UPPSALA UNIVERSITET
Matematiska institutionen
Dennis A. Hejhal

Prov i matematik
Teknisk fysik
Komplex analys, 3 p
2007–10–15

Tillåtna hjälpmedel: formelblad (på baksidan).
Lösningarna skall åtföljas av förklarande text/figurer. Varje uppgift ger maximalt 5 poäng. Den som har godkänt på duggan från 070917 behöver ej lösa uppgift 1, och den som har godkänt på duggan från 071010 behöver ej lösa uppgift 2.
Skrivtid: *14.00–19.00. (An English version of this exam is available - ask if you prefer that.)*

1. Lös ekvationen $\tan z = -2i$. (För full poäng krävs att svaret är angivet på formen $a + bi$, där a och b är reella.)

2. Utveckla funktionen
$$f(z) = \frac{z}{z^2+4}$$
i en Laurentserie i området $\{z : 2 < |z - 4i| < 6\}$.

3. Bestäm konvergensradien för Taylorserieutvecklingen kring $z = 5\pi i$ av funktionen
$$f(z) = \frac{(1-e^{iz})\sin((1+i)z)}{\sin^2\left(\frac{z}{2}\right)}.$$

4. Låt C vara den positivt orienterade fyrkanten med hörn i punkterna 2, $2i$, -2 och $-2i$. Låt vidare
$$f(a) = \oint_C \frac{e^z}{z^4 + a^2 z^2}\,dz.$$
Beräkna $f(a)$ för $a > 0$, $a \neq 2$.

5. Beräkna medelst residuekalkyl integralen
$$\int_{-\infty}^{\infty} \frac{4+\sin(2x)}{x^2+x+1}\,dx.$$
(För full poäng skall alla gränsvärdesoperationer på inblandade integraler noggrannt förklaras.)

6. Bestäm alla funktioner $f(z)$, som är analytiska i området $\{z : |z| < 2\}$ och vars realdel $u(x,y)$ satisfierar differentialekvationen
$$\frac{\partial u}{\partial y} = 2u,$$
samt för vilka $f(0) = 0$ och $f(\pi/2) = i$. (Svaret skall vara uttryckt endast i termer av z.)

7. Bestäm antalet rötter till ekvationen $z^6 + 5z^4 + 8 = -i\cos z$ i området $\{z : |z| < 2\}$.

8. Bestäm alla funktioner $f(z)$ som är analytiska i området $\{z : 0 < |z| < \infty\}$ och som uppfyller $|f(z)| \leq |z|^2 e^{2y}$ samt $f(\pi) = 9i$ där. (Här är $z = x + iy$, $x, y \in \mathbb{R}$.)

(V.g.v.)

Trigonometriska formler

$$\sin 2x = 2\sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= 1 - 2\sin^2 x = 2\cos^2 x - 1$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin^2 \tfrac{1}{2}x = \tfrac{1}{2}(1 - \cos x)$$

$$\cos^2 \tfrac{1}{2}x = \tfrac{1}{2}(1 + \cos x)$$

$$\sin x \sin y = \tfrac{1}{2}(\cos(x-y) - \cos(x+y))$$

$$\sin x \cos y = \tfrac{1}{2}(\sin(x+y) + \sin(x-y))$$

$$\cos x \cos y = \tfrac{1}{2}(\cos(x+y) + \cos(x-y))$$

Maclaurinutvecklingar

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + O(x^{n+1})$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + O(x^{2n+1})$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + O(x^{2n+2})$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + O(x^{n+1})$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + O(x^{2n+1})$$

$$(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots + \binom{\alpha}{n}x^n + O(x^{n+1})$$

UPPSALA UNIVERSITET
Matematiska institutionen
Dennis A. Hejhal

Prov i matematik
Teknisk fysik
Komplex analys, 3 p
2007–10–15

Allowed help materials: formula sheet on reverse side. Solutions should be accompanied by explanatory text/figures. Each problem is worth 5 points. Those who passed dugga 070917 can skip problem 1, and those who passed dugga 071010 can skip problem 2.
Exam time: *14.00–19.00. (A Swedish version of this exam is available – ask if you prefer that.)*

1. Solve the equation $\tan z = -2i$. (For full credit, the answer should be expressed in the form $a + bi$, a and b real.)

2. Expand the function
$$f(z) = \frac{z}{z^2+4}$$
as a Laurent series on the region $\{z : 2 < |z - 4i| < 6\}$.

3. Determine the radius of convergence of the Taylor series expansion of the function
$$f(z) = \frac{(1-e^{iz})\sin((1+i)z)}{\sin^2\left(\frac{z}{2}\right)}.$$
about the point $z = 5\pi i$.

