26. Note that each irrational $x \in \mathbb{R}$ is represented by exactly one ideal, namely $I_x = I^x$. Each rational $x \in \mathbb{R}$, on the other hand, is represented by four distinct ideals: I_x, I^x, and

$$I_x^- = \{[a,b] : a < x \leq b\} \text{ and } I_x^+ = \{[a,b] : a \leq x < b\}.$$

We know by Theorem 3.23 that $\mathbb{R}$ cannot be homeomorphically represented by the maximal elements in $\mathcal{R}$ since $\mathbb{R}$ is a connected space.

We close this part by mentioning a theorem which says that if we want an upwards closed algebraic retract domain representation of a space X then we may as well use a neighbourhood system. (In particular they exist.)

Theorem 3.32. (Blanck [5]) *A topological space X has an upwards closed algebraic retract domain representation if, and only if, X is a regular Hausdorff space.*

A domain representation of X obtained from a neighbourhood system is said to be a *standard algebraic representation* of X.

Next we consider the problem of lifting continuous functions to the representing domains.

Theorem 3.33. *Let X and Y be regular Hausdorff spaces with neighbourhood systems P and Q, respectively. Let $(\bar{P}, \bar{P}^R, \nu)$ and $(\bar{Q}, \bar{Q}^R, \mu)$ be the domain representations of X and Y obtained from P and Q. Suppose $f\colon X \to Y$ is a continuous function. Then there is a continuous function $\bar{f}\colon \bar{P} \to \bar{Q}$ such that for all $I \in \bar{P}^R$,*

$$\mu(\bar{f}(I)) = f(\nu(I)),$$

i.e., $\bar{f}$ is a lifting or representation of f.

Proof. Given continuous $f\colon X \to Y$ define $\bar{f}\colon P \to \bar{Q}$ by

$$\bar{f}(F) = \{G \in Q\colon f[F] \subseteq G^\circ\}.$$

It is easily verified that $\bar{f}(F)$ is an ideal and that $\bar{f}$ is monotone. We also denote by $\bar{f}$ its unique continuous extension to all of $\bar{P}$. In fact for $I \in \bar{P}$,

$$\bar{f}(I) = \{G \in Q\colon (\exists F \in I)(f[F] \subseteq G^\circ)\}.$$

Suppose $I \in \bar{P}^R$ and $\nu(I) = x$. Then $I_x \subseteq I$ and hence $\bar{f}(I_x) \subseteq \bar{f}(I)$. Thus it suffices to show that $I_{f(x)} \subseteq \bar{f}(I_x)$.

Let $G \in I_{f(x)}$. Then $f(x) \in G^\circ$ and $x \in f^{-1}[G^\circ]$. But then there is $F \in I_x$ such that $F \subseteq f^{-1}[G^\circ]$. This shows that $G \in \bar{f}(I_x)$. □

Corollary 3.34. *Let $A = (A; \sigma_1, \ldots, \sigma_k)$ be a topological Σ-algebra and suppose A is a regular Hausdorff space. Then A has an algebraic retract domain representation which is upwards closed, dense and total.*

Proof. This follows from the theorem since if P is a neighbourhood system for A then P^n is a neighbourhood system for A^n. □

The ring operations on $\mathbb{R}$ are continuous so the ring of real numbers has a nice domain representation. In Section 5 we will show that the domain representation for the ring $\mathbb{R}$ obtained from P_2 in Example 3.26 is effective.

3.4 Type structures

In this section we briefly return to the category **PER(Dom)** discussed in Section 3.1. Here we restrict ourselves to the case when **Dom** is the category of algebraic domains with continuous functions as morphisms.

The category **PER(Dom)** is cartesian closed as noted by Proposition 3.6. This means that we can build type structures over a domain with a per $(D, \equiv)$. Recall that if $(D, \equiv_D)$ and $(E, \equiv_E)$ are domains with pers then $\equiv_{[D\to E]}$ defined by

$$f \equiv_{[D\to E]} g \iff \forall x, y(x \equiv_D y \implies f(x) \equiv_E g(y))$$

is a per on $[D \to E]$ and

$$([D \to E], \equiv_{[D\to E]})$$

is the exponent (function space) in **PER(Dom)**.

We also define a per $\equiv_{D\times E}$ on $D \times E$ by

$$(x,y) \equiv_{D\times E} (z,w) \quad \Longleftrightarrow \quad x \equiv_D z \text{ and } y \equiv_E w.$$

The set of *finite type symbols* TS is defined inductively by $o \in \mathrm{TS}$; and if $\sigma, \tau \in \mathrm{TS}$ then $(\sigma \times \tau) \in \mathrm{TS}$ and $(\sigma \to \tau) \in \mathrm{TS}$.

Given a domain with per $(D_o, \equiv_o)$ we obtain a typestructure

$$\begin{cases} \mathcal{D}_o & = & (D_o, \equiv_o) \\ \mathcal{D}_{\sigma\times\tau} & = & (D_\sigma \times D_\tau, \equiv_{\sigma\times\tau}) \\ \mathcal{D}_{\sigma\to\tau} & = & ([D_\sigma \to D_\tau], \equiv_{\sigma\to\tau}). \end{cases}$$

Exercise 3.35. Let D and E be algebraic domains and suppose $f\colon D \to E$ and $g\colon D \to E$ are continuous. Show that if $x, y \in D$ then $x \sqcap y \in D$, where $x \sqcap y =$ infimum of x and y. Show also that $(f \sqcap g)(x) = f(x) \sqcap g(x)$.

We say that a per $\equiv$ on D is *upwards closed* if

$$x \equiv x \text{ and } x \sqsubseteq y \implies x \equiv y.$$

Consider the type structure above when the per $\equiv_\sigma$ is ignored. Then we obtain a *partial* type structure of domains in that it contains partial elements, e.g., $\perp$ in D_o. There is a natural way to isolate the *total* elements of this partial type structure as follows. A pair of elements is total if each component is total, and a function is total if it takes total elements to total elements. We thus obtain the following type structure of total elements.

$$\begin{cases} \bar{D}_o & = & \{x \in D_o : x \equiv_o x\} \\ \bar{D}_{\sigma\times\tau} & = & \bar{D}_\sigma \times \bar{D}_\tau \\ \bar{D}_{\sigma\to\tau} & = & \{f \in [D_\sigma \to D_\tau] : f[\bar{D}_\sigma] \subseteq \bar{D}_\tau\}. \end{cases}$$

Here is a useful connection between total elements and the corresponding pers. Note that it holds even when density is not assumed. The proposition indicates that our concepts are correct.

Proposition 3.36. (Longo-Moggi [26]) *Assume that $\equiv_o$ is upwards closed in the type structire $\{\mathcal{D}_\sigma\}_{\sigma\in\mathrm{TS}}$. Then for each $\sigma \in \mathrm{TS}$ and $x, y \in D_\sigma$,*

$$x \equiv_\sigma y \Longleftrightarrow x \sqcap y \in \bar{D}_\sigma$$

Proof. By an easy induction on σ. $\square$

When studying a type structure of domains with pers there are two natural questions to be considered.

Lifting: Let $(D, \equiv_D)$ and $(E, \equiv_E)$ be domains with pers. Then $\bar{D}/{\equiv_D}$ and $\bar{E}/{\equiv_E}$ are topological spaces given the quotient topology from the Scott topology. Does a continuous function $f\colon \bar{D}/{\equiv_D} \to \bar{E}/{\equiv_E}$ have a continuous representation or lifting $\bar{f}\colon D \to E$?

Density: Is the per $\equiv_D$ dense in D, i.e., does every non-empty open subset of D contain an element x such that $x \equiv_D x$? For algebraic domains this equivalent to the question if for each $a \in D_c$ there is $x \in D$ such that $a \sqsubseteq x$ and $x \equiv_D x$.

Exercise 3.37.

(i) Let $(D, \equiv)$ be a dense domain with per. Show that every continuous function $f\colon \bar{D}/{\equiv} \to \mathbb{N}$ (where $\mathbb{N}$ is given the discrete topology) has a continuous lifting $\bar{f}\colon D \to \mathbb{N}_\perp$.

(ii) Let $(D, \equiv)$ be an algebraic domain with per and assume D is separable and $\equiv$ is upwards closed. Show that every continuous function $f\colon \bar{D}/{\equiv} \to \mathbb{N}$ has a continuous lifting $\bar{f}\colon D \to \mathbb{N}_\perp$.

Example 3.38. (Ershov [15]) Let $D_o = \mathbb{N}_\perp$ and let $n \equiv_o m \Longleftrightarrow n = m \in \mathbb{N}$. Then

$$\bar{D}_\sigma / \equiv_\sigma \cong Ct(\sigma)$$

where $\{Ct(\sigma)\}_{\sigma \in \mathrm{TS}}$ is the type structure of the Kleene-Kreisel total continuos functionals ([21, 22]). Moreover the two type structures $\{\bar{D}_\sigma / \equiv_\sigma\}_\sigma$ and $\{Ct(\sigma)\}_\sigma$ are naturally isomorphic in that they respect the evaluation functionals. The type structure satisfies both lifting and density.

Example 3.39. Consider the neighbourhood system P for $\mathbb{R}$ given in Example 3.26 (ii) and let $\mathcal{R}$ be the ideal completion of P. Let also $\mathcal{R}$ denote the standard domain representation $\mathcal{R} = (\mathcal{R}, \mathcal{R}^R, \nu)$ for $\mathbb{R}$. Then define for $x, y \in \mathcal{R}$,

$$x \equiv_o y \Longleftrightarrow x, y \in \mathcal{R}^R \text{ and } \nu(x) = \nu(y).$$

Then $(\mathcal{R}, \equiv)$ is a domain with per. Note that $\mathcal{R}^R / \equiv \; \cong \; \mathbb{R}$ and that $\equiv$ is upwards closed and dense. Consider the type structure over $(\mathcal{R}, \equiv)$. It has been shown for the first few levels in [7] and in general by D. Normann [31] that the type structure is dense and has liftings. Note that the type structure obtained from the neighbourhood system given in Example 3.26 (i) is not dense. The topology of $\overline{[\mathcal{R} \to \mathcal{R}]} / \equiv_{\mathcal{R} \to \mathcal{R}}$ is the *compact-open* topology (see di Gianantonio [16] and Blanck [5]).

We close this section by mentioning that also transfinite type structures have been studied. Thus, for example, Waagbø [45] has given a domain representation of Martin-Löf type theory, building on work by Palmgren and Stoltenberg-Hansen [32] and Normann [30].

4 Effective domains

Until now we have, tacitly, identified computability with continuity. This identification, though of course not generally valid, may be defended as follows. There are rather general theorems, such as Ceitin's theorem (see Ceitin [10] and Moschovakis [29]), which state that computable functions between effective metric spaces (satisfying natural conditions) are continuous. On the other hand it is an empirical fact that "natural" functions in analysis shown to be continuous are also computable. Nonetheless we need a precise notion of computability to even make such statements. In addition the effective theory is interesting by itself.

In this section we will impose and study notions of computability or effectivity on domains. The type of effectivity we consider is based on what may be described as "concrete" computability, i.e., based on computations which may in principle be performed on a digital computer. Put differently, our computability theory is driven by the partial recursive functions. We use the Mal'cev-Ershov-Rabin theory of numberings in order to extend computability from the natural numbers to other structures, such as domains.

We know from 2.3.1 that an algebraic cpo is completely determined by its cusl of compact elements and, more generally, a continuous cpo is determined by a base. Moreover continuous functions between continuous cpos are completely determined by their behaviour on the bases of the cpos (see Proposition 2.27). Thus it suffices to compute on a base of the cpo. A further argument is that a base for a cpo is an approximation structure with respect to the way below relation and that we compute on cpos by computing on the approximations.

We assume some very basic knowledge of recursion theory. This can be found in any text on recursion theory and also in Stoltenberg-Hansen et al. [39]. Here are a few conventions and facts. Throughout we let $\{W_e\}_{e\in\omega}$ be a standard numbering of the recursively enumerable (r.e.) sets. Let W be an r.e. set. An *enumeration* of W is a total recursive function $\lambda n.W^n$, where W^n is (a canonical index for) a finite set, such that the following hold:

(i) $m \leq n \implies W^m \subseteq W^n$, and

(ii) $W = \bigcup_{n\in\omega} W^n$.

Each r.e. set has an enumeration. In fact, there is a total recursive function $\lambda en.W_e^n$ such that for each e, $\lambda n.W_e^n$ is an enumeration of W_e.

Let A be a set. A *numbering* of A is a surjective function $\alpha\colon \omega \to A$. It should be thought of as a coding of A by natural numbers. A subset $S \subseteq A$ is *α-semidecidable* if $\alpha^{-1}(S)$ is r.e. and S is *α-decidable* if $\alpha^{-1}(S)$ is recursive.

