Solutions For Exercise Sheet 7 - Applied Logic:
Modal Logic
October 21, 2008

This note presents solutions to some of the exercises on Exercise Sheet 7.

Exercise 3

Show that the following formulas are valid in the class of all relational frames

1. $\Diamond\phi \leftrightarrow \neg\Box\neg\phi$,
2. $\Box(\phi \wedge \psi) \leftrightarrow \Box\phi \wedge \Box\psi$,
3. $\Diamond(\phi \vee \psi) \leftrightarrow \Diamond\phi \vee \Diamond\psi$.

Solution. We solve 1.

Let $\mathcal{M} = (W, R, L)$ be any Kripke model. If $w \in W$ satisfies $w \Vdash \Diamond\varphi$, there is $v \in W$ with wRv and $v \Vdash \varphi$. Hence $v \nVdash \neg\varphi$ and so we cannot have $w \Vdash \Box\neg\varphi$, i.e. $w \nVdash \Box\neg\varphi$. But this is just $w \Vdash \neg\Box\neg\varphi$.

If $w \Vdash \neg\Box\neg\varphi$ then $w \nVdash \Box\neg\varphi$ and so there must be some $v \in W$ with wRv and $v \nVdash \neg\varphi$, i.e. $v \Vdash \varphi$. But this is just $v \Vdash \Diamond\varphi$. $\Box$

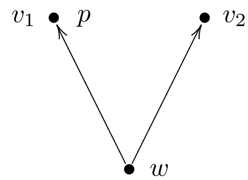
Exercise 5

Show that the following formulas are non-valid by constructing a counterexample in each case:

1. $\Box\bot$,
2. $\Diamond p \to \Box p$,
3. $p \to \Box\Diamond p$,
4. $\Diamond\Box p \to \Box\Diamond p$,
5. $\Box p \to p$.

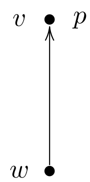
Solution. We solve 2 and 4.

For 2, consider the following model $\mathcal{M}_1$:



Then $\mathcal{M}_1, w \Vdash \Diamond p$ since $v_1 \Vdash p$, but $w \nVdash \Box p$ since $v_2 \nVdash p$. Hence, $\mathcal{M}_1 \not\models \Diamond p \to \Box p$.

For 4, consider the following model $\mathcal{M}_2$:



Then $w \Vdash \Diamond\Box p$ since $v \Box p$ (trivially, since v has no R-successors). But $w \nVdash \Box\Diamond p$ since v has no R-successor. $\Box$

Exercise 6

Show the following

1. Frame-validity of B: $\phi \to \Box\Diamond\phi$ corresponds to symmetry of R.

2. Frame-validity of D: $\Box\phi \to \Diamond\phi$ corresponds to R being serial.

Solution. We solve 2.

Suppose $\mathcal{F} = (W, R)$, $\mathcal{F} \models \Box\varphi \to \Diamond\varphi$ and let $w \in W$. We define a model $\mathcal{M} = (W, R, L)$, based on $\mathcal{F}$, by defining the labelling function L by setting $L(x) = \{p\}$ for all $x \in W$. Then clearly $\mathcal{M}, w \Vdash \Box p$, and since $\mathcal{F} \models \Box p \to \Diamond p$ we have $w \Vdash \Diamond p$. That is, there is $v \in W$ with wRv (and $v \Vdash p$). This shows that R must be serial.

Conversely, suppose $\mathcal{F} = (W, R)$ with R serial. Let $\mathcal{M}$ be any model based on $\mathcal{F}$ and suppose $w \in W$ with $\mathcal{M}, w \Vdash \Box\varphi$, then since R is serial there is $v \in W$ with wRv and hence we must have $v \Vdash \varphi$. That is, $w \Vdash \Diamond\varphi$. Hence we see that $\mathcal{F} \models \Box\varphi \to \Diamond\varphi$. $\Box$

Exercise 10

Consider a modal language with two boxes [1] and [2]. Show that $p \to [2]\langle 1\rangle p$ is valid on precisely those frames for the language that satisfy the condition

$$\forall xy(xR_2y \to yR_1x).$$

What sort of frames does $p \to [1]\langle 1\rangle p$ define?

Solution. Suppose $\mathcal{F} = (W, R_1, R_2)$ satisfies $(\forall x, y \in W)(xR_2y \to yR_1x)$ and let $\mathcal{M}$ be any model based on $\mathcal{F}$. Suppose $x \in W$ with $\mathcal{M}, x \Vdash p$ and suppose xR_2y, then yR_1x and hence $y \Vdash \langle 1\rangle p$. But then $x \Vdash [2]\langle 1\rangle p$ since y was arbitrary with xR_2y. Hence $x \Vdash p \to [2]\langle 1\rangle p$ and we have $\mathcal{F} \models p \to [2]\langle 1\rangle p$.