4. Let C be the positively oriented, square-shaped path with vertices at 2, $2i$, -2 and $-2i$. Let
$$f(a) = \oint_C \frac{e^z}{z^4 + a^2z^2}\,dz.$$
Evaluate $f(a)$ for $a > 0$, $a \neq 2$.

5. Using residues, calculate the integral
$$\int_{-\infty}^{\infty} \frac{4+\sin(2x)}{x^2+x+1}\,dx.$$
(For full credit, you should carefully explain any underlying limit operations involving integrals.)

6. Find all analytic functions $f(z)$ on $\{z : |z| < 2\}$ whose real part $u(x,y)$ satisfies the differential equation
$$\frac{\partial u}{\partial y} = 2u,$$
and for which $f(0) = 0$ and $f(\pi/2) = i$. (Your final answer should be expressed only in terms of z.)

7. Determine the number of roots that the equation $z^6 + 5z^4 + 8 = -i\cos z$ has in the region $\{z : |z| < 2\}$.

8. Find all functions $f(z)$ which are analytic on $\{z : 0 < |z| < \infty\}$ and which satisfy $|f(z)| \leq |z|^2 e^{2y}$ and $f(\pi) = 9i$ there. (Here $z = x + iy$, $x, y \in \mathbb{R}$.)

(over)

Solutions — 15 Oct 2007 — Sketch

Hejhal

① We want $\frac{\sin z}{\cos z} = \frac{1}{i}\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}} = -2i$. Let $u = e^{iz}$. Get $\frac{1}{i}\frac{u^2-1}{u^2+1} = -2i$, hence $\frac{u^2-1}{u^2+1} = 2$, hence $u^2 = -3$. So, $u = e^{iz} = \pm i\sqrt{3}$. This leads to $iz = \ln(\sqrt{3}) + i\arg(\pm i\sqrt{3})$.
So, $iz = \ln\sqrt{3} + i(\pm\frac{\pi}{2} + 2n\pi) \Rightarrow z = -i\ln(\sqrt{3}) \pm \frac{\pi}{2} + 2n\pi$.
Since $-\frac{\pi}{2} = \frac{\pi}{2} - \pi$, we finally get $z = -i\ln(\sqrt{3}) + \frac{\pi}{2} + m\pi$.

② $f = \frac{1}{2}\left[\frac{1}{z+2i} + \frac{1}{z-2i}\right]$. Set $z = 4i + h$. Get:

$$f = \frac{1}{2}\left[\frac{1}{6i+h} + \frac{1}{2i+h}\right] = \frac{1}{2}\left[\frac{1}{6i(1+\frac{h}{6i})} + \frac{1}{h(1+\frac{2i}{h})}\right]$$
$$= \frac{1}{12i}\left[1 - \frac{h}{6i} + \left(\frac{h}{6i}\right)^2 - \left(\frac{h}{6i}\right)^3 \pm \cdots\right]$$
$$+ \frac{1}{2h}\left[1 - \frac{2i}{h} + \left(\frac{2i}{h}\right)^2 - \left(\frac{2i}{h}\right)^3 \pm \cdots\right]$$

provided $\left|\frac{2i}{h}\right| < 1$ and $\left|\frac{h}{6i}\right| < 1$. But $2 < |h| < 6$ by hypothesis. To conclude, just write $h = z - 4i$.

③ We must find the non-removable singularity of $f(z)$ closest to $5\pi i$. The singularities are at $z = 2n\pi$.
Note that $\sin(\frac{z}{2})$ vanishes with multiplicity 1 at $2n\pi$.
Indeed, derivative $= \frac{1}{2}\cos(\frac{z}{2}) = \frac{1}{2}\cos(n\pi) \neq 0$. So, $2n\pi$ is a double zero for $\sin^2(\frac{z}{2})$. At $2n\pi$, $1 - e^{iz} = 0$ with multiplicity 1. Indeed, derivative is $-ie^{iz} = -i \neq 0$.
$\sin[(1+i)z]$ has simple zeros at $z = \frac{k\pi}{1+i}$. So, for $f(z)$, we get:

$z = 0$ $\quad f = \frac{(\text{simple})(\text{simple})}{\text{double}} \Rightarrow$ removable sing

$z = 2n\pi$, $n \neq 0$ $\quad f = \frac{(\text{simple})(\text{non zero})}{\text{double}} \Rightarrow$ get simple pole.

$\pm 2\pi$ are closest to $5\pi i$. Get $R^* = \sqrt{25\pi^2 + 4\pi^2} = \pi\sqrt{29}$.