Let B be a set with a numbering β. Then a function $f\colon A \to B$ is said to be *(α,β)-computable* if there is a recursive function $\bar{f}\colon \omega \to \omega$ such that for each n,

$$f(\alpha(n)) = \beta(\bar{f}(n)).$$

We say that $\bar{f}$ *tracks* f.

We develop the theory for continuous cpos. The treatment for algebraic cpos is in certain places slightly simpler and the reader may for convenience think of a domain as an algebraic cpo and of its base as the cusl of compact elements.

Definition 4.1. A continuous cpo $D = (D; \sqsubseteq, \perp)$ is *weakly effective* if D has a base B for which there is a surjective function

$$\alpha: \omega \to B$$

such that the relation $\alpha(n) \ll \alpha(m)$ is a recursively enumerable relation on ω.

Terminology: In this section we use "domain" to denote a continuous cpo unless otherwise specified.

We denote a domain weakly effective under a numbering α by (D, α). Implicit in this notation is a fixed base $B = \alpha[\omega]$. We will use the notation B for such a base. Thus for $x \in D$, we let $\mathrm{approx}_\alpha(x) = \{a \in B: a \ll x\}$, the set of approximations of x w.r.t. the base B determined by α. The condition on the numbering in Definition 4.1 is usually stated as $\ll$ is α-*semidecidable.*

Computable elements are those that can be effectively approximated and effective functions are those whose values can be effectively approximated from effective approximations of the arguments.

Definition 4.2. Let (D, α) and (E, β) be weakly effective domains.

(i) An element $x \in D$ is α-*computable* if the set

$$\{n \in \omega : \alpha(n) \ll x\} = \alpha^{-1}(\mathrm{approx}_\alpha(x))$$

is r.e. An r.e. index for the set $\alpha^{-1}(\mathrm{approx}_\alpha(x))$ is an *index* for x.

(ii) A continuous function $f: D \to E$ is (α, β)-*effective* if the relation

$$\beta(m) \ll f(\alpha(n))$$

is r.e. An r.e. index for the set $\{\langle m, n\rangle: \beta(m) \ll f(\alpha(n))\}$ is an *index* for f.

The intuition in part (ii) is that the approximations to $f(x)$ are generated simultaneously with the approximations to x.

Notation $D_{k,\alpha} = \{x \in D: x\ \alpha\text{-computable}\}$.

Proposition 4.3. *Let (D, α), (E, β) and (F, γ) be weakly effective domains and suppose $f: D \to E$ and $g: E \to F$ are (α, β)-effective and (β, γ)-effective, respectively.*

(i) *If $x \in D$ is α-computable then $f(x)$ is β-computable.*

(ii) *$h = g \circ f$ is (α, γ)-effective.*

Proof. (i) Let $W = \{n: \alpha(n) \ll x\}$. Then W is r.e. and

$$\beta(m) \ll f(x) \iff (\exists n \in W)(\beta(m) \ll \alpha(n))$$

by Proposition 2.27. (ii) is proved similarly. □

Observe that the proof is *uniform*. This means there is a total recursive function which given indices for x and f computes an index for $f(x)$. Similarly, there is a recursive function which given indices for f and g computes an index for $g \circ f$.

We now show that $D_{k,\alpha}$ has a natural numbering.

Theorem 4.4. *Let (D, α) be a weakly effective domain with base $B = \alpha[\omega]$. Then there is a numbering $\bar{\alpha}: \omega \to D_{k,\alpha}$ such that*

(i) *the inclusion mapping $\iota: B \to D_{k,\alpha}$ is $(\alpha, \bar{\alpha})$-computable; and*

(ii) *the relation $\alpha(n) \ll \bar{\alpha}(m)$ is r.e., i.e., $\mathrm{approx}_\alpha(\bar{\alpha}(m))$ is α-semidecidable uniformly in m.*

Proof. Let $s: \omega \to \omega$ be a total recursive function such that

$$W_{s(e)} = \{n: (\exists m \in W_e)(\alpha(n) \ll \alpha(m))\}$$

and let $\lambda n. \ll^n$ be a recursive enumeration of $\ll$.

We define total recursive functions $f(e, n)$ and $r(e, n)$, a recursive relation $R(e, n)$ and a recursive enumeration of finite sets V_e^n. Set $f(e, 0) = n_\perp$, where $\alpha(n_\perp) = \perp$, $r(e, 0) = 0$, and $V_e^0 = \{n_\perp\} = \{f(e, 0)\}$. Let

$$R(e, n) \iff (\exists k \in W_{s(e)}^n)(\alpha[V_e^n \cup W_{s(e)}^{r(e,n)}] \ll^n \alpha(k)).$$

Now define

$$f(e, n+1) = \begin{cases} (\text{some } k \in W_{s(e)}^n)(\alpha[V_e^n \cup W_{s(e)}^{r(e,n)}] \ll^n \alpha(k)) & \text{if } R(e, n) \\ f(e, n) & \text{if } \neg R(e, n); \end{cases}$$

$$r(e, n+1) = \begin{cases} r(e, n) + 1 & \text{if } R(e, n) \\ r(e, n) & \text{if } \neg R(e, n); \end{cases}$$

and

$$V_e^{n+1} = V_e^n \cup \{f(e, n+1)\}.$$

These functions are clearly recursive. Let $V_e = \bigcup_n V_e^n$. Then V_e is r.e. with an r.e. index computed uniformly from e. Let $t: \omega \to \omega$ be a total recursive function such that

$$W_{t(e)} = \{m \in \omega: (\exists k \in V_e)(\alpha(m) \ll \alpha(k))\}.$$

It is not hard to show, using interpolation when appropriate, that $\alpha[W_{t(e)}]$ is an ideal, that $\alpha[V_e]$ is a $\ll$-chain and that $\bigsqcup \alpha[W_{t(e)}] = \bigsqcup \alpha[V_e]$.

Define $\bar{\alpha}: \omega \to D_{k,\alpha}$ by

$$\bar{\alpha}(e) = \bigsqcup \alpha[W_{t(e)}] = \bigsqcup \alpha[V_e].$$

Then $\bar{\alpha}$ is a surjective numbering of $D_{k,\alpha}$.

Let $\iota: B \to D_{k,\alpha}$ be the inclusion mapping and let g be a total recursive function such that $W_{g(n)} = \{m: \alpha(m) \ll \alpha(n)\}$. Then

$$\bar{\alpha}(g(n)) = \bigsqcup \alpha[W_{tg(n)}] = \bigsqcup \alpha[W_{g(n)}] = \alpha(n)$$

since $W_{tg(n)} = W_{g(n)}$. Thus ι is $(\alpha, \bar{\alpha})$-computable.

Finally, it follows that $\alpha(n) \ll \bar{\alpha}(e)$ is r.e. since

$$\alpha(n) \ll \bar{\alpha}(e) = \bigsqcup \alpha[W_{t(e)}] \iff (\exists m \in W_{t(e)})(\alpha(n) \ll \alpha(m)).$$

□

A numbering γ of $D_{k,\alpha}$ satisfying (i) and (ii) of the theorem, as does $\bar{\alpha}$, is said to be a *constructive numbering* of $D_{k,\alpha}$. We say that $\bar{\alpha}$ is the *canonical numbering* of $D_{k,\alpha}$ obtained from α.

From the proof of the theorem we easily abstract the following important facts.

Lemma 4.5. *Let (D, α) be a weakly effective domain and let $\bar{\alpha}$ be the canonical numbering of $D_{k,\alpha}$ obtained from α.*

(i) *There is a total recursive function $t: \omega \to \omega$ such that for each e, $W_{t(e)} = \alpha^{-1}(\mathrm{approx}_\alpha(\bar{\alpha}(e)))$ and $\bar{\alpha}(t(e)) = \bar{\alpha}(e)$.*

(ii) *For each $x \in D_{k,\alpha}$, $W_e = \alpha^{-1}(\mathrm{approx}_\alpha(x)) \implies \bar{\alpha}(e) = x$.*

(iii) *If $\alpha[W_e]$ is directed then $\bar{\alpha}(e) = \bigsqcup \alpha[W_e]$.*

The following approximations will be used later.

Lemma 4.6. *Let (D, α) be a weakly effective domain with base $B = \mathrm{Im}(\alpha)$. There is a function $\tilde{\alpha}: \omega^2 \to B$ and a total recursive function $p: \omega^2 \to \omega$ such that for each $e \in \omega$,*

(i) $m \leq n \Longrightarrow \tilde{\alpha}(e,m) \ll \tilde{\alpha}(e,n)$;

(ii) $\bar{\alpha}(e) = \bigsqcup_n \tilde{\alpha}(e,n)$; *and*

(iii) $\tilde{\alpha}(e,n) = \alpha(p(e,n))$.

Proof. We use the notation in the proof of Theorem 4.4. The function $r(t(e),n)$ is unbounded so we define a recursive function q by $q(0) = 0$ and

$$q(n+1) = (\text{least } k > q(n))R(t(e),k).$$

Let $p(e,n) = f(t(e),q(n))$ and $\tilde{\alpha}(e,n) = \alpha(p(e,n))$. □

Definition 4.7. Let (D,α) be a weakly effective domain. A constructive numbering γ of $D_{k,\alpha}$ is *recursively complete* if there is a total recursive function h such that for each e,

$$\gamma[W_e]\text{directed } \Longrightarrow \gamma h(e) = \bigsqcup \gamma[W_e].$$

Theorem 4.8. *Let (D,α) be a weakly effective domain and let $\bar{\alpha}$ be the canonical numbering of $D_{k,\alpha}$.*

(i) *$\bar{\alpha}$ is recursively complete.*

(ii) *If γ is a recursively complete constructive numbering of $D_{k,\alpha}$ then* $\mathrm{id}\colon D_{k,\alpha} \to D_{k,\alpha}$ *is $(\bar{\alpha},\gamma)$-computable and $(\gamma,\bar{\alpha})$-computable.*

Part (ii) of the theorem says that $\bar{\alpha}$ and γ are equivalent as numberings of $D_{k,\alpha}$, i.e., $D_{k,\alpha}$ is *stable* with respect to recursively complete numberings.

Proof. (i) Let W_e be r.e. such that $\bar{\alpha}[W_e]$ is directed. Then

$$\alpha(n) \ll \bigsqcup \bar{\alpha}[W_e] \Longleftrightarrow (\exists k \in W_e)(\alpha(n) \ll \bar{\alpha}(k)).$$

Thus there is a total recursive function h such that

$$W_{h(e)} = \{n\colon (\exists k \in W_e)(\alpha(n) \ll \bar{\alpha}(k))\},$$

since $\bar{\alpha}$ is constructive. By Lemma 4.5 $\bar{\alpha}(h(e)) = \bigsqcup \bar{\alpha}[W_e]$.

The proof of (ii) is similar, again using Lemma 4.5. □

Example 4.9.

(i) Let (D,α) be a weakly effective flat domain. Then $D_{k,\alpha} = D$.

(ii) Let $\mathcal{P}$ be the domain of all partial functions from $\mathbb{N}$ into $\mathbb{N}$ ordered by graph inclusion. Let α be a standard numbering of the set $\mathcal{P}_c$ of finite functions. Then $\mathcal{P}_{k,\alpha}$ is the set of partial recursive functions. The numbering $\bar{\alpha}$ is a standard numbering of the partial recursive functions satisfying the universal property and the s-m-n theorem.

(iii) Let $\wp(\omega)$ be the domain of subsets of ω ordered by inclusion and let α be the canonical numbering of the finite sets. Then $\wp(\omega)_{k,\alpha}$ is the family of r.e. sets.

Exercise 4.10. Let $\{\cdot\}\colon \omega \to \mathcal{P}_{k.\alpha}$ be defined by $\{e\}(n) \simeq U(\langle e, n\rangle)$ where U is a partial recursive universal function. (This means that for each partial recursive function f there is e such that $f(n) \simeq U(\langle e, n\rangle)$). Show that $\{\cdot\}$ is a constructive numbering of $\mathcal{P}_{k,\alpha}$. Show further that if the s-m-n theorem also holds for $\{\cdot\}$ then $\{\cdot\}$ is recursively complete. Deduce a form of Roger's isomorphism theorem which states that all enumerations of the partial recursive functions which are universal and satisfy the s-m-n theorem are recursively equivalent.

We are now going to consider the equivalence of a function between weakly effective domains being computable and being effective in the sense of Definition 4.2. One direction is simple.