Conversely, suppose $\mathcal{F} \models p \to [2]\langle 1\rangle p$ and let $x \in W$. Define a model $\mathcal{M} = (W, R, L)$ by setting $L(w) = \{p\}$ if $w = x$ and $L(w) = \emptyset$ if $w \neq x$. Hence $w \Vdash p$ if and only if $w = x$. Now, if xR_2y, then $y \Vdash \langle 1\rangle p$ and hence there is $z \in W$ with yR_1z and $z \Vdash p$, i.e. $z = x$ and we have yR_1x. Hence $\mathcal{F}$ must satisfy $\forall xy(xR_2y \to yR_1x)$. □

Exercise 12*

Consider a language with two boxes [1] and [2]. Prove that the class of frames in which $R_1 = R_2^*$, where R_2^* is the reflexive transitive closure of R_2, is defined by the formulas

1. $\langle 1\rangle p \to (p \vee \langle 1\rangle(\neg p \wedge \langle 2\rangle p))$,
2. $\langle 1\rangle p \leftrightarrow (p \vee \langle 2\rangle\langle 1\rangle p)$.

How is this related to PDL?

Solution. Suppose $R_1 = R_2^*$, then xR_1y if and only if $x = y$ or there are $x_0, \ldots, x_n$ such that $x = x_0$, $x_iR_2x_{i+1}$ and x_nR_2y. To see that $\mathcal{F}$ satisfies 1, suppose $\mathcal{M}, x \Vdash \langle 1\rangle p$ (in some model $\mathcal{M}$ based on $\mathcal{F}$). Then either $x \Vdash p$ or there are $x = x_0, \ldots, x_n, y \in W$ with $x_iR_2x_{i+1}$, x_nR_2y and $y \Vdash p$. If $x \nVdash p$ then clearly xR_1x_n and we may assume $x_n \nVdash p$ (why?), so that $x \Vdash \langle 1\rangle(\neg p \wedge \langle 2\rangle p)$. We also have $\mathcal{F} \models \langle 1\rangle p \leftrightarrow (p \vee \langle 2\rangle\langle 1\rangle p)$ since for any model $\mathcal{M}$ based on $\mathcal{F}$ we have $\mathcal{M}, x \Vdash \langle 1\rangle p$ iff $\exists y$ such that $x = y$ or $\exists x_0, \ldots, x_n$ such that $x = x_0R_2x_1R_2 \ldots R_2x_nR_2y$ and $y \Vdash p$ iff $x \Vdash p$ or $\exists x_0, \ldots, x_n$ with $n \geq 1$, $x_0 = x$, $x_iR_2x_{i+1}$, xR_2x_1 and x_1R_1y with $y \Vdash p$ iff $x \Vdash p \vee \langle 2\rangle\langle 1\rangle p$.

Conversely, suppose that $\mathcal{F} = (W, R_1, R_2)$ satisfies 1 and 2. Then,

R_1 is reflexive: Let $x \in w$ and set $L(w) = \{p\}$ iff $w = x$ and $L(w) = \emptyset$ iff $w \neq x$. Hence $x \Vdash p \vee \langle 2\rangle\langle 1\rangle p$, and by 2 we then have $x \Vdash \langle 1\rangle p$, i.e. xR_1x.

$R_2 \subseteq R_1$: Suppose xR_2y, and set $L(y) = \{p\}$ and $L(w) = \emptyset$ if $w \neq y$. Then $x \Vdash \langle 2\rangle\langle 1\rangle p$ since R_1 is reflexive. Hence $x \Vdash p \vee \langle 2\rangle\langle 1\rangle p$ and by 2 we have $x \Vdash \langle 1\rangle p$, i.e. xR_1y.

If xR_2y and yR_2z then xR_1z: Set $L(z) = \{p\}$ and $L(w) = \emptyset$ when $w \neq z$. Then $x \Vdash \langle 2\rangle\langle 1\rangle p$, since xR_2yR_2z i.e. xR_2yR_1 since $R_2 \subseteq R_1$. Hence $x \Vdash \langle 1\rangle p$ by 2, i.e. xR_1z.

Now, the three clauses above show that if $x = y$ or there are $x_0, \ldots, x_n \in W$ such that $x = x_0R_2 \ldots R_2x_n = y$ then xR_1y.

We need to show the converse as well, so suppose for a contradiction that there is no finite sequence $x_0, \ldots, x_n \in W$ such that $x = x_0R_2 \ldots R_2x_n = y$, $x \neq y$ but xR_1y. We define $L(w) = \{p\}$ iff there is a finite sequence $x_0, \ldots, x_n$ such that $w = x_0R_2 \ldots R_2x_n = y$. Then $y \Vdash p$ so $x \Vdash \langle 1\rangle p$ and by 1 we have that there is $z \in W$ with xR_1z, $z \nVdash p$ and there is $z' \in W$ with zR_2z' and $z' \Vdash p$, i.e. there is a finite sequence $x_0, \ldots, x_n$ such that $z' = x_0R_2 \ldots R_2x_n = y$ and zR_2z' but $z \nVdash p$. This is clearly a contradiction and hence we must conclude that there are $x_0, \ldots, x_n \in W$ with $x = x_0R_2 \ldots R_2x_n = y$, i.e. $R_1 = R_2^*$.

