Exercise Sheet 7 - Applied Logic:
Modal Logic
October 15, 2008

Exercise 1

Using $\Box\phi$ to mean 'Agent A knows ϕ', represent the following statements with modal formulas:

1. If ϕ is true, then it is consistent with what A knows, that A knows it.
2. If it is consistent with what A knows that ϕ, and it is consistent with what A knows that ψ, then it is consistent with what A knows that $\phi \wedge \psi$.
3. If A knows ϕ, then it is consistent with what A knows that ϕ.
4. If it is consistent with what A knows that it is consistent with what A knows that ϕ, then it is consistent with what A knows that ϕ.

Which of these statements seems plausible principles concerning knowledge and consistency.

Exercise 2

Suppose $\Diamond\phi$ is interpreted as 'ϕ is permissible'. How should we interpret $\Box\phi$? Give some examples of formulas that seem plausible under this interpretation. Should $\Box(\Box p \to p) \to \Box p$ be one such example?

Exercise 3

Show that the following formulas are valid in the class of all relational frames

1. $\Diamond\phi \leftrightarrow \neg\Box\neg\phi$,
2. $\Box(\phi \wedge \psi) \leftrightarrow \Box\phi \wedge \Box\psi$,
3. $\Diamond(\phi \vee \psi) \leftrightarrow \Diamond\phi \vee \Diamond\psi$.

Exercise 4

Consider the basic temporal language and frames $(\mathbb{Z}, <)$, $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$, where $<$, in each case, is the usual less-than relation. Which of the following formulas are valid on these frames?

1. $GGp \to p$,
2. $(p \wedge Hp) \to FHp$.

Exercise 5

Show that the following formulas are non-valid by constructing a counterexample in each case:

1. $\Box\bot$,
2. $\Diamond p \to \Box p$,
3. $p \to \Box\Diamond p$,
4. $\Diamond\Box p \to \Box\Diamond p$,
5. $\Box p \to p$.

Exercise 6

Show the following

1. Frame-validity of B: $\phi \to \Box\Diamond\phi$ corresponds to symmetry of R.

2. Frame-validity of D: $\Box\phi \to \Diamond\phi$ corresponds to R being serial.

Exercise 7

Show that the rules of proof of the system **K**, as explained in the lecture notes, preserve validity. Furthermore, discuss for each rule whether it preserves global and/or local truth.

Exercise 8

Give **K**-proofs of the formulas $(\Box p \wedge \Diamond q) \to \Diamond(p \wedge q)$ and $\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$.

Exercise 9

Let $\mathbf{F}$ be a class of frames. Show that $\Lambda_{\mathbf{F}} = \{\varphi \mid \mathcal{F} \models \varphi \text{ for all } \mathcal{F} \in \mathbf{F}\}$ is a normal modal logic.

Exercise 10

Consider a modal language with two boxes [1] and [2]. Show that $p \to [2]\langle 1 \rangle p$ is valid on precisely those frames for the language that satisfy the condition

$$\forall xy(xR_2y \to yR_1x).$$

What sort of frames does $p \to [1]\langle 1 \rangle p$ define?

Exercise 11

Consider a language with three boxes $[1], [2]$ and $[3]$. Show that the modal formula $\langle 3 \rangle p \leftrightarrow \langle 1 \rangle \langle 2 \rangle p$ is valid on a frame for this language if and only if the frame satisfies the condition

$$\forall xy(xR_3y \leftrightarrow \exists z(xR_1z \wedge zR_2y)).$$

Exercise 12*

Consider a language with two boxes [1] and [2]. Prove that the class of frames in which $R_1 = R_2^*$, where R_2^* is the reflexive transitive closure of R_2, is defined by the formulas

1. $\langle 1 \rangle p \to (p \vee \langle 1 \rangle(\neg p \wedge \langle 2 \rangle p))$,
2. $\langle 1 \rangle p \leftrightarrow (p \vee \langle 2 \rangle \langle 1 \rangle p)$.

How is this related to PDL?

Exercise 13*

Suppose $\mathcal{T} = (T, <)$ is a bidirectional frame (where we write $y < x$ instead of $x \check{<} y$) such that $<$ is transitive, irreflexive and satisfies $\forall xy(x < y \vee x = y \vee y < x)$. Show that

$$\mathcal{T} \models \{G(Gp \to p) \to Gp, H(Hp \to p) \to Hp\}$$

implies that $\mathcal{T}$ is finite.

Exercise 14**

Show that Grzegorczyk's formula

$$\Box(\Box(p \to \Box p) \to p) \to p$$

characterizes the class of frames $\mathcal{F} = (W, R)$ satisfying

- (i) R is reflexive,
- (ii) R is transitive,
- (iii) there are no infinite paths $x_0 R x_1 R x_2 R \ldots$ such that for all i, $x_i \neq x_{i+1}$.

Many of the exercises are taken from the book *Modal Logic* by Patrick Blackburn, Maarten de Rijke and Yde Venema, which is an excellent book if you want to learn more about Modal Logic.