Proposition 4.11. *Let (D, α) and (E, β) be weakly effective domains. If $f\colon D \to E$ is (α, β)-effective then $f|_{D_{k,\alpha}}\colon D_{k,\alpha} \to E_{k,\beta}$ is $(\bar{\alpha}, \bar{\beta})$-computable.*

Proof. Considering the functions from Lemma 4.6 we have

$$f(\bar{\alpha}(e)) = f(\bigsqcup_{n\in\omega} \tilde{\alpha}(e, n)) = \bigsqcup_{n\in\omega} f\tilde{\alpha}(e, n).$$

But $\tilde{\alpha}(e, n) = \alpha(p(e, n))$ where p is total recursive, so

$$\beta^{-1}(\mathrm{approx}_\beta(f\bar{\alpha}(e))) = \{m\colon \exists n(\beta(m) \ll f(\alpha(p(e, n))))\}.$$

The right hand side is r.e. uniformly in e so there is a total recursive function g such that $W_{g(e)} = \{m\colon \exists n(\beta(m) \ll f(\alpha(p(e, n))))\}$. But then

$$\bar{\beta}(g(e)) = f\bar{\alpha}(e),$$

i.e., g recursively tracks $f|_{D_{k,\alpha}}$. □

Note that an index for g is obtained uniformly from an index for f.

We are now going to show the converse of the above proposition. We are, however, going to show something stronger, namely, that each $(\bar{\alpha}, \bar{\beta})$-computable function $f\colon D_{k,\alpha} \to E_{k,\beta}$ has an (α, β)-effective (and hence continuous) extension $\bar{f}\colon D \to E$.

First we need a recursion theoretic lemma, a consequence of the second recursion theorem.

Lemma 4.12. (Berger) *Let V and W be r.e. sets such that W contains all r.e. indices for V. Let r be a total recursive function and let $\lambda q.V^q$ be a recursive enumeration of V. Then there is $e \in W$ and $q \in \omega$ such that*

$$W_e = V^q \cup W_{r(q)}.$$

In other words, if W contains all r.e. indices for V then W also contains an index for $W_{r(q)}$ for some q, modulo a finite subset of V.

Proof. Let $\lambda q.W^q$ be an enumeration of W such that $W^0 = \emptyset$. Define W_e using the second recursion theorem by

$$x \in W_e \iff \exists q[(x \in V^q \wedge e \notin W^q) \vee (x \in W_{r(q)} \wedge e \in W^{q+1} - W^q)].$$

Suppose $e \notin W$. Then

$$x \in W_e \iff \exists q(x \in V^q) \iff x \in V,$$

i.e., $W_e = V$. But then $e \in W$ by assumption, so we conclude $e \in W$. Let q be such that $e \in W^{q+1} - W^q$. Then

$$x \in W_e \iff x \in V^q \vee x \in W_{r(q)}.$$

$\square$

We now connect effectivity with the Scott topology.

Definition 4.13. Let (D, α) be a weakly effective domain.

(i) $U \subseteq D$ is α-*effectively open* if there is an r.e. set W such that

$$U = \bigcup_{e \in W} \Uparrow\alpha(e).$$

An r.e. index for W is an α-*index* for U as an effectively open set.

(ii) $U \subseteq D_{k,\alpha}$ is α-*effectively open* in $D_{k,\alpha}$ if there is an α-effectively open set V such that $U = D_{k,\alpha} \cap V$.

Theorem 4.14. (Ershov) *Let (D, α) be a weakly effective domain. Then $U \subseteq D_{k,\alpha}$ is $\bar{\alpha}$-semidecidable if, and only if, U is α-effectively open in $D_{k,\alpha}$.*

Proof. Suppose U is α-effectively open and let W be an r.e. set such that $U = \bigcup_{e \in W}(D_{k,\alpha} \cap \Uparrow\alpha(e))$. Then

$$\bar{\alpha}(n) \in U \iff (\exists e \in W)(\alpha(e) \ll \bar{\alpha}(n))$$

so U is $\bar{\alpha}$-semidecidable.

For the converse suppose $U \subseteq D_{k,\alpha}$ is $\bar{\alpha}$-semidecidable and consider $W = \{e \in \omega : \alpha(e) \in U\}$. Note that W is r.e. since the inclusion $\iota : B \to D_{k,\alpha}$ is $(\alpha, \bar{\alpha})$-computable and U is $\bar{\alpha}$-semidecidable. Suppose we have shown

(i) U is upwards closed with respect to $\ll$ in $D_{k,\alpha}$, and

(ii) for each $x \in U$ there is $a \in B \cap U$ such that $a \ll x$.

Then $U = \bigcup_{e \in W}(D_{k,\alpha} \cap \Uparrow\alpha(e))$, i.e., U is α-effectively open.

To prove (i) we show

$$x \in U, y \in D_{k,\alpha} \text{ and } x \sqsubseteq y \implies y \in U.$$

Suppose $x = \bar{\alpha}(m)$ and $y = \bar{\alpha}(n)$ and let $W = \bar{\alpha}^{-1}(U)$. We assume $m \in W$ and $x \sqsubseteq y$ and we show $n \in W$. Let t be the total recursive function from Lemma 4.5 and consider $W_{t(m)} = \alpha^{-1}(\mathrm{approx}_\alpha(x))$. Suppose $W_e = W_{t(m)}$. Then

$$\bar{\alpha}(e) = \bar{\alpha}(t(m)) = \bar{\alpha}(m) = x \in U,$$

i.e., $e \in W$. Thus W contains all r.e. indices for $W_{t(m)}$. Let $r(q)$ be the constant function with value $t(n)$. Then by Lemma 4.12, there are numbers e and q such that $e \in W$ and $W_e = W^q_{t(m)} \cup W_{t(n)}$. But

$$W_{t(m)} = \alpha^{-1}(\mathrm{approx}_\alpha(x)) \subseteq \alpha^{-1}(\mathrm{approx}_\alpha(y)) = W_{t(n)}$$

so $W_e = W_{t(n)}$. Again by Lemma 4.5

$$y = \bar{\alpha}(n) = \bar{\alpha}(t(n)) = \bar{\alpha}(e)$$

and $e \in W$, so $y \in U$.

To prove (ii) we again let $W = \bar{\alpha}^{-1}(U)$ and $x = \bar{\alpha}(m) \in U$. Let p be the total recursive function from Lemma 4.6 and let r be a total recursive function such that $W_{r(q)} = \{p(t(m), q)\}$ for $q \in \omega$. Let

$$W^q_{t(m)} = \{k \leq q : \alpha(k) \ll^q \alpha(p(t(m), q))\}.$$

By the properties of t and p, we see that $\lambda q. W^q_{t(m)}$ is a recursive enumeration of $W_{t(m)}$. By Lemma 4.12 there is $e \in W$ and $q \in \omega$ such that

$$W_e = W^q_{t(m)} \cup W_{r(q)} = W^q_{t(m)} \cup \{p(t(m), q)\}.$$

But $\alpha(p(t(m), q))$ is top element in $\alpha[W_e]$. In particular $\alpha[W_e]$ is directed and hence

$$\bar{\alpha}(e) = \bigsqcup \alpha[W_e] = \alpha(p(t(m), q)) = \tilde{\alpha}(t(m), q) \ll \bar{\alpha}(t(m)) = \bar{\alpha}(m) = x.$$

By the above, $\bar{\alpha}(e)$ is the sought element. $\square$

Note that the proof of the theorem is uniform.

Remark 4.15. The Rice-Shapiro theorem follows from the theorem above when applied to the domain $\wp(\omega)$.

Exercise 4.16. Show that $A \subseteq D_{k,\alpha}$ is $\bar{\alpha}$-decidable if, and only if, $A = \emptyset$ or $A = D_{k,\alpha}$. This is a generalisation of Rice's theorem.

Definition 4.17. Let (D, α) and (E, β) be weakly effective domains and let $D' \subseteq D$ and $E' \subseteq E$. Then a function $f: D' \to E'$ is *effectively continuous* if there is a total recursive function $g: \omega \to \omega$ such that

$$f^{-1}[E' \cap \Uparrow\beta(e)] = \bigcup_{m \in W_{g(e)}} (D' \cap \Uparrow\alpha(m)).$$

We leave the following proposition as a straightforward exercise.

Proposition 4.18. *Let (D, α) and (E, β) be weakly effective domains and let $D' \subseteq D$ and $E' \subseteq E$ containing the bases determined by α and β. Suppose $f: D' \to E'$ is continuous.*

(i) *f is effectively continuous if, and only if, the relation $\beta(m) \ll f(\alpha(n))$ is r.e.*

(ii) *f has a unique continuous extension to $\bar{f}: D \to E$ and $\bar{f}$ is (α, β)-effective if, and only if, f is effectively continuous.*

Here follows a generalisation of the Myhill-Shepherdson theorem.

Theorem 4.19. *Let (D, α) and (E, β) be weakly effective domains and let $f: D_{k,\alpha} \to E_{k,\beta}$. Then f is $(\bar{\alpha}, \bar{\beta})$-computable if, and only if, there is an (α, β)-effective function $\bar{f}: D \to E$ such that $\bar{f}|_{D_{k,\alpha}} = f$.*

Proof. One direction is Proposition 4.11. For the other direction we assume $f: D_{k,\alpha} \to E_{k,\beta}$ is $(\bar{\alpha}, \bar{\beta})$-computable. Let $\hat{f}$ be the total recursive function tracking f. Then

$$\bar{\alpha}(n) \in f^{-1}(\Uparrow\beta(m) \cap E_{k,\beta}) \iff \beta(m) \ll f(\bar{\alpha}(n)) \iff \beta(m) \ll \bar{\beta}\hat{f}(n).$$

The latter relation is r.e. since $\bar{\beta}$ is a constructive numbering. Thus the set $f^{-1}(\Uparrow\beta(m) \cap E_{k,\beta})$ is open by Theorem 4.14. By the uniformity of that theorem we compute an index for the open set from m, i.e., f is effectively continuous. Now the result follows from Proposition 4.18. $\square$

We see from the above theorem that the rather weak hypothesis that the way below relation $\ll$ is semidecidable nonetheless suffices to obtain strong results. However, we cannot build effective type structures. For this we need a stronger notion of domain (since the categories of continuous and algebraic cpos are not cartesian closed) and a stronger notion of effectivity. Here we give one such notion, for simplicity restricting ourselves to consistently complete algebraic cpos, i.e., algebraic domains.

Definition 4.20. An algebraic domain $D = (D; \sqsubseteq, \bot)$ is *effective* if there is a numbering $\alpha: \omega \to D_c$ such that the following relations are recursive:

(i) $\alpha(m) \sqsubseteq \alpha(n)$;

(ii) $\exists k(\alpha(m), \alpha(n) \sqsubseteq \alpha(k))$; and

(iii) $\alpha(m) \sqcup \alpha(n) = \alpha(k)$.

An effective algebraic domain is a pair (D, α) where D is an algebraic domain and α is an effective numbering of D_c in the above sense.

Let **EADOM** be the category of effective algebraic domains whose morphisms from (D, α) to (E, β) consists of the (α, β)-effective functions from D to E.

Theorem 4.21. *The category* **EADOM** *is cartesian closed.*

Proof. Straightforward from the results in Section 2.3.2 □

5 Effective domain representations

Our motivation for the study of domain representability was its use together with the theory of effective domains in investigating in a uniform way the effective content of topological algebras. In this section we survey some results of this approach indicating that it is a general method. A further conclusion is that various notions of computability via "concrete" computations coincide, thus showing that the theory of "concrete" computability is stable (see Stoltenberg-Hansen and Tucker [44]).

5.1 Effective topological algebras

The general method we pursue to study the effective properties of a generally uncountable topological algebra A is to find an *effective* domain D representing A in the sense of Definition 3.1 and then measure the effectivity of A by means of the effectivity of the representing domain D. Thus the effectivity of A is dependent on the domain representation D and its effectivity. In practice, given an algebra A one finds a *computable* or *effective* structure P of "concrete" approximations for A which is such that the ideal completion $\bar{P}$ of P is a domain representation of A.

Definition 5.1. Let A be a topological Σ-algebra.

(i) A is *(weakly) effectively domain representable* by (D, D_A, ν, α) when (D, D_A, ν) is a domain representation of A and (D, α) is a (weakly) effective Σ-domain.

(ii) A is *effectively domain representable* if A is representable by an effective domain.

Next we consider the set of computable elements of a weakly effectively domain representable algebra A, analogous to the set $D_{k,\alpha}$ of computable elements in a weakly effective domain (D, α).

Definition 5.2. Let A be a topological Σ-algebra weakly effectively domain representable by (D, D_A, ν, α). Then the set $A_{k,\alpha}$ of (D, D_A, ν, α)-*computable elements* of A is the set

$$A_{k,\alpha} = \{x \in A : \nu^{-1}(x) \cap D_{k,\alpha} \neq \emptyset\}.$$

We suppress the reference to (D, D_A, ν, α) whenever possible and simply write $A_{k,\alpha}$ (or A_k when also α is suppressed) for the set of computable elements, the effective domain representation being understood.

