Komplex Analys

A few notes on complex integrals. Integrals aren't hard if you remember our goal is to try to apply the theorems – rather than "breaking down" and plugging in the parametric equation of the given contour γ.

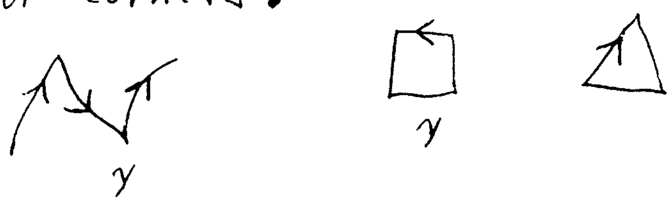
Curve: $z = x(t) + iy(t) = z(t)$, $a \leq t \leq b$ $\left(\begin{array}{c} x(t), y(t) \\ \text{CONTINUOUS} \end{array} \right)$
t either $\nearrow$ or $\downarrow$

Closed Curve: $z(a) = z(b)$

Simple Closed Curve: a closed curve that does not intersect itself

ALL CURVES IN OUR CLASS ARE PIECEWISE SMOOTH; i.e. $z'(t)$ is nonzero and continuously varying except for a finite number of corners.

$$\boxed{z'(t) = \frac{dx}{dt} + i\frac{dy}{dt}}$$
tangent vector

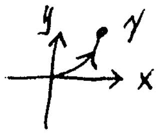


Let $H(z)$ be continuous along γ. Then:

$$\int_\gamma H(z)\,dz \overset{\text{def.}}{=} \int_a^b H[z(t)]\,z'(t)\,dt \quad \left\{ \begin{array}{l} \text{ordinary calculus} \\ \text{integral} \end{array} \right\}$$

when γ has $t \nearrow a$ to b. If $t \downarrow b$ to a, use $\int_b^a$.

Example: $z = t + it^2$ $\left(\begin{array}{c} x = t \\ y = t^2 \end{array} \right)$ for $0 \leq t \leq 1$, $t \nearrow$.



Evaluate $\int_\gamma z\,dz$. EASY!!

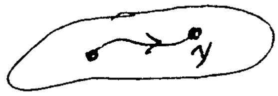
$$I = \int_0^1 (t + it^2)(1 + 2it)\,dt = \int_0^1 [t - 2t^3 + i(t^2 + 2t^2)]\,dt$$
$$= \frac{1}{2} - 2\left(\frac{1}{4}\right) + i\left(\frac{1}{3} + 2\left(\frac{1}{3}\right)\right)$$
$$= 0 + i(1) = \underline{i}$$

Suppose that we now write $H = A + iB$. Then,

$$\int_\gamma H(z)\,dz = \int_a^b [A+iB](x'+iy')\,dt$$
$$= \int_a^b (Ax' - By')\,dt + i\int_a^b (Bx' + Ay')\,dt$$
$$\overset{\text{def.}}{=} \int_\gamma (A\,dx - B\,dy) + i\int_\gamma (B\,dx + A\,dy).$$

The last 2 integrals are <u>line integrals</u>, as in multi-variable calculus. We can always convert $\int_\gamma H\,dz$ to such (<u>if</u> we prefer to use them).

Let γ be a curve. Let γ be contained in domain D. Let $K(z)$ be <u>analytic</u> on D. Let γ be $z = z(t)$.



<u>Claim:</u>

$$\lim_{\substack{h\to 0\\ (\text{real})}} \frac{K[z(t+h)] - K[z(t)]}{h} = K'[z(t)]\cdot z'(t)$$

analytic function type derivative

<u>Proof</u>

Choose any specific t_0. Write $\begin{cases} z(t_0) = z_0 \\ z(t_0+h) = z_0 + \Delta z \end{cases}$.

But: (*)

$$K(z_0 + \ell) = K(z_0) + [K'(z_0) + \varepsilon(\ell)]\cdot \ell \qquad \boxed{\ell \text{ complex},\ \ell \neq 0}$$

where $\lim_{\ell\to 0} \varepsilon(\ell) = 0$.