④ 

Write $\frac{1}{z^2(z^2+a^2)} = \frac{1}{a^2}\left[\frac{1}{z^2} - \frac{1}{z^2+a^2}\right]$. Then,

$$f(a) = \oint_C \frac{e^z}{a^2}\left[\frac{1}{z^2} - \frac{1}{(z+ia)(z-ia)}\right] dz .$$

We use the CRT. Let $g(z)$ be the integrand. Get:

$0 < a < 2 \Rightarrow f(a) = 2\pi i\,[\mathrm{Res}(0) + \mathrm{Res}(ia) + \mathrm{Res}(-ia)]$

$a > 2 \Rightarrow f(a) = 2\pi i\,[\mathrm{Res}(0)]$.

Since $e^z = 1 + z + \frac{z^2}{2!} + \cdots$, clearly by inspection $\mathrm{Res}(0) = \frac{1}{a^2}$.

When ia is relevant, it is a <u>simple</u> pole. Get:

$$\mathrm{Res}(g; ia) = -\frac{e^{ia}}{a^2}\,\frac{1}{ia+ia} = -\frac{e^{ia}}{a^2}\,\frac{1}{2ia} .$$

Similarly,
$$\mathrm{Res}(g; -ia) = -\frac{e^{-ia}}{a^2}\,\frac{1}{-ia-ia} = -\frac{e^{-ia}}{a^2}\,\frac{1}{-2ia} .$$

In toto, then,

$$f(a) = \begin{cases} 2\pi i\left[\frac{1}{a^2} - \frac{e^{ia}}{a^2 \cdot 2ia} + \frac{e^{-ia}}{a^2 \cdot 2ia}\right], & 0 < a < 2 \\ 2\pi i\left[\frac{1}{a^2}\right], & a > 2 \end{cases}$$

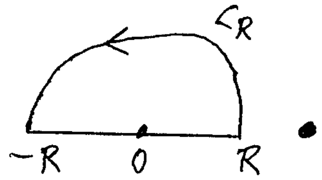
$$= \begin{cases} \frac{2\pi i}{a^2}\left[1 - \frac{e^{ia} - e^{-ia}}{2ia}\right], & 0 < a < 2 \\ \frac{2\pi i}{a^2}[1], & a > 2 \end{cases}$$

$$= \frac{2\pi i}{a^2}\begin{cases} 1 - \frac{\sin(a)}{a}, & 0 < a < 2 \\ 1, & 2 < a < \infty \end{cases} .$$

⑤ Do NOT take $f(z) = \frac{4 + \sin(2z)}{z^2 + z + 1}$! This does <u>NOT</u> work!

Notice instead that

$$\text{integral} = \underline{\mathrm{Re}} \int_{-\infty}^{\infty} \frac{4 - ie^{2ix}}{x^2 + x + 1}\,dx .$$

Take $f(z) = \frac{4 - ie^{2iz}}{z^2 + z + 1}$ and apply CRT to 

.

$z^2+z+1=0 \Leftrightarrow z=\frac{-1\pm i\sqrt{3}}{2}$. Let $z_1=\frac{-1+i\sqrt{3}}{2}$.

$f(z)$ has a simple pole at z_1. (Simply factor z^2+z+1.)

By CRT,

$$\int_{-R}^{R}\frac{4-ie^{2ix}}{x^2+x+1}\,dx+\int_{C_R}\frac{4-ie^{2iz}}{z^2+z+1}\,dz = 2\pi i\, \mathrm{Res}(\text{at } z_1)$$

$$= 2\pi i\,\frac{A(z_1)}{B'(z_1)} \qquad \boxed{\begin{array}{c}545\\ p.309\\ Ex\ 2\end{array}}$$

$\{$where $A=4-ie^{2iz}$, $B=z^2+z+1\}$.

Observe that $\frac{4-ie^{2iz_1}}{2z_1+1}=\frac{4-ie^{2i\left(\frac{-1+i\sqrt{3}}{2}\right)}}{2\left(\frac{i\sqrt{3}}{2}\right)}=\frac{4-ie^{-i}e^{-\sqrt{3}}}{i\sqrt{3}}$.

So,

$$(*)\quad \int_{-R}^{R}\frac{4-ie^{2ix}}{x^2+x+1}\,dx+\int_{C_R} f\,dz=\frac{2\pi}{\sqrt{3}}\left[4-ie^{-\sqrt{3}}(\cos 1-i\sin 1)\right].$$

Since z^2+z+1 has degree 2, we expect $\lim_{R\to\infty}\int_{C_R} f(z)\,dz=0$.