A Σ-algebra A is said to have a *numbering with recursive operations* if there is a surjection $\beta : \Omega \to A$, where $\Omega \subseteq \omega$, such that each operation in A is β-computable. By a k-ary operation σ on A being β*-computable* we mean that there is a *partial* recursive function $\hat{\sigma}$ such that

$$n_1, \ldots, n_k \in \Omega \implies \hat{\sigma}(n_1, \ldots, n_k) \text{ defined,}$$

and for $n_1, \ldots, n_k \in \Omega$,

$$\sigma(\beta(n_1), \ldots, \beta(n_k)) = \beta(\hat{\sigma}(n_1, \ldots, n_k)),$$

that is $\hat{\sigma}$ *tracks* σ with respect to β in the usual way.

We say that (A, β) is a *numbered algebra with recursive operations* if β is a numbering of A with recursive operations. Note that we put *no* requirement on the complexity of the code set Ω nor on the (relative) complexity of the equality relation.

Proposition 5.3. *Let $(A; \sigma_1, \ldots, \sigma_q)$ be a topological Σ-algebra weakly effectively domain representable by (D, D_A, ν, α).*

(i) *$A_{k,\alpha}$ is a subalgebra of A.*

(ii) *$A_{k,\alpha}$ is a numbered algebra with recursive operations with a numbering $\tilde{\alpha}$ induced by α.*

Proof. Let $\bar{\alpha}: \omega \to D_{k,\alpha}$ be the canonical constructive numbering of $D_{k,\alpha}$ obtained from α. Let $\Omega_A = \bar{\alpha}^{-1}(D_{k,\alpha} \cap D_A)$ and define $\tilde{\alpha}: \Omega_A \to A_{k.\alpha}$ by

$$\tilde{\alpha}(n) = \nu(\bar{\alpha}(n)).$$

Then $\tilde{\alpha}$ is a surjective numbering of $A_{k,\alpha}$.

Now (i) follows immediately from Proposition 4.3 (i). For the proof of (ii) let σ be an m-ary operation tracked by the effective operation $\hat{\sigma}$ on D. Then by Proposition 4.11 there is a recursive function $f: \omega^m \to \omega$ tracking $\hat{\sigma}$ on $D_{k,\alpha}$. Thus for each $n_1, \ldots, n_m \in \Omega_A$,

$$\begin{aligned} \sigma(\tilde{\alpha}(n_1), \ldots, \tilde{\alpha}(n_m)) &= \nu(\hat{\sigma}(\bar{\alpha}(n_1), \ldots, \bar{\alpha}(n_m))) \\ &= \nu(\bar{\alpha}(f(n_1, \ldots, n_m))) \\ &= \tilde{\alpha}(f(n_1, \ldots, n_m)), \end{aligned}$$

showing that f also tracks σ with respect to $\tilde{\alpha}$. $\square$

Now we introduce two notions of effectivity for functions between weakly effectively domain representable topological spaces.

Definition 5.4. Let A and B be topological spaces, weakly effectively domain representable by (D, D_A, ν_A, α) and (E, E_B, ν_B, β), respectively.

(i) A continuous function $f: A \to B$ is said to be (α, β)-*effective* if there is an (α, β)-effective continuous function $\bar{f} : D \to E$ representing f, that is $\bar{f}[D_A] \subseteq E_B$ and for each $x \in D_A$, $f(\nu_A(x)) = \nu_B(\bar{f}(x))$.

(ii) A function $f: A_{k,\alpha} \to B_{k,\beta}$ is $(\tilde{\alpha}, \tilde{\beta})$-*computable*, where $\tilde{\alpha}$ and $\tilde{\beta}$ are the numberings obtained in Proposition 5.3, if there is a partial recursive function $\hat{f}$ such that $\Omega_A \subseteq \mathrm{dom}(\hat{f})$ and for all $n \in \Omega_A$,

$$f(\tilde{\alpha}(n)) = \tilde{\beta}(\hat{f}(n)),$$

that is $\hat{f}$ *tracks* f with respect to $\tilde{\alpha}$ and $\tilde{\beta}$.

Note that in (ii) we do not require f to be continuous. In many important situations it is the case that each $(\tilde{\alpha}, \tilde{\beta})$-computable function is continuous. Examples are Ceitin's theorem ([10, 29]) for recursive metric spaces which are recursively separable and have limit algorithms, and the Kreisel-Lacombe-Shoenfield theorem ([23]) on recursive operators on the set of total recursive functions. Berger [9] has proved a generalisation of the Kreisel-Lacombe-Shoenfield theorem to algebraic domains which via carefully chosen domain representations also imply Ceitin's theorem.

Example 5.5. Inverse limits and ultrametric algebras.

Recall the discussion in Section 3.2. Let $\{\equiv_n\}_{n\in\omega}$ be a family of separating congruences on a Σ-algebra A. In order for $\hat{A} = \lim_{\rightarrow} A/{\equiv_n}$ to be effectively domain representable it clearly suffices that $\equiv_n$ is decidable uniformly in n. In case $\equiv_n$ is semidecidable uniformly in n then $\hat{A}$ is weakly effectively domain representable. Examples (i), (ii) and (iii) in 3.10 are effectively domain representable (for (iii) we need to assume R is a computable ring, i.e., the ring operations are computable and equality is decidable).

In order to obtain a weakly effective domain representation of an ultrametric space it suffices that the space is a weakly effective metric space (see Definition 5.18).

Exercise 5.6.

(i) Let $2^\omega = \{f \mid f: \omega \to \{0, 1\}\}$, the Cantor set. Show that 2^ω is effectively domain representable such that $(2^\omega)_k$ is the set of (characteristic functions of the) recursive sets.

(ii) Let X be a complete effective ultrametric space and let $f: X \to X$ be an effective contraction mapping. Show that the fixed point of f is computable.

5.2 Locally compact Hausdorff spaces

In this section we analyse the effectivity of standard algebraic domain representations from Section 3.3 for locally compact Hausdorff (and hence regular) spaces. We restrict our attention to effective domain representations. The reader is invited to make a similar analysis of weakly effective domain representations using continuous domains or using the technique described in Stoltenberg-Hansen and Tucker [44].

Let X be a locally compact Hausdorff space and let P be a neighbourhood system for X in the sense of Definition 3.24 consisting of compact sets (except possibly X). We call such a P for a *compact neighbourhood system*.

A compact neighbourhood system is said to be *α-computable* if α is a numbering of P satisfying the conditions in Definition 4.20. It follows that if P is α-computable then $\bar{P}$, the ideal completion of P, is an effective domain.

The extra needed hypothesis on a computable compact neighbourhood system is that the relation $F \subseteq G^\circ$ must be semidecidable.

Below we assume P is an α-computable compact neighbourhood system of X and we consider the standard algebraic domain representation $(\bar{P}, \bar{P}^R, \nu, \alpha)$ of X.

A first observation is that the ideals I_x play an important role.

Lemma 5.7. *Suppose the relation $F \subseteq G^\circ$ is α-semidecidable for $F, G \in P$. Let $f\colon \bar{P} \to \bar{P}$ be the function defined by*

$$f(I) = \{G \in P : (\exists F \in I)(F \subseteq G^\circ)\}.$$

(i) *f is a continuous (α, α)-effective function and $f[\bar{P}^R] \subseteq \bar{P}^R$.*

(ii) *f restricted to $\bar{P}^R_{k,\alpha}$ is $\bar{\alpha}$-computable.*

Proof. The proof of (i) is immediate using the added hypothesis. Part (ii) follows from Proposition 4.11 □

Corollary 5.8. *Suppose the relation $F \subseteq G^\circ$ is α-semidecidable for $F, G \in P$. Then $x \in X_{k,\alpha}$ if, and only if, the ideal $I_x = \{G \in P : x \in G^\circ\} \in \bar{P}_{k,\alpha}$.*

Proof. For the non-trivial direction assume $x \in X_{k,\alpha}$. Let $I \in \bar{P}^R$ be an α-computable ideal and let f be the effective function from Lemma 5.7. Then $f(I) = I_x$ and hence I_x is computable by the effectivity of f. □

Next we give a sufficient condition for continuous functions between locally compact Hausdorff spaces to be effective. The crucial condition is the semidecidability of whether, for given compact sets, the image of one is in the interior of the other.

Proposition 5.9. *Let X and Y be locally compact Hausdorff spaces with cusls P and Q of compact neighbourhood systems, respectively. Suppose P*

is α-computable and Q is β-computable. Let $f: X \to Y$ be a continuous function such that the relation

$$f[F] \subseteq G^{\circ}$$

is (α, β)-semidecidable for all (compact) $F \in P$ and $G \in Q$. Then f is (α, β)-effective.

Proof. Let $\bar{f}: \bar{P} \to \bar{Q}$ be the continuous function representing f, defined in the proof of Theorem 3.33. Thus, for compact $F \in P$,

$$\bar{f}(F) = \{G \in Q : f[F] \subseteq G^{\circ}\},$$

that is

$$\downarrow G \sqsubseteq \bar{f}(F) \Longleftrightarrow f[F] \subseteq G^{O}.$$

This relation is by assumption (α, β)-semidecidable, so $\bar{f}$ is (α, β)-effective. $\square$

In the remaining part of this section we consider the effective content of the ring of real numbers $\mathbb{R}$. The set P_2 of Example 3.26 is a compact neighbourhood system for $\mathbb{R}$. Let $\mathcal{R} = \bar{P}_2$ and let α be a standard numbering making $\mathcal{R}$ into en effective domain. Then $(\mathcal{R}, \mathcal{R}^R, \nu, \alpha)$ is the standard domain representation of $\mathbb{R}$ that we consider.

The effective theory of $\mathbb{R}$ has been much studied and is well understood. Our purpose here is to show that the standard effective theory of $\mathbb{R}$ is equivalent with that obtained from domain representability. A nice general reference for recursive analysis is Pour-El and Richards [35].

Definition 5.10. An element $x \in \mathbb{R}$ is a *computable real* if there is a computable sequence of rational numbers (r_n) such that for each n,

$$|r_n - x| < 2^{-n}.$$

Proposition 5.11. *The set $\mathbb{R}_{k,\alpha}$ is precisely the set of computable reals.*

Proof. Suppose $x \in \mathbb{R}$ is a computable real. Let (r_n) be a computable sequence of rationals converging to x with the modulus prescribed by Definition 5.10. For each n, let $F_n = [r_n - 2^{-n}, r_n + 2^{-n}]$. Then $x \in F_n^{\circ}$ and the set $\{F_n : n \in \omega\}$ generates a computable ideal, in fact I_x. Thus $x \in \mathbb{R}_{k,\alpha}$.

For the converse inclusion suppose $x \in \mathbb{R}_{k,\alpha}$. Then an $\tilde{\alpha}$-index for x is an $\bar{\alpha}$-index for some ideal $I \in \mathcal{R}^R$ such that $\bigcap I = \{x\}$. By Lemma 4.5 we uniformly obtain an r.e. index for $\alpha^{-1}[I]$. For each n we effectively search through I and find $[a_n, b_n] \in I$ such that $b_n - a_n < 2^{-n}$, and set $r_n = (b_n + a_n)/2$. Then (r_n) is a computable sequence of rational numbers witnessing that x is a computable real. $\square$

An index for a computable real x is an index for a computable sequence (r_n) of rationals converging to x in the manner prescribed by Definition 5.10. Thus we obtain a numbering

$$\beta : \Omega \to \mathbb{R}_{k,\alpha}$$

where Ω is the set of such indices. The above proof is uniform in the sense that given a β-index for a computable real x then we can compute an $\tilde{\alpha}$-index for x, and conversely. To be precise, there are partial recursive functions f and g such that if e is a β-index for a computable real x then $f(e)$ is defined and $f(e)$ is an $\tilde{\alpha}$-index for x. Similarly, if e is an $\tilde{\alpha}$-index for x then $g(e)$ is defined and $g(e)$ is a β-index for x. Of course, it is not recursively decidable whether or not a number e is an index of a computable real. In fact, Ω is a $\prod_2^0$ set.

We now proceed to show that a function on $\mathbb{R}$ is computable in the sense of recursive analysis if, and only if, it is effective in our sense.

The notion of a *computable* or *recursive* function in recursive analysis is a sensitive one. Here we consider the generally accepted notion, originally due to Grzegorczyk [17] and Lacombe [25].

Definition 5.12. A sequence (x_n) of real numbers is *computable* if there is a computable double sequence (r_{nk}) of rational numbers such that

$$|r_{nk} - x_n| \leq 2^{-k} \quad \text{for all } k \text{ and } n.$$

Definition 5.13. A function $f\colon \mathbb{R} \to \mathbb{R}$ is *computable* if the following hold.

