UPPSALA UNIVERSITET
Matematiska institutionen
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Prov i matematik
Teknisk fysik
Komplex analys, 3 p
2008–01–11

Allowed help materials: formula sheet on reverse side. Solutions should be accompanied by explanatory text/figures. Each problem is worth maximum 5 points.
Exam time: *14.30–19.30. (A Swedish version of this exam is available.)*

1. Solve the equation $\tan z = i/2$. (For full credit, the answer should be expressed in the form $a + bi$, a and b real.)

2. Expand the function

$$f(z) = \frac{2z}{z^2 - 4}$$

in a Laurent series of the form $\sum_{n=-\infty}^{\infty} c_n(z - 2 - 2i)^n$ which converges at $z = 0$. For which z in the complex plane will your expansion be absolutely convergent?

3. Determine all singularities of the function

$$g(z) = \frac{(z-3)^2 \sin(i\pi(z+1))}{1 + \sin\frac{\pi z}{2}}$$

in the complex plane and specify their types.

4. Let C denote the positively oriented unit circle and let

$$f(a) = \oint_C \frac{e^{iz}}{z^2(a^2z^2+1)}\,dz.$$

Evaluate $f(a)$ for all $a \in \mathbb{C}$ such that $|a| \neq 1$.

5. Using residues, calculate the integral

$$\int_0^\infty \frac{x \sin x}{(x^2+4)^2}\,dx.$$

(For full credit, you should carefully explain any underlying limit operations involving integrals.)

6. Find all analytic functions $f(z)$ on $\{z : |z| < 5\}$ whose real part $u(x, y)$ satisfies the differential equation

$$\frac{\partial u}{\partial y} = -2u,$$

and for which $f(0) = 0$ and $f(3\pi/2) = 6i$. (Your final answer should be expressed only in terms of z.)

7. Find all functions $f(z)$ which are analytic on $\{z : 0 < |z| < \infty\}$ and which satisfy $f(\pi) = 1/2$ and $|f(z)| \leq \frac{2}{|z|}e^{2y}$ whenever $|z| = 2^k$ and k is an integer (positive or negative).

8. Prove that $z - 2 + 2e^{-z}$ has exactly one zero in the right half-plane $\{z : \operatorname{Re} z > 0\}$ and that this zero is real. Hint: consider the line $x = \epsilon$.