□

Exercise 13*

Suppose $\mathcal{T} = (T, <)$ is a bidirectional frame (where we write $y < x$ instead of $x \check{<} y$) such that $<$ is transitive, irreflexive and satisfies $\forall xy(x < y \vee x = y \vee y < x)$. Show that

$$\mathcal{T} \models \{G(Gp \to p) \to Gp, H(Hp \to p) \to Hp\}$$

implies that $\mathcal{T}$ is finite.

Solution. Suppose $\mathcal{T} = (T, <)$ satisfies $\mathcal{T} \models \Gamma$, where $\Gamma := \{G(Gp \to p) \to Gp, H(Hp \to p) \to Hp\}$. Define a labelling L of T by setting

$$L(t) = \{p\} \Longleftrightarrow_{def} t\uparrow, t\downarrow \text{ are finite.}$$

Here, $t\uparrow := \{s \in T \mid t < s\}$ and $t\downarrow := \{s \in T \mid s < t\}$. Since $\mathcal{T} \models \Gamma$ we have for every $t \in T$ that either $t \Vdash Gp$ or $t \nVdash G(Gp \to p)$. If $t \Vdash Gp$ then either $t\uparrow = \emptyset$ or there is $t' > t$ with $t' \Vdash p$, i.e. $t'\uparrow$ and $t'\downarrow$ are finite and so T is finite. If $t \nVdash G(Gp \to p)$ then there is $t' > t$ such that $t' \Vdash Gp$ but $t' \nVdash p$ and then we must have $t'\uparrow = \emptyset$. Suppose now we have $t \in T$ with $t\uparrow = \emptyset$. Since $\mathcal{T} \models \Gamma$ we have $t \Vdash Hp$ or $t \nVdash H(Hp \to p)$. Similarly as above we then have either $t\downarrow = \emptyset$ in which case T is finite, or there is $t' < t$ with $t'\downarrow = \emptyset$.

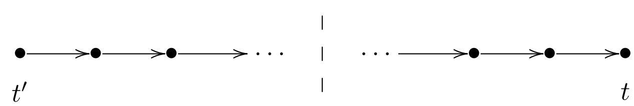
So suppose that we have $t, t' \in T$ with $t' < t$ and $t\uparrow = t'\downarrow = \emptyset$. Define a new labelling L of T by

$$L(s) = \{p\} \Longleftrightarrow_{def} s = t'.$$

Suppose that for every $s \in T$ such that $t' < s < t$ there is $s' \in T$ with $t' < s' < s$. Then $t \Vdash H(Hp \rightarrow p)$, since if $s < t$ we have either $s = t'$ in which case $s \Vdash Hp \rightarrow p$ or we have $t' < s < t$ and then there is s' with $t' < s' < s$ so that $s \nVdash Hp$, i.e. $s \Vdash Hp \rightarrow p$. But then $t \Vdash Hp$, i.e. $s < t$ implies $s = t'$. So either T is finite or t' has an immediate successor t'', i.e. $t' < t''$ and if $t' < s$ then $t'' = s$ or $t'' < s$. We denote this relation by $t \prec t''$.

Similarly we can show that every element $s \in T$ such that $s < t$ has an immediate successor, and analogously that every element $s \in T$ with $t' < s$ has an immediate predecessor s', i.e. $s' \prec s$.

We can now draw the following picture of $\mathcal{T}$:



where each arrow denotes the immediate successor relation $\prec$.

Now, finally we define a last labelling L of T by

$$L(s) = \{p\} \Longleftrightarrow_{def} \exists x_0, \ldots, x_n \in T : x = x_0 \prec \ldots \prec x_n = t.$$

Then $t' \Vdash G(Gp \rightarrow p)$ since if $t' < s$, $s = t$ (in which case $s \Vdash p$ and so $s \Vdash Gp \rightarrow p$) or $t' < s < t$. In the latter case s has an immediate successor s'. If $s \Vdash Gp$ we have $s' \Vdash p$ and so there are $x_0, \ldots, x_n$ with $s' = x_0 \prec \ldots \prec x_n = t$. But then we have $y_0 = s$, $y_1 = x_0, \ldots, y_{n+1} = x_n = t$ and $s = y_0 \prec \ldots \prec y_{n+1} = t$ so $s \Vdash p$. Hence $s \Vdash Gp \rightarrow p$. But then $t' \Vdash Gp$, and since t' has an immediate successor t'', we then have $t'' \Vdash p$, i.e. there is a finite chain $t' \prec t'' \prec \ldots \prec t$. Hence T must be finite.

$\square$

Many of the exercises are taken from the book *Modal Logic* by Patrick Blackburn, Maarten de Rijke and Yde Venema, which is an excellent book if you want to learn more about Modal Logic.