(i) If (x_n) is a computable sequence of real numbers then the sequence $(f(x_n))$ is computable.

(ii) There is a recursive function $d\colon \omega^2 \to \omega$ such that for all natural numbers N and M and for each $x, y \in [-M, M]$,

$$|x - y| < \frac{1}{d(M,N)} \implies |f(x) - f(y)| < 2^{-N}.$$

Condition (ii) assures that a computable function on $\mathbb{R}$ is continuous. Thus, in order to show that such a function f is effective, it suffices by Proposition 5.9 to show that the relation $f([a,b]) \subseteq [c,d]^\circ$ is α-semidecidable, where $f([a,b])$ denotes the image of $[a,b]$ under f. Of course, $f([a,b])$ is again a compact interval by continuity.

Lemma 5.14. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a computable function. Then f is (α, α)-effective.*

Proof. We sketch the argument. It is well-known from recursive analysis that given recursive real numbers $a \le b$ we can compute indices for recursive real numbers $m \le M$ such that $f([a,b]) = [m, M]$. (Note, however, that one cannot in general compute where a maximum or minimum is taken.) It is also well-known and easy to see that the $<$ relation on the recursive reals is semidecidable. It follows that for $[a,b], [c,d] \in \mathcal{R}_c$ the relation

$$f([a,b]) \subseteq [c,d]^\circ$$

is α-semidecidable, and hence that f is (α, α)-effective. □

Now assume that $f\colon \mathbb{R} \to \mathbb{R}$ is effective. Thus f is represented by an effective continuous function $\bar{f}\colon \mathcal{R} \to \mathcal{R}$. By Lemma 5.7 we may assume that $\bar{f}(I) = I_{f(x)}$ whenever I is a total ideal representing x. We shall show that f is computable.

To prove condition (i) let (x_n) be a computable sequence of reals and let (w_{nk}) be an associated computable double sequence of rationals such that $|w_{nk} - x_n| < 2^{-k}$ for each n and k. Then the ideal I_{x_n} is computable, uniformly in n, since it is generated by

$$[w_{nk} - 2^{-k}, w_{nk} + 2^{-k}] \quad \text{for } k \in \omega.$$

By the effectivity of $\bar{f}$, the ideal $I_{f(x_n)} = \bar{f}(I_{x_n})$ is computable, uniformly in n. Thus, given a number t, we search effectively for $[c,d] \in I_{f(x_n)}$ such that $|d - c| < 2^{-t}$. In this way we obtain a computable double sequnce (s_{nt}) by letting

$$s_{nt} = (c + d)/2.$$

Clearly $|s_{nt} - f(x_n)| < 2^{-t}$ for each n and t, so $(f(x_n))$ is a computable sequence.

It remains to prove the existence of a recursive modulus function for f. Suppose we are given positive natural numbers M and N. For each $k \in \omega$, partition $[-M, M]$ into k subintervals

$$[a_{ki}, a_{ki+1}] \quad i = 0, \ldots, k-1$$

of equal length.

Claim. There is k such that for each $i = 0, \ldots, k-1$ there is $[c,d] \in \bar{f}([a_{ki}, a_{ki+1}])$ such that $|d - c| < 2^{-N}$.

Suppose the claim is false for N. Then for each k we choose i_k, $0 \le i_k \le k-1$, such that

$$[c,d] \in \bar{f}([a_{ki_k}, a_{ki_k+1}]) \implies |d - c| \ge 2^{-N}.$$

Choose $x_k \in [a_{ki_k}, a_{ki_k+1}]$. Then, by compactness, the set $\{x_k : k \in \omega\}$ has a limit point x in $[-M, M]$. Suppose first that $x \ne \pm M$. Choose

$[c,d] \in I_{f(x)} = \bar{f}(I_x)$ such that $|d-c| < 2^{-N}$. Thus for some $[a,b] \in I_x$, $[c,d] \in \bar{f}([a,b])$. But then there is k sufficiently large so that $[a_{ki_k}, a_{ki_k+1}] \subseteq [a,b]$ and hence

$$[c,d] \in \bar{f}([a_{ki_k}, a_{ki_k+1}])$$

which is a contradiction. In case $x = M$ we consider the ideal I^L_x, generated by $\{[a,M] : a < M\}$, in place of I_x to obtain a contradiction. The case $x = -M$ is handled similarly, completing the proof of the claim.

Given M and N we compute a number k which witnesses the claim. The computation is performed by an effective search, using the effectivity of $\bar{f}$. Then we define $d\colon \omega^2 \to \omega$ by

$$d(M,N) = \text{ least number} > k/2M.$$

Thus d is a total recursive function with an index obtained uniformly from an index of $\bar{f}$.

Suppose $x, y \in [-M,M]$ and $|x-y| < 1/d(M,N)$. Then for the k computed above, $|x-y| < 2M/k$ so there is i such that

$$|x - a_{ki}| < 2M/k \quad \text{ and } \quad |y - a_{ki}| < 2M/k.$$

Say, without loss of generality, that $x \in [a_{ki}, a_{ki+1}]$. Then there is $[c,d] \in \bar{f}([a_{ki}, a_{ki+1}])$ such that $|d-c| < 2^{-N}$. It is easy to see that

$$f[[a_{ki}, a_{ki+1}]] \subseteq [c,d]$$

so $|f(x) - f(a_{ki})| < 2^{-N}$. The same holds for y and hence we have

$$|f(x) - f(y)| \;\leq\; |f(x) - f(a_{ki})| + |f(a_{ki}) - f(y)| < 2^{-N} + 2^{-N} = 2^{-N+1}.$$

We have proved the following theorem.

Theorem 5.15. *A function $f\colon \mathbb{R} \to \mathbb{R}$ is computable in the sense of recursive analysis if, and only if, f is effective.*

5.3 Metric spaces

Some early analysis of the effective content of metric spaces are Lacombe [25] and Moschovakis [29]. There is also an important constructive analysis of metric spaces in Ceitin [10].

Definition 5.16. A metric space (X,d) is *recursive* in the sense of Moschovakis if

(i) there is a surjective numbering $\alpha\colon \Omega_\alpha \to X$, where $\Omega_\alpha \subseteq \omega$;

(ii) $d\colon X \times X \to \mathbb{R}_k$, where $\mathbb{R}_k$ is the set of recursive real numbers; and

(iii) the distance function d is (α, ρ)-computable, where $\rho: \Omega_\rho \to \mathbb{R}_k$ is a standard numbering.

This is a very general definition. Its weak point is that calculations with distances are limited to those possible with recursive reals. Nonetheless it is possible to give a weakly effective domain representation to (the completion of) a recursive metric space along the lines given below. We shall not pursue this here.

An alternate definition is possible that strengthens the computability of the space and which is more appropriate for examples.

By an ordered field K we mean a field $K = (K; +, -, \times, 0, 1, \leq)$. The field K is *computable* if there is a numbering of K from ω such that all the operations and the relation $\leq$ (and hence $=$) are recursive. It is known that if K is a computable ordered field then its real closure is computable as an ordered field (Madison [27]). Furthermore, if K is archimedian then K can be computably embedded into $\mathbb{R}_k$ (Lachlan and Madison [24]).

Definition 5.17. A metric space (X, d) is *computable* if

(i) there is a numbering $\alpha: \omega \to X$ with decidable equality;

(ii) $d: X \times X \to K$, for some computable archimedian ordered field K;

(iii) the distance function d is (α, γ)-computable, where $\gamma: \omega \to K$ is a computable numbering of K.

These two definitions determine two general definitions of effective metric spaces.

Definition 5.18.

(i) A metric space (X, d) is *weakly effective* if there exists a dense subspace A of X such that (A, d) is recursive in the sense of Moschovakis.

(ii) A metric space (X, d) is *effective* if there exists a dense subspace A of X such that (A, d) is computable.

The existence of a recursive or computable dense subset A of X allows us to define the *computable elements* of the metric space; these are the elements of X that can be approximated by computable Cauchy sequences of elements from A with computable modulus functions. To define the set X_k of computable elements of X we embed the space X in the metric completion A^* of A. So we may assume that $A \subseteq X \subseteq A^*$. From the numbering $\alpha: \Omega_\alpha \to A$ we can construct a canonical numbering $\alpha_k: \Omega_{\alpha_k} \to A_k$ of the set A_k of computable elements in the completion A^*. Then we set $X_k = X \cap A_k$, and give it the numbering α_k restricted to $\alpha_k^{-1}(X_k)$.

Example 5.19. The majority of examples of interest are effective metric spaces (rather than weakly effective metric spaces), including: (i) the Euclidean spaces $\mathbb{R}^n$; (ii) the space $C[0,1]$ of continuous functions $[0,1] \to \mathbb{R}$ with the sup norm; and (iii) L^p spaces for rational $p \geq 1$.

We now describe how to obtain an effective algebraic domain representation of an effective metric space.

Let (X, d) be a metric space with a dense subset A. A *formal closed ball* is a "notation" $F_{a,r}$, where $a \in A$ and $r \in \mathbb{Q}_+$, the set of non-negative rational numbers. The formal ball is a name or syntax for a closed ball and we may write it semantically by

$$F_{a,r} = \{x \in X : d(a,x) \leq r\}.$$

Two formal balls are *consistent*,

$$F_{a,r} \uparrow F_{b,s} \quad \text{if } d(a,b) \leq r + s.$$

We say that $F_{b,s}$ is *formally contained* in $F_{a,r}$,

$$F_{a,r} \sqsubseteq F_{b,s} \quad \text{if } d(a,b) + s \leq r.$$

A set $\{F_{a_1,r_1}, \ldots, F_{a_n,r_n}\}$ of formal balls is *permissible* if the balls are pairwise consistent and no ball is contained within another, i.e., for $1 \leq i \leq j \leq n$, $F_{a_i,r_i} \uparrow F_{a_j,r_j}$ and it is not the case that $F_{a_i,r_i} \sqsubseteq F_{a_j,r_j}$ or $F_{a_j,r_j} \sqsubseteq F_{a_i,r_i}$. We use the notation σ, τ for permissible sets.

Let P be the set of all permissible sets of formal balls. We need to extend the relation $\sqsubseteq$ to permissible sets:

$$\sigma \sqsubseteq \tau \Longleftrightarrow (\forall F_{a,r} \in \sigma)(\exists F_{b,s} \in \tau)(F_{a,r} \sqsubseteq F_{b,s}).$$

We note that consistency is characterised by

$$\sigma \uparrow \tau \Longleftrightarrow (\forall F_{a,r} \in \sigma)(\forall F_{b,s} \in \tau)(F_{a,r} \uparrow F_{b,s}).$$

Given consistent permissible sets σ and τ, the supremum $\sigma \sqcup \tau = g(\sigma, \tau)$ where g removes those formal balls in $\sigma \cup \tau$ properly contained in others.

The following is immediate from the construction above. But note that we need to consider sets of formal balls in order to be able to compute the supremum operation.

Lemma 5.20. *If (A, d) is a computable metric space then the obtained structure $P = (P; \sqsubseteq, \uparrow, \sqcup, \perp)$ is a computable cusl with a numbering α obtained from the numbering of A.*

We now let $D = \bar{P}$, the ideal completion of P. Thus (D, α) is an effective domain.

An ideal $I \in D$ is *converging* if for any $\varepsilon > 0$ there exists $\{F_{a,r}\} \in I$ such that $r < \varepsilon$. We define $x \in A^*$ to be *approximated* by the ideal I if $(\forall \sigma \in I)(\forall F_{a,r} \in \sigma)(x \in F_{a,r})$. A convergent ideal I approximates exactly one element x in A^*; we write $I \to x$. Let $D_X = \{I \in D : I \to x \in X\}$. The function $\nu_X \colon D_X \to X$ defined by

$$\nu_X(I) = x \Longleftrightarrow I \to x$$

is a quotient mapping. Using this construction one may now verify

Theorem 5.21. *Each effective metric space X has an effective domain representation D such that the computable elements X_k obtained from the metric coincides uniformly with the computable elements induced by D, i.e., $X_k = X_{k,D}$.*

The situation with computable functions is more difficult. We just state the following theorem which is, essentially, Theorem 3.4.33 in Blanck [3]. It uses Berger's generalisation in [9] of the Kreisel-Lacombe-Shoenfield theorem.

Theorem 5.22. *Let X and Y be effective metric spaces. Then there exists a semieffective domain representation D of X consisting of permissible sets of formal balls such that together with a standard effective formal ball representation E of Y, the following are equivalent for any function $f \colon X_k \to Y_k$:*

(i) *the function $f \colon X_k \to Y_k$ is computable; and*

(ii) *there is a continuous extension of f to $f \colon X \to Y$ that is effective with respect to the domain representations D and E of the metric spaces X and Y.*

By a semieffective domain we mean one where the consistency relation on the compact elements need not be decidable. The semieffective domain representation in the theorem is obtained by taking the dense part of a standard effective formal ball representation of X.