To prove this, notice that: $\boxed{|e^{2iz}|=e^{-2y}} \leftarrow$ AT MOST 1 for $y>0$

$$\left|\int_{C_R} f(z)\,dz\right| \leq \int_{C_R}\left|\frac{4-ie^{2iz}}{z^2+z+1}\right|ds \leq \int_{C_R}\frac{4+e^{-2y}}{|z|^2\left|1+\frac{1}{z}+\frac{1}{z^2}\right|}\,ds$$

$$\leq \int_{C_R}\frac{5(1.1)}{R^2}\,ds=\frac{5(1.1)\pi R}{R^2}.$$

So, by letting $R\to\infty$ in $(*)$, we get

$$\int_{-\infty}^{\infty}\frac{4-ie^{2ix}}{x^2+x+1}\,dx=\frac{2\pi}{\sqrt{3}}\left[4-e^{-\sqrt{3}}\sin 1\right]-\frac{2\pi i}{\sqrt{3}}e^{-\sqrt{3}}\cos 1.$$

Therefore, by taking Re parts,

$$\int_{-\infty}^{\infty}\frac{4+\sin(2x)}{x^2+x+1}\,dx=\frac{2\pi}{\sqrt{3}}\left[4-e^{-\sqrt{3}}\sin 1\right].$$

$\boxed{\text{OK}}$

(6) $\frac{\partial u}{\partial y} = 2u \Rightarrow u = A(x)e^{2y}$. (real-valued: $A(x)$) Must have u harmonic on $\{|z|<2\}$. Need: $u_{xx}+u_{yy} = A''(x)e^{2y} + 4A(x)e^{2y} = 0$.

So, $A''(x)+4A(x)=0$. Hence $A(x) = \alpha\cos(2x)+\beta\sin(2x)$.

So, $u = \alpha\cos(2x)e^{2y} + \beta\sin(2x)e^{2y}$. Notice that

$$e^{-2iz} = e^{-2ix}e^{2y} = e^{2y}(\cos 2x - i\sin 2x).$$

So, $u = \mathrm{Re}[\alpha e^{-2iz} + \beta i e^{-2iz}] = \mathrm{Re}[(\alpha+i\beta)e^{-2iz}]$. By our basic theorem from class,

$$\mathrm{Re}[f(z)] = \mathrm{Re}[(\alpha+i\beta)e^{-2iz}] \Rightarrow f = (\alpha+i\beta)e^{-2iz} + iK,$$

for some real K. Ie, $f = \mathscr{C}e^{-2iz}+iK$. But, now,

$f(0)=0 \Rightarrow \mathscr{C} = -iK$. $f = iK(1-e^{-2iz})$.

$f(\frac{\pi}{2}) = i \Rightarrow i = iK(2) \Rightarrow K = \frac{1}{2}$. $f = \frac{i}{2}(1-e^{-2iz})$.

(7) $|z|<2$ has edge $C: \{|z|=2\}$. Use Rouché's thm.

Need $z^6+5z^4+8+i\cos z = 0$. Put:

$F \leftarrow$ bigger
$E \leftarrow$ smaller

$$F(z) = 5z^4, \quad E(z) = z^6+8+i\cos z.$$

ALONG C, $|F(z)| = 5\cdot 16 = 80$. Also, on C,

$$|E(z)| \leq 64+8+|\cos z|.$$

But,

$$|\cos z| = \left|\frac{e^{iz}+e^{-iz}}{2}\right| \leq \frac{e^{-y}+e^{y}}{2} \leq \frac{e^{2}+e^{-2}}{2} \quad (\text{on } C)$$

$\{$since $e^{y}+e^{-y}$ increases for $0\leq y\leq 2\}$

$$\leq \frac{9+1}{2} = 5. \quad (\text{NOT } 1 !!!)$$

So, on C, $|E| \leq 77 < 80$. By Rouché,

$\{$number roots $F+E=0\} = \{$number of roots of $F=0\}$
$= 4$.

(8) $|e^{2iz}| = e^{-2y}$, $|e^{-2iz}| = e^{2y}$ clearly. Accordingly, on $\{0<|z|<\infty\}$, $|f(z)e^{2iz}| \leqq |z|^2$. Hence:

$$\left|\frac{f(z)e^{2iz}}{z^2}\right| \leqq 1.$$

Let $g(z) = f(z)e^{2iz}/z^2$ on $\{0<|z|<\infty\}$. $g(z)$ is analytic. By Riemann's thm, $z=0$ is a <u>removable</u> singularity. So, g is analytic on $\{|z|<\infty\}$. But $|g(z)| \leqq 1 \Rightarrow$ $g(z) \equiv C$ by Liouville's thm. And $|C| \leqq 1$.

So, $f(z) = Cz^2e^{-2iz}$. $f(\pi) = 9i \Rightarrow 9i = C\pi^2(1)$
$\Rightarrow C = 9i/\pi^2$.

Get:

$$f(z) = \frac{9i}{\pi^2} z^2 e^{-2iz}.$$