(*) See S&S, p. 70, prob 2

Hence: (let $\ell = \Delta z$)

$$K(z_0 + \Delta z) - K(z_0) = [K'(z_0) + \varepsilon(\Delta z)]\,\Delta z$$

and, so,

$$\frac{K(z_0+\Delta z) - K(z_0)}{h} = [K'(z_0) + \varepsilon(\Delta z)]\frac{\Delta z}{h}.$$

Let $\underset{\text{(real)}}{h} \to 0$. Get:

$$\text{LIMIT} = [K'(z_0) + 0]\, z'(t_0)$$

$$= K'[z(t_0)]\, z'(t_0).$$

QED

Thus, there is a useful chain rule:

$$(\star) \qquad \underset{\uparrow \text{ real}}{\frac{d}{dt}} K[z(t)] = K'[z(t)] \cdot z'(t).$$

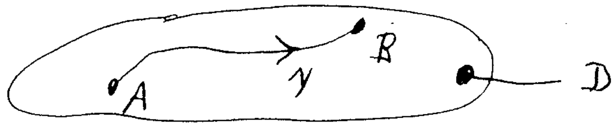
See Saff/Snider p. 174 middle

They SKIPPED the proof!!

Example: old path $z = t + it^2$

$$\underset{\substack{\uparrow \\ \text{real}}}{\frac{d}{dt}} e^{b(t+it^2)} = b e^{b(t+it^2)} \cdot (1 + 2it), \quad \text{for } 0 \leq t \leq 1.$$

$\boxed{K(z) = e^{bz}}$



Corollary

Let D be a domain. Let γ be a curve in D. Suppose that we can find an analytic function $K(z)$ on D such that $H(z) = K'(z)$.

Then:

$$\int_\gamma H(z)\,dz = \int_\gamma K'(z)\,dz = K(B) - K(A).$$

(Looks like "baby" calculus.)

Proof

Write $z = z(t)$, $a \leq t \leq b$, $t \nearrow$ for γ. So, by $(\star)$,

$$\int_\gamma H(z)\,dz = \int_a^b K'[z(t)]\,z'(t)\,dt = \int_a^b \underset{\uparrow \text{ real}}{\frac{d}{dt}} K[z(t)]\,dt.$$

Finally, then, by elementary calculus,

$$\int_\gamma H(z)\,dz = K[z(b)] - K[z(a)] = K(B) - K(A). \quad \blacksquare$$

<u>Example:</u> Take our old arc γ $z = t + it^2$ and $\int_\gamma z\,dz$.

Notice that $K(z) = \dfrac{z^2}{2}$ works on $D = \mathbb{C}$.

So:

$$\int_\gamma z\,dz = \frac{z^2}{2}\Big]_0^{1+i} = \frac{(1+i)^2 - 0^2}{2} = \frac{2i}{2} = \underline{\underline{i}}.$$

The problem in this class is that most of the time we <u>cannot</u> "see" $K(z)$. So, we have to rely on other <u>theorems</u> instead!

One basic theorem — <u>closely related to</u> the corresponding theorem in calculus for line integrals — is this:

<u>THM</u> (fund. thm of contour integration)

Let D be a domain. Let $H(z)$ be continuous on D. The following are <u>equivalent</u>:

(a) $\int_\gamma H(z)\,dz$ is independent of path for γ in D

↳ i.e. depends only on <u>endpoints</u>

(b) $\int_C H(z)\,dz = 0$ for <u>every</u> <u>closed</u> path C in D

(b′) $\int_C H(z)\,dz = 0$ for <u>every</u> <u>simple</u> <u>closed</u> path C in D

(c) there exists <u>some</u> analytic function $K(z)$ <u>on D</u> such that $K'(z) = H(z)$.

Let us prove, for instance, that (a) and (c) are equivalent.

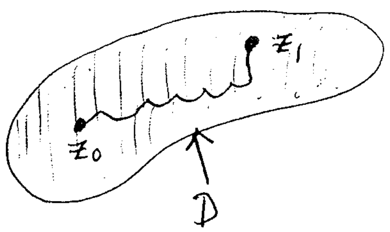
Suppose condition (c) holds. To prove (a), just write

$$\int_\gamma H(z)\,dz = \int_\gamma K'(z)\,dz = K(B) - K(A) \quad \text{by the Corollary.}$$

This answer depends only on A & B, <u>not</u> shape of γ.

OK

We now prove that (a) $\Rightarrow$ (c). Select any $z_0 \in D$.