The implication (i) implies (ii) has a form of Ceitin's Theorem as a corollary.

6 Bifinite domains

A main advantage of domain theory is the ease with which one can build type structures. Thus we know, e.g., that the category of cpos with continuous functions as morphisms is cartesian closed. This means that the cartesian product of finitely many cpos is a cpo and the set of continuous functions between two cpos is a cpo. However, for computability we want to consider subclasses or subcategories of cpos and these are not always cartesian closed. Finite cartesian products are not problematic. The problem is the function space. Thus it is wellknown that the category of algebraic cpos is not cartesian closed, whereas the category of consistently complete algebraic (or continuous) cpos is cartesian closed (Theorem 2.33). We leave as a not completely trivial exercise to construct an algebraic cpo D such that $[D \to D]$ is not algebraic.

A possible solution would be to restrict oneself to the categories of consistently complete algebraic (or continuous) cpos, and this is often done. There is, however, one important domain construction, the Plotkin power domain described in Section 7, under which the class of consistently complete algebraic cpos is not closed. Fortunately there is a larger subcategory of the algebraic cpos, the bifinite domains, which is cartesian closed and which is also closed under the Plotkin power domain construction. It turns out, as shown by Jung [20], the the category of bifinite domains is a maximal cartesian closed subcategory of the algebraic cpos. When restricting to the category of countably based algebraic cpos, then the countably based bifinite domains make up the largest cartesian closed subcategory, i.e. it contains all other cartesian closed subcategories (see Smyth [37]). A countably based bifinite domain is also called an *sfp*-object, indicating that it is the limit of a sequence of finite partial orders.

This section closely follows the presentation given in Hamrin [19].

6.1 Basic definitions

To motivate our definition let $(P; \sqsubseteq)$ be a partial order and let us consider $\sqsubseteq$ as an information ordering. Suppose $A \subseteq P$. When is A sufficiently well-structured so that A contains witnesses to all the consistent pieces of information in A?

Definition 6.1. Let $(P; \sqsubseteq, \perp)$ be a partial order with a least element.

(i) $B \subseteq P$ is a *complete set* (in P) if

$$(\forall C \subseteq B)(\forall x \sqsupseteq C)(\exists b \in B)(C \sqsubseteq b \sqsubseteq x).$$

(ii) A family $\mathcal{F} = \{B_i : i \in I\}$ of finite subsets of P is a *complete cover* of P if each B_i is complete and for each $A \subseteq_f^* P$ there is $i \in I$ such that $A \subseteq B_i$.

As usual, $C \sqsubseteq b$ means $(\forall c \in C)(c \sqsubseteq b)$ and $A \subseteq_f^* B$ means that A is a nonempty finite subset of B. Note that if B is complete then $\perp \in B$.

The following immediate observation is crucial.

Lemma 6.2. *Suppose $A \subseteq P$ is a finite non-empty complete set and let $x \in P$. If $A \cap {\downarrow}x \neq \emptyset$ then* $\max(A \cap {\downarrow}x)$ *exists and is a member of A.*

What we require of a bifinite domain is that each finite subset of compact elements be *covered* by a finite complete set of compact elements.

Definition 6.3. D is a *bifinite domain* if D is an algebraic cpo and D_c has a complete cover.

By the algebraicity of a bifinite domain D the covering property holds for each $x \in D$.

For a partial order $(P; \sqsubseteq)$ and $A \in \mathcal{P}^*_f(P)$ we let $\mathrm{mub}(A)$ be the set of minimal upper bounds of A.

Suppose D is a bifinite domain. Then for each $A \in \mathcal{P}^*_f(D_c)$ there is a finite complete set $B \supseteq A$. Clearly this set contains $\mathrm{mub}(A)$ so $\mathrm{mub}(A)$ is finite. (Why is $\mathrm{mub}(A) \subseteq D_c$?) Furthermore, if $A \sqsubseteq x$ then there is $a \in \mathrm{mub}(A)$ such that $A \sqsubseteq a \sqsubseteq x$, again since B is complete. Now we iterate the mub operation as follows:

(i) $\mathrm{mub}^0(A) = \mathrm{mub}(A)$.

(ii) $\mathrm{mub}^{n+1}(A) = \bigcup\{\mathrm{mub}(C) : C \in \mathcal{P}_f(\mathrm{mub}^n(A))\}$.

And then we set $\mathrm{mc}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{mub}^n(A)$.

Note that each $\mathrm{mub}^n(A) \subseteq B$ and hence $\mathrm{mc}(A)$ is finite. In addition $\mathrm{mc}(A)$ is complete. For if $C \subseteq \mathrm{mc}(A)$ then $C \subseteq \mathrm{mub}^n(A)$ for some n and hence for each $x \sqsupseteq C$ there is $a \in \mathrm{mub}^{n+1}(A)$ which witnesses the completeness.

Proposition 6.4. *Let D be a bifinite domain. Then $\mathcal{F} = \{\mathrm{mc}(A) : A \in \mathcal{P}^*_f(D_c)\}$ is a complete cover of D_c. Furthermore $\mathrm{mc}(A)$ is the least complete set containing $A \subseteq^*_f D_c$, so $\mathcal{F}$ is the finest complete cover of D_c.*

Remark 6.5. A consistently complete algebraic cpo D is bifinite. For if $A \in \mathcal{P}^*_f(D_c)$ then $\mathrm{mc}(A) = \mathrm{mub}^1(A) = \{\bigsqcup B : B \subseteq A\}$.

Exercise 6.6. Give an example of a bifinite domain which is not consistently complete.

There is an equivalent characterisation of bifinite domains in terms of finite projections.

Definition 6.7. Let D be a cpo. A continuous function $p\colon D \to D$ is a *projection* if $p \sqsubseteq \mathrm{id}_D$ and p is idempotent. A projection p is *finite* if its image $\mathrm{im}(p)$ is finite.

We leave the following lemma as an exercise.

Lemma 6.8. *Let p and q be finite projections on a cpo D. Then*

(i) $p \sqsubseteq_{[D\to D]} q \iff \mathrm{im}(p) \subseteq \mathrm{im}(q)$;

(ii) $\mathrm{im}(p) \subseteq D_c$;

(iii) $p \in [D \to D]_c$.

Here is the characterisation in terms of finite projections.

Proposition 6.9. *Let D be a cpo. Then D is a bifinite domain if, and only if, there exists a directed family $(p_i)_{i\in I}$ of finite projections such that*

$$\bigsqcup_{i\in I} p_i = \mathrm{id}_D.$$

Proof. For the only if direction let $\mathcal{F}$ be a complete cover of D_c and define, for $A \in \mathcal{F}$,

$$p_A(x) = \bigsqcup \{a \in A : a \sqsubseteq x\}.$$

Then the conclusion holds for $(p_A)_{A\in\mathcal{F}}$.

For the converse let $\mathcal{F} = \{\mathrm{im}(p_i) : i \in I\}$. The algebraicity of D follows from Lemma 6.8. $\square$

We now show that the function space between bifinite domains is bifinite. There are several ways to do this. Let us first sketch the result using finite projections.

Theorem 6.10. *Let D and E be bifinite domains. Then $[D \to E]$ is a bifinite domain.*

Proof. Let $(p_i)_{i\in I}$ and $(q_j)_{j\in J}$ be families of finite projections witnessing that D and E are bifinite. For $(i, j) \in I \times J$ define $F_{ij}\colon [D \to E] \to [D \to E]$ by $F_{ij}(h) = q_j \circ h \circ p_i$. It is routine to verify that F_{ij} is continuous and idempotent and that $\bigsqcup_{ij} F_{ij} = \mathrm{id}_{[D\to E]}$. The image $\mathrm{im}(F_{ij})$ is finite since it is determined by $\mathrm{im}(q_j)$ and $\mathrm{im}(p_i)$, which are finite by assumption. $\square$

In order to consider the effectivity of a function space $[D \to E]$ of bifinite domains D and E we need a finitary characterisation of the compact elements in $[D \to E]$ in terms of the compact elements in D and E.

Definition 6.11. Let D and E be bifinite domains.

(i) A non-empty finite set

$$\{(a_i, b_i) : i \in I\} \subseteq^*_f D_c \times E_c$$

is said to be *joinable* if the set $\{a_i : i \in I\}$ is complete and $a_i \sqsubseteq a_j \implies b_i \sqsubseteq b_j$.

(ii) Suppose $u = \{(a_i, b_i) : i \in I\}$ is joinable and let $A = \{a_i : i \in I\}$. Then $s_u\colon [D \to E] \to [D \to E]$ is defined by

$$s_u(x) = b_i \iff a_i = \max(A \cap {\downarrow} x).$$

Note that s_u is well defined by Lemma 6.2 since $A \cap {\downarrow} x \neq \emptyset$ for each x.

Lemma 6.12. *Let D and E be bifinite domains and let u be joinable.*

(i) s_u *is continuous.*

(ii) $s_u = \bigsqcup\{\langle a; b\rangle : (a, b) \in u\}$.

(iii) $[D \to E]_c = \{s_u : u \subseteq D_c \times E_c,\ u \text{ joinable}\}$.

Proof. (i) and (ii) are routine given Lemma 6.2. To show (iii) we first note that each step function $\langle a; b\rangle$ is compact and hence by (ii) that s_u is compact when u is joinable. Now suppose $f \in [D \to E]_c$. By the argument of Proposition 2.30,

$$f = \bigsqcup\{\langle a; b\rangle : \langle a, b\rangle \sqsubseteq f, a \in D_c, b \in E_c\}.$$

Consider a finite set $\{(a_i, b_i) : i \in I\}$ such that each $\langle a_i, b_i\rangle \sqsubseteq f$. Let $A = \{a_i : i \in I\}$ and $B = \{b_i : i \in I\}$ and let $\bar{A} \supseteq A$ and $\bar{B} \supseteq B$ be finite complete sets of compact elements. For each $a \in \bar{A}$ let $b_a = \max(\bar{B} \cap {\downarrow} f(a))$ and let $u = \{(a, b_a) : a \in \bar{A}\}$.

Clearly s_u is joinable. For suppose $a_1, a_2 \in \bar{A}$ and $a_1 \sqsubseteq a_2$. Then $f(a_1) \sqsubseteq f(a_2)$ and hence $b_{a_1} \sqsubseteq b_{a_2}$. Furthermore $\langle a; b\rangle \sqsubseteq f$ for $(a, b) \in u$, since then $b \leq f(a)$. By (ii) we then have $s_u \sqsubseteq f$.

By an analogous argument we see that the set

$$\{s_u : s_u \sqsubseteq f \text{ and } s_u \text{ joinable}\}$$

is directed and its supremum is f. It follows that $f = s_u$ for some joinable u since f is compact. □

Remark 6.13. It is easily seen from the above proof that it suffices to consider the joinable sets obtained from a complete cover of D_c in order to obtain all of $[D \to E]_c$.

6.2 Effective bifinite domains

We shall briefly consider a notion of effective bifinite domains. It should be an extension of the effectivity of consistently complete algebraic cpos. We shall simply effectivise our definition of a bifinite domain.

Suppose $\alpha: \omega \to A$ is a numbering of a set A. Then let $\alpha^*: \omega \to \mathcal{P}_f^*(A)$ be the numbering defined by $\alpha^*(e) = \alpha[K_e]$, where $K_e \subseteq \omega$ is the non-empty finite subset with canonical index e.

Definition 6.14. A bifinite domain D is an effective bifinite domain if there is a numbering $\alpha: \omega \to D_c$ such that

(i) the relation $\alpha(n) \sqsubseteq \alpha(m)$ is recursive, i.e. $\sqsubseteq$ is α-decidable, and

(ii) there is a complete cover $\mathcal{F}$ of D_c such that $\mathcal{F}$ is α^*-decidable.

We leave the following as an exercise.

Proposition 6.15. *Let D be a bifinite domain and let α be a numbering of D_c such that $\sqsubseteq$ is α-decidable. Then (D, α) is an effective bifinite dumain if, and only if,* $\mathrm{mub}\colon \mathcal{P}_f^*(D_c) \to \mathcal{P}_f^*(D_c)$ *is (α^*, α^*)-computable.*

Let (D, α) be an effective bifinite domain. Then by the above we get that mub^n is (α^*, α^*)-computable for each n, uniformly in n. It follows that mc is (α^*, α^*)-computable since $\mathrm{mc}(A) = \mathrm{mub}^n(A)$ where n is such that $\mathrm{mub}^n(A) = \mathrm{mub}^{n+1}(A)$. Thus we can also conclude that the relation "A is complete" for $A \in \mathcal{P}_f^*(D_c)$ is α^*-decidable, since

$$A \text{ is complete} \iff \mathrm{mc}(A) = A.$$

Proposition 6.16. *Let (D, α) be an effective bifinite domain. Then*

(i) $\mathrm{mc}\colon \mathcal{P}_f^*(D_c) \to \mathcal{P}_f^*(D_c)$ *is (α^*, α^*)-computable;*

(ii) *The relation "A is complete" is α^*-decidable.*

The category of effective bifinite domains is cartesian closed.

Theorem 6.17. *Let (D, α) and (E, β) be effective bifinite domains. Then $[D \to E]$ is an effective bifinite domain with a numbering obtained uniformly from α and β.*

Proof. From an α^*-decidable complete cover $\mathcal{F}$ of D_c we obtain (by Remark 6.13) an $(\alpha \times \beta)^*$-decidable family $\mathcal{U} \subseteq D_c \times E_c$ of joinable sets such that

$$[D \to E]_c = \{s_u : u \in \mathcal{U}\}.$$

Thus we obtain in a standard way a numbering $\gamma\colon \omega \to [D \to E]_c$, uniformly from α, β and $\mathcal{F}$. The relation $\sqsubseteq$ on $[D \to E]_c$ is γ-decidable since

$$s_u \sqsubseteq s_v \iff (\forall (a, b) \in u)(b = \bot_E \vee (\exists (c, d) \in v)(c \sqsubseteq a \wedge b \sqsubseteq d)).$$

We are to construct a γ^*-decidable complete cover of $[D \to E]_c$. Let $\mathcal{G}$ be a β^*-decidable complete cover of E_c. Then, using the notation in the proof of Proposition 6.9, $(p_A)_{A \in \mathcal{F}}$ and $(q_B)_{B \in \mathcal{G}}$ are directed families of projections witnessing that D and E are bifinite. Using the notation and proof of Theorem 6.10 it suffices, by Proposition 6.9, to show that the family

$$\{\mathrm{im}(F_{AB}) : A \in \mathcal{F}, B \in \mathcal{G}\}$$

is γ^*-decidable. For this it suffices to show that for all u joinable with respect to $\mathcal{F}$,

$$s_u \in \mathrm{im}(F_{AB}) \iff (\exists v \subseteq A \times B)(v \text{ joinable } \wedge \pi_0(v) = A \wedge s_u = s_v),$$

where π_0 is the first projection function.

We leave the straight forward proof of the "if" direction as an exercise. To prove the "only if" direction let $s_u \in \mathrm{im}(F_{AB})$ where $u = \{(c, d_c) : c \in C\}$ is joinable and $C \in \mathcal{F}$. For $a \in A$ let $c_a = \max(C \cap \downarrow a) \in C$. Note that $a \sqsubseteq a' \implies c_a \sqsubseteq c_{a'}$ and hence $d_{c_a} \sqsubseteq d_{c_{a'}}$. Thus the set $v = \{(a, d_{c_a}) : a \in A\}$ is joinable. For $(a, d_{c_a}) \in v$ we have

$$d_{c_a} = \langle c_a; d_{c_a} \rangle (a) \sqsubseteq s_u(a)$$

and hence $s_v \sqsubseteq s_u$.

For $c \in C$ let $c_A = \max(C \cap \downarrow p_A(c))$. We have for $c \in C$,

$$\begin{aligned} F_{AB}(s_u)(c) &= q_B s_u p_A(c) \\ &= q_B \langle c_A; d_{c_A} \rangle p_A(c) \\ &= q_B(d_{c_A}). \end{aligned}$$

But $F_{AB}(s_u)(c) = s_u(c) = d_c$ and hence $d_c = q_B(d_{c_A}) \sqsubseteq d_{c_A}$. But $c_A \sqsubseteq c$ so $d_{c_A} \sqsubseteq d_c$, that is equality holds. In particular $d_c \in B$ and hence $v \subseteq A \times B$.

To prove $s_u \sqsubseteq s_v$ let $(c, d_c) \in u$. Then $c_A \sqsubseteq p_A(c) \sqsubseteq c$ and, in fact, $c_A = c_{p_A(c)}$. From the above, $d_c = d_{c_A}$, and hence

$$d_c \sqsubseteq \langle p_A(c); d_{c_A} \rangle (c) \sqsubseteq s_v(c)$$

which proves that $s_u \sqsubseteq s_v$.

□

7 Power domains

Power domains were introduced by Plotkin [33] in order to give a semantics for finitely branching non-deterministic or parallel programs. Assume that each run or possible outcome of a class of non-deterministic programs has an interpretation in a fixed domain D. Then an interpretation of a non-deterministic program in this class would be the *set* of interpretations of all possible outcomes of the program. Thus an appropriate domain to interpret this class of non-deterministic programs should be something analogous to the power set of D, the *power domain* of D.

For simplicity we will in this section restrict ourselves to algebraic cpos. We will define three types of power domains of which the Plotkin power domain (also called the convex power domain) is the most important.

For a set S we let

$$\mathcal{P}_f^*(S) = \{A \subseteq S : A \text{ finite and } A \neq \emptyset\}.$$

Definition 7.1. Let S be a preordered set under $\sqsubseteq$ and let $\mathcal{F}$ be a family of non-empty subsets of $\mathcal{F}$. Let A and B be in $\mathcal{F}$.

(i) $A \sqsubseteq_l B \iff (\forall a \in A)(\exists b \in B)(a \sqsubseteq b)$.

(ii) $A \sqsubseteq_u B \iff (\forall b \in B)(\exists a \in A)(a \sqsubseteq b)$.

(iii) $A \sqsubseteq_{EM} B \iff A \sqsubseteq_l B$ and $A \sqsubseteq_u B$.

The relations defined above are clearly preorders on $\mathcal{F}$. The preorder $\sqsubseteq_{EM}$ is known as the *Egli-Milner* ordering.

Definition 7.2. Let D be an algebraic cpo.

(i) The *lower* or *Hoare power domain* $\mathcal{P}_l(D)$ of D is the ideal completion of $(\mathcal{P}^*_f(D_c), \sqsubseteq_l)$.

(ii) The *upper* or *Smyth power domain* $\mathcal{P}_u(D)$ of D is the ideal completion of $(\mathcal{P}^*_f(D_c), \sqsubseteq_u)$.

(iii) The *Plotkin power domain* $\mathcal{P}_P(D)$ of D is the ideal completion of $(\mathcal{P}^*_f(D_c), \sqsubseteq_{EM})$.

Thus the power domains of algebraic cpos are algebraic cpos whose compact elements are determined by the elements in $\mathcal{P}^*_f(D_c)$.

The upper power domain is discussed in Smyth [36].

Similar definitions are made for a continuous domain D by considering the ideal completion of $(B, \ll)$, where $\ll$ is the way below relation on a basis B for D.

The following proposition is straight forward.

Proposition 7.3. *Let $(S, \sqsubseteq)$ be a preordered set and let A and B be non-empty subsets of S.*

(i) $A \sqsubseteq_l B \iff {\downarrow}A \sqsubseteq_l B$.

(ii) $A \sqsubseteq_l B \iff {\downarrow}A \subseteq {\downarrow}B$.

(iii) $A \sqsubseteq_u B \iff A \sqsubseteq_u {\uparrow}B$.

(iv) $A \sqsubseteq_u B \iff {\uparrow}B \subseteq {\uparrow}A$.

In particular it follows that $A \sim_l {\downarrow}A$ and $A \sim_u {\uparrow}A$, where, as usual, $A \sim B$ means $A \sqsubseteq B$ and $B \sqsubseteq A$.

Exercise 7.4. Determine $\mathcal{P}_l(\mathbb{N}_\perp)$ and $\mathcal{P}_u(\mathbb{N}_\perp)$.

Proposition 7.5. *Let D be an algebraic cpo.*

(i) *$\mathcal{P}_l(D)$ is consistently complete.*

(ii) *If D is bifinite then $\mathcal{P}_u(D)$ is consistently complete.*

Proof. Given $A, B \in \mathcal{P}_f^*(D_c)$ it is clear that $[A]_l \sqcup [B]_l = [A \cup B]_l$, where $[A]_l$ denotes the compact ideal in $\mathcal{P}_l(D)$ determined by A. This proves (i). For (ii) let

$$C = \bigcup \{\mathrm{mub}(a, b) : a \in A, b \in B, \text{ and } a, b \text{ consistent}\}.$$

If A and B are consistent then $C \neq \emptyset$. Furthermore C is finite since D is assumed bifinite. It is now straight forward to see that $[A]_u \sqcup [B]_u = [C]_u$. □

In particular consistent completeness is preserved by the upper and lower power domain constructions. However, this is not true for the Plotkin power domain construction.

Exercise 7.6. Let D be the four element diamond lattice and let $E = D \times D$. Then E is consistently complete. Show that $\mathcal{P}_P(E)$ is not consistently complete.

The bifinite domains are preserved under the Plotkin power domain construction.

Theorem 7.7. *If D is a bifinite domain then so is $\mathcal{P}_P(D)$.*

Proof. Let $(p_i)_{i \in I}$ be a directed family of finite projections on D such that

$$\bigsqcup_{i \in I} p_i = \mathrm{id}_D \,.$$

Define $\bar{p}_i \colon \mathcal{P}_P(D) \to \mathcal{P}_P(D)$ by, for $\{a_1, \ldots, a_n\} \in \mathcal{P}_f^*(D_c)$,

$$\bar{p}_i(\downarrow\{a_1, \ldots, a_n\}) = \downarrow\{p_i(a_1), \ldots, p_i(a_n)\}.$$

It is routine to verify that $(\bar{p}_i)_{i \in I}$ is a well-defined directed family of finite projections on $\mathcal{P}_P(D)$ such that $\bigsqcup_{i \in I} \bar{p}_i = \mathrm{id}_{\mathcal{P}_P(D)}$. □

The Plotkin power domain is also called the *convex* power domain for the following reason. Suppose $(P, \sqsubseteq)$ is a preorder. For $A \subseteq P$ let

$$\mathrm{cvx}_P(A) = \{p \in P : (\exists q, r \in A)(q \sqsubseteq p \sqsubseteq r)\},$$

the *convex hull of* A in P. A set $A \subseteq P$ is said to be *convex* if $\mathrm{cvx}_P(A) = A$. For subsets A and B it is easily verified that

$$A \sim_{EM} B \iff \mathrm{cvx}_P(A) = \mathrm{cvx}_P(B).$$

Proposition 7.8. *Let D be an algebraic cpo and let $\mathcal{C}(D_c) = \{\mathrm{cvx}_{D_c}(A) : A \in \mathcal{P}_f^*(D_c)\}$. Then the Plotkin power domain $\mathcal{P}_P(D)$ is isomorphic to the ideal completion of $(\mathcal{C}(D_c), \sqsubseteq_{EM})$.*

The Plotkin power domain construction preserves effective bifinite domains, uniformly.

Theorem 7.9. *Let (D, α) be an effective bifinite domain. Then the Plotkin power domain $\mathcal{P}_P(D)$ is an effective bifinite domain with a numbering obtained uniformly from α.*

Proof. The set of compact elements $\mathcal{P}_P(D)_c$ is numbered by $\alpha^*\colon \omega \to \mathcal{P}_f^*(D_c)$, the canonical numbering obtained from α. The order relation $\sqsubseteq$ on $\mathcal{P}_P(D)_c$ is clearly α^*-decidable using the α-decidable ordering of D_c.

Let $\mathcal{F}$ be a complete cover of D_c. We claim that

$$\mathcal{G} = \{\mathcal{P}_f^*(A) : A \in \mathcal{F}\}$$

is a complete cover of $\mathcal{P}_P(D)_c$. (Here we work on representations of the elements in $\mathcal{P}_P(D)_c$.) Note that $\mathcal{G}$ is obviously α^{**}-decidable. For $A \in \mathcal{F}$ and $C \subseteq_f^* D_c$ let

$$B_C = \{\max(A \cap \downarrow c) : c \in C\}.$$

Then $B_C \subseteq A$ since A is complete. Furthermore $B_C = \max(\mathcal{P}_f^*(A) \cap \downarrow C)$ and hence $\mathcal{P}^*(A)$ is complete in $\mathcal{P}_P(D)$. To see that $\mathcal{G}$ is a complete cover suppose $B_1, \ldots, B_k \subseteq_f^* D_c$. Then choose $A \in \mathcal{F}$ such that $\cup_{i=1}^k B_i \subseteq A$. $\square$

The Plotkin power domain construction has been used by Edalat [11, 12] and Blanck [6] to study computability of compact subsets of metric spaces and iterated function systems on metric spaces.

We just cite a few results in order to illustrate the use of the Plotkin power domain in relation to domain representability.