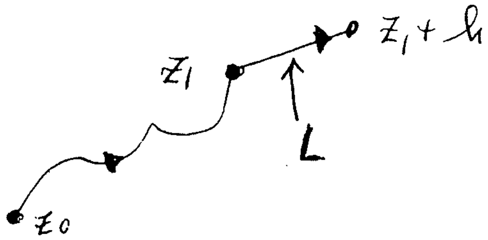


To <u>define</u> K at z_1, select <u>any</u> path z_0 to z_1 in D. Write

$$K(z_1) = \int_{\substack{z_0 \\ \text{that path}}}^{z_1} H(z)\,dz$$

This is a good legitimate definition! depends only on z_1

To define K at $z_1 + h$, we follow the given path to z_1 then just go on a <u>line segment</u> z_1 to $z_1 + h$.



Notice that $K(z_1+h) - K(z_1) = \int_{z_1,\,L}^{z_1+h} H(z)\,dz$.

Parametric equation of line segment is:

$$z = z_1 + ht, \quad 0 \leq t \leq 1, \quad t \nearrow.$$

So:

$$\Delta K = \int_0^1 H(z_1 + ht)\cdot h \cdot \underset{\text{real}}{dt}.$$

So,

$$\frac{\Delta K}{h} - H(z_1) = \int_0^1 H(z_1+ht)\,dt - H(z_1)$$

$$= \int_0^1 [H(z_1+ht) - H(z_1)]\,dt \, .$$

$\uparrow$ real

For <u>any</u> complex function $g(t)$, one gets

$$\left|\int_0^1 g(t)\,dt\right| \leq \int_0^1 |g(t)|\,dt$$

by triangle inequality. (Proof given on next page.)

So,

$$\left|\frac{\Delta K}{h} - H(z_1)\right| \leq \int_0^1 |H(z_1+ht) - H(z_1)|\,dt \, .$$

For $h \to 0$, we have

$$|H(z_1+ht) - H(z_1)| < \varepsilon \, , \quad 0 \leq t \leq 1$$

<u>since</u> $H(z)$ is <u>continuous</u> <u>at</u> <u>z_1</u>. So,

$$\left|\frac{\Delta K}{h} - H(z_1)\right| \leq \int_0^1 \varepsilon\,dt = \varepsilon \, .$$

Ie,

$$\lim_{h\to 0} \frac{\Delta K}{h} = \lim_{h\to 0} \frac{K(z_1+h) - K(z_1)}{h} = \underline{H(z_1)} \, .$$

The function K thus works!! $\Leftarrow$ $K'(z) = H(z)$ at any z_1

Ie, (a) $\Rightarrow$ (c). OK

$d\phi = P\,dx + Q\,dy$ $\phi_x = P, \ \phi_y = Q$	The function K is the analog of the potential function ϕ for line integrals.

Proof that $\left| \int_0^1 g(t)\,dt \right| \leq \int_0^1 |g(t)|\,dt$. $\boxed{g(t) \text{ complex}}$

<u>Proof #1</u> (by Δ inequality):

The left-hand side is approximately $\left| \sum_{k=1}^{N} g(t_k^*)\,\Delta t_k \right|$ for N large (by Riemann sums). This expression is $\leq \sum_{k=1}^{N} |g(t_k^*)|\,\Delta t_k$ by Δ inequality. Let $N \to \infty$. Get

$$\left| \int_0^1 g(t)\,dt \right| \leq \int_0^1 |g(t)|\,dt. \qquad \text{qed}$$

no Δ inequality!

<u>Proof #2</u> (the slick proof):

If $\int_0^1 g(t)\,dt = 0$, we are done. Let $\int_0^1 g(t)\,dt = Re^{i\phi}$, with $R > 0$. Notice that

$$R = e^{-i\phi} \int_0^1 g(t)\,dt = \int_0^1 e^{-i\phi} g(t)\,dt.$$

Since R is real,

$$R = \int_0^1 \underline{\underline{\mathrm{Re}}}\left(e^{-i\phi} g(t)\right) dt.$$

But, trivially,

$$\mathrm{Re}\left(e^{-i\phi} g(t)\right) \leq \left|e^{-i\phi} g(t)\right| = |g(t)|.$$

So,

$$R \leq \int_0^1 |g(t)|\,dt.$$

But, $R = \left| \int_0^1 g(t)\,dt \right|$ by definition. Done!