Theorem 7.10. (Blanck [6]) *Let (D, D^R, ν) be a standard effective domain representation of a complete effective metric space X and let $E = \mathcal{P}_P(D)$. Then there is $E^R \subseteq E$ and μ such that (E, E^R, μ) is an effective domain representation of $\mathcal{H}(X)$, the set of compact subsets of X.*

The topology induced on $\mathcal{H}(X)$ is the one given by the Hausdorff metric on $\mathcal{H}(X)$. Recall that the Hausdorff metric d_H on $\mathcal{H}(X)$ is given by

$$d_H(K, K') = \max(\sup_{x \in K} d(x, K'), \sup_{y \in K'} d(y, K)),$$

where $d(x, K) = \inf_{y \in K} d(x, y)$ and d is the metric on X.

To get a feeling for the Plotkin power domain construction in terms of domain representability the reader should consider the standard interval domain for the real numbers and take the Plotkin power domain of it. Then construct an ideal which represents the Cantor set.

An application area of this theory is that of Iterated Function Systems (IFS).

Theorem 7.11. (Blanck [6]) *An effective hyperbolic IFS on an effective complete metric space has a unique effective non-empty compact attractor.*

8 Representation of non-continuous functions

There are important phenomena in computing that are not continuous. Suppose we model a *stream* of data as a function from time into a set of data. Let us think of time as being continuous. Then it is reasonable to model time by the real number line $\mathbb{R}$ or a final segment of $\mathbb{R}$. If the data set is discrete, i.e., has the discrete topology, then the only continuous functions or streams are the constant functions. The reason is that the reals $\mathbb{R}$ is a connected space. (Streams that takes values in a discrete data set are sometimes called *signals*.) Thus, in order to represent such a stream space, or a space of signals, by a function space domain, we need a way to represent a non-continuous function by a continuous function between domains. We know by Proposition 3.5 that this is impossible. We have to settle for representing non-continuous functions *approximately*. For a discussion of streams, stream transformers and domain representations see Blanck, Stoltenberg-Hansen and Tucker [7].

Another example is that of solid geometry. Suppose we want to represent a solid body in space. If the space is modelled by $\mathbb{R}^n$, which is usual, then the solid body is modelled by a function $f\colon \mathbb{R}^n \to \{0,1\}$, where $f(x) = 0$ if x is in the solid body and $f(x) = 1$ otherwise. That is, a solid body is represented by its characteristic function. Again we meet the problem that no such function is continuous and hence cannot be represented exactly. For a discussion of modelling solid geometry see Edalat and Lieutier [14] and of constructive volume geometry see Blanck, Stoltenberg-Hansen and Tucker [8].

An analogous problem is how to represent a relation on a topological space. A canonical example is the space of real numbers $\mathbb{R}$. How do we represent the $\leq$ relation?

Let X be a topological space and let $\mathbb{B} = \{\text{true}, \text{false}\}$ be the discrete boolean space. An n-ary relation P on X can be identified with its *characteristic function* $c_P\colon X \to \mathbb{B}$ defined by

$$c_P(a_1,\ldots,a_n) = \begin{cases} \text{true}, & \text{if } P(a_1,\ldots,a_n); \\ \text{false}, & \text{if } \neg P(a_1,\ldots,a_n). \end{cases}$$

The idea is to represent the possibly non-continuous characteristic function continuously in such a way that it gives exact values at points of continuity and possibly only proper approximations at points of discontinuity. We know that this is the best possible.

Let (D, D^R, ν) be a domain representation of X and let P be an n-ary relation on X. Define $\bar{c}_P : D_c^n \to \mathbb{B}_\perp$ by

$$\bar{c}_P(\vec{a}) = \begin{cases} \text{true}, & \text{if } (\forall \vec{x} \in (D^R)^n)(\vec{a} \sqsubseteq \vec{x} \implies P(\nu(\vec{x}))); \\ \text{false}, & \text{if } (\forall \vec{x} \in (D^R)^n)(\vec{a} \sqsubseteq \vec{x} \implies \neg P(\nu(\vec{x}))); \\ \perp, & \text{otherwise.} \end{cases}$$

$\bar{c}_P$ is clearly monotone and hence extends uniquely to a continuous function

$$\bar{c}_P \colon D^n \to \mathbb{B}_\perp.$$

We say that $\bar{c}_P$ *represents* c_P or P *approximately.*

Example 8.1. Consider the standard interval representation $\mathcal{R}$ of the reals $\mathbb{R}$ (Example 3.31), and the relation $\leq$ on $\mathbb{R}$. Then

$$\bar{c}_\leq([a,b],[c,d]) = \begin{cases} \text{true}, & \text{if } b \leq c; \\ \text{false}, & \text{if } d < a; \\ \perp, & \text{otherwise.} \end{cases}$$

Note that $\bar{c}_\leq$ is effective. If $x < y$ in $\mathbb{R}$ then $\bar{c}_\leq(I_x, I_y) = \text{true}$ and if $y < x$ in $\mathbb{R}$ then $\bar{c}_\leq(I_x, I_y) = \text{false}$. In case $x = y$ then $\bar{c}_\leq(I_x, I_x) = \perp$.

The function $c_\leq \colon \mathbb{R}^2 \to \mathbb{B}$ is continuous on

$$\{(x,y) : x \neq y\} \subseteq \mathbb{R}^2,$$

and discontinuous on the diagonal. Thus $\bar{c}_\leq$ represents $c_\leq$ exactly on points of continuity. At points of discontinuity $\bar{c}_\leq$ only provides the trivial approximation of the value of $c_\leq$.

It is well-known that $\leq$ is not decidable or even semidecidable on the recursive reals $\mathbb{R}_k$, the problem being that equality on $\mathbb{R}_k$ is not semidecidable. (Equality is cosemidecidable, i.e., $\neq$ is semidecidable.) This is reflected by the discontinuity of $c_\leq$.

We now want to generalise the continuous representation of relations to continuous representations of non-continuous functions. The idea is the same. We want our representation to be exact at points of continuity and as good as possible in terms of approximations at points of discontinuity. Here is the definition.

Definition 8.2. Let (D, D^R, ν_X) and (E, E^R, ν_Y) be domain representations of the topological spaces X and Y, respectively. Then a function $f \colon X \to Y$ (not necessarily continuous) is said to be *represented approximately* by (or *lifts approximately* to) $\bar{f} \colon D \to E$ if

(i) $\bar{f}$ is continuous,

(ii) $(\forall x \in D^R)(f \text{ continuous at } \nu_X(x) \implies \bar{f}(x) \in E^R \text{ and } f\nu_X(x) = \nu_Y \bar{f}(x))$, and

(iii) $(\forall x \in D^R)(f \text{ not continuous at } \nu_X(x) \implies (\exists y \in \nu_Y^{-1}[f\nu_X(x)])(\bar{f}(x) \sqsubseteq y))$.

The following example illustrates the notion above.

Example 8.3. The floor function $\lfloor\cdot\rfloor : \mathbb{R} \to \mathbb{Z}$ is discontinuous at precisely the integer points. We shall represent the floor function by a continuous domain function in the above precise sense.

Let $\mathcal{R}$ be the standard interval domain representation of $\mathbb{R}$. We could choose $\mathbb{Z}_\perp$ as a representation of $\mathbb{Z}$ and proceed as in the case of representing relations. However we can do much better if we choose a domain representation with more care. Let $E = \mathcal{P}_u(\mathbb{Z}_\perp)$, the upper (or Smyth) power domain of $\mathbb{Z}_\perp$. By Exercise 7.4, E can be identified with $\mathcal{P}_f^*(\mathbb{Z}) \cup \{\mathbb{Z}\}$ ordered by reverse inclusion$\supseteq$. Thus the singleton sets $\{n\}$, for $n \in \mathbb{Z}$, are the maximal elements. Letting E^R be the set of maximal elements we obtain a domain representation of $\mathbb{Z}$.

Define $f : \mathcal{R}_c \to E$ by

$$f([a,b]) = \{m \in \mathbb{Z} : \lfloor a\rfloor \leq m \leq \lfloor b\rfloor\}$$

and $f(\mathbb{R}) = \mathbb{Z}$ (i.e., f is strict). Clearly, f is monotone and hence extends uniquely to a continuous function $f : \mathcal{R} \to E$.

Let $x \in \mathbb{R}$ and let $I_x \in \mathcal{R}^R$ be the smallest ideal representing x. If x is not an integer then $f(I_x) = \{\lfloor x\rfloor\}$. Thus f represents the floor function exactly for all points of continuity. Now consider an integer m. Recall the four different representations of m described in Example 3.31. It is easily seen that

$$\begin{aligned} f(I_m) &= \{m-1, m\}, \\ f(I_m^-) &= \{m-1, m\}, \\ f(I_m^+) &= \{m\}, \text{ and} \\ f(I^m) &= \{m\}. \end{aligned}$$

It follows that f represents the floor function approximately in the sense of Definition 8.2. However, thanks to our choice of representation for $\mathbb{Z}$ we are able to obtain much information also at points of discontinuity. This illustrates the importance of choosing appropriate representations of the data types. Had we chosen $\mathbb{Z}_\perp$ to represent $\mathbb{Z}$ then the representation of the floor function would provide no information at points of discontinuity.

We close this section by showing that, under rather general conditions, there is a best continuous approximate representation of an arbitrary function, if there is one at all.

Theorem 8.4. *Let (D, D^R, ν_X) and (E, E^R, ν_Y) be domain representations of X and Y, respectively. Assume that D^R is dense in D, and that (E, E^R, ν_Y) is upwards closed and local. Let $f : X \to Y$ be a function and assume that f has one approximate representation in $[D \to E]$. Then there is a best approximate representation $\bar{f} \in [D \to E]$ in the sense of the domain ordering.*

Proof. Let $\mathcal{A}_f = \{\bar{f} \in [D \to E] \colon \bar{f}$ represents f approximately$\}$. We show that $\mathcal{A}_f$ is directed. $\mathcal{A}_f \neq \emptyset$ by assumption. Suppose $\bar{f}_1, \bar{f}_2 \in \mathcal{A}_f$. We first show that $\bar{f}_1$ and $\bar{f}_2$ are consistent. For this it suffices by the density of D^R to show that $\bar{f}_1(x)$ and $\bar{f}_2(x)$ are consistent for each $x \in D^R$.

Fix $x \in D^R$ and assume f is continuous at $\nu_X(x)$. Thus $\bar{f}_1(x), \bar{f}_2(x) \in E^R$ and

$$\nu_Y(\bar{f}_1(x)) = \nu_Y(\bar{f}_2(x)) = f(\nu_X(x)).$$

But by the hypotheses on (E, E^R, ν_Y) the supremum $\bar{f}_1(x) \sqcup \bar{f}_2(x)$ exists and

$$\nu_Y(\bar{f}_1(x) \sqcup \bar{f}_2(x)) = f(\nu_X(x)).$$

Now assume f is discontinuous at $\nu_X(x)$. Then there are $y_1, y_2 \in E^R$ such that $\nu_Y(y_1) = \nu_Y(y_2) = f(\nu_X(x))$ and $\bar{f}_1(x) \sqsubseteq y_1$ and $\bar{f}_2(x) \sqsubseteq y_2$. Again, by the assumptions on (E, E^R, ν_Y),

$$\bar{f}_1(x) \sqcup \bar{f}_2(x) \sqsubseteq y_1 \sqcup y_2 \in E^R$$

and $\nu(y_1 \sqcup y_2) = f(\nu_X(x))$. We conclude that $\bar{f}_1$ and $\bar{f}_2$ are consistent and $\bar{f}_1 \sqcup \bar{f}_2 \in \mathcal{A}_f$, that is, $\mathcal{A}_f$ is directed.

Let $\bar{f} = \bigsqcup \mathcal{A}_f$. We need to show that $\bar{f} \in \mathcal{A}_f$. Let $x \in D^R$ be such that f is continuous at $\nu_X(x)$ and let $\bar{g} \in \mathcal{A}_f$. Then $\bar{g}(x) \sqsubseteq \bar{f}(x)$ and $\bar{g}(x) \in E^R$ so $\bar{f}(x) \in E^R$ and

$$\nu_Y(\bar{f}(x)) = \nu_Y(\bar{g}(x)) = f(\nu_X(x)).$$

Now suppose f is discontinuous at $\nu_X(x)$. Let $\bar{y} \in E^R$ be the maximal element such that $\nu_Y(\bar{y}) = f(\nu_X(x))$. The assumptions on (E, E^R, ν_Y) imply that such $\bar{y}$ exists. Thus $\bar{g}(x) \sqsubseteq \bar{y}$ and hence

$$\bar{f}(x) = \bigsqcup \{\bar{g}(x) : \bar{g} \in \mathcal{A}_f\} \sqsubseteq \bar{y},$$

proving that $\bar{f} \in \mathcal{A}_f$. □ □

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