Komplex Analys Dugga — 22 Sept 2008
(D. A. Hejhal) — 13:15 – 14:00

This dugga is closed book, closed notes, no calculators. There are 3 problems; each is worth 5 points. For full credit in the problems, you <u>must</u> show your work. Simply giving the correct answer will yield <u>at most</u> 50% credit.

$\boxed{1}$ Find the <u>principal value</u> of $(-1-i)^{4+i}$. The answer must be placed in the form $a+bi$, where a and b are real.

Note: assume $-\pi < \operatorname{Arg}(w) \leq \pi$ for the principal value of $\arg(w)$, as in Saff & Snider.

$\boxed{2}$ Find the solution of $\cos(z) = 8i$ in the <u>upper</u> half-plane $\{\operatorname{Im}(z) > 0\}$ which lies <u>closest</u> to the point $z = 3\pi$.

(The answer should be placed in the form $a+bi$, where a and b are real.)

$\boxed{3}$ Find all functions $f(z)$ which are analytic on $\mathbb{C}$ and which satisfy

$$\left\{ \begin{array}{l} \operatorname{Im} f(z) = e^{2y}\cos(2x) + xy \\ \text{and } f(0) = 4+i \end{array} \right\}.$$

For full credit, your answer must be in terms of <u>z</u> (not x, y).

Dugga — Sketch of Solutions (2008)

$\boxed{1}$

$(-1-i)^{4+i} = \exp[(4+i)\,\mathrm{Log}(-1-i)]$ is the principal value we seek. But,

$$\mathrm{Log}(-1-i) = \ln|-1-i| + i\,\mathrm{Arg}(-1-i)$$
$$= \ln\sqrt{2} + i\left(-\tfrac{3}{4}\pi\right).$$

So,

$$\text{answer} = e^{(4+i)(\ln\sqrt{2} - \frac{3}{4}\pi i)}$$
$$= e^{4\ln\sqrt{2} - 3\pi i}\, e^{i(\ln\sqrt{2} - \frac{3}{4}\pi i)}$$
$$= (\sqrt{2})^4 e^{-3\pi i}\, e^{i\ln\sqrt{2}}\, e^{\frac{3}{4}\pi}$$
$$= -4e^{\frac{3}{4}\pi}\left[\cos(\ln\sqrt{2}) + i\sin(\ln\sqrt{2})\right]$$
$$= -4e^{\frac{3}{4}\pi}\cos(\ln\sqrt{2}) - 4ie^{\frac{3}{4}\pi}\sin(\ln\sqrt{2}).$$

$\boxed{2}$ Must first solve $\cos(z) = 8i$. Let $u = e^{iz}$.

Get

$$\frac{u+u^{-1}}{2} = 8i \Rightarrow u^2+1 = 16iu \Rightarrow$$
$$u^2 - 16iu + 1 = 0 \Rightarrow u = \frac{16i \pm \sqrt{-256-4}}{2}.$$

So,

$$e^{iz} = i(8 \pm \sqrt{65}).$$ (Note that $8-\sqrt{65} < 0$.)

2

<u>Case I</u>

$$iz = \log[i(8+\sqrt{65})] = \ln(8+\sqrt{65}) + i\frac{\pi}{2} + 2n\pi i$$

$$z = -i\ln(8+\sqrt{65}) + \frac{\pi}{2} + 2n\pi$$

These are in LOWER half-plane, hence irrelevant.

<u>Case II</u>

$$iz = \log[-i(\sqrt{65}-8)] = \ln(\sqrt{65}-8) - i\frac{\pi}{2} + 2n\pi i$$

$$z = -i\ln(\sqrt{65}-8) - \frac{\pi}{2} + 2n\pi .$$

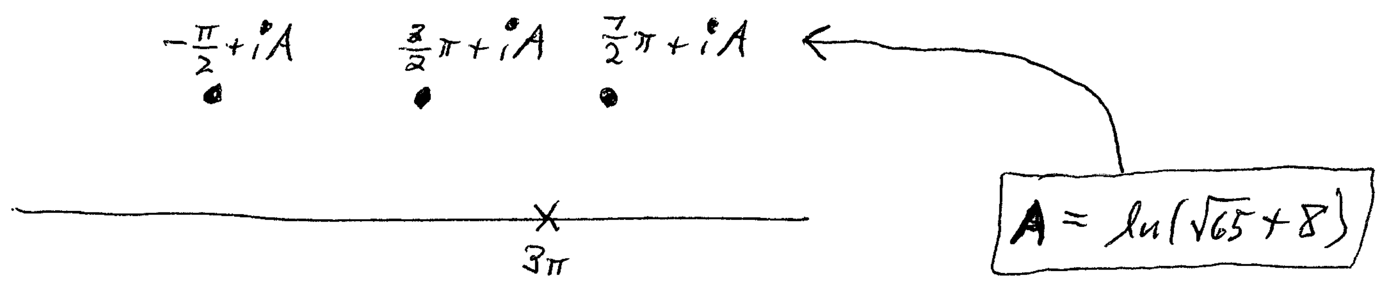
But, $0 < \sqrt{65}-8 < 1$ since

$$1 = (\sqrt{65}+8)(\sqrt{65}-8) .$$

So, in fact,

$$z = i\ln(\sqrt{65}+8) - \frac{\pi}{2} + 2n\pi \quad \text{here.}$$

Clearly $\operatorname{Im}(z) > 0$. We want to get as close to $z = 3\pi$ as possible! Clearly:



yields $\quad z = (3.5)\pi + i\ln(\sqrt{65}+8)$. //

$\boxed{3}$ $\text{Im } f = e^{2y}\cos(2x) + xy$, $f(0) = 4+i$.

Method I (guess)

$$z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$$

$$e^{2iz} = e^{2ix-2y} = e^{-2y}(\cos 2x + i\sin 2x)$$

$$e^{-2iz} = e^{-2ix+2y} = e^{2y}(\cos 2x - i\sin 2x)$$

$$ie^{-2iz} = ie^{2y}\cos(2x) + e^{2y}\sin(2x)$$

$$\text{Im}\left(\frac{z^2}{2}\right) = xy \qquad \text{Im}\left(ie^{-2iz}\right) = e^{2y}\cos(2x)$$

So: $\text{Im}[f] = \text{Im}\left[\frac{z^2}{2} + ie^{-2iz}\right]$. by guessing

By THM, get:

$$f = \frac{z^2}{2} + ie^{-2iz} + A, \quad A \text{ real.}$$

Plug in $z = 0$.

$$4+i = i + A \Rightarrow A = 4.$$

$$f = ie^{-2iz} + \frac{z^2}{2} + 4. \quad //$$

Method II

$$v = e^{2y}\cos(2x) + xy$$

C-R: $u_x = v_y = 2e^{2y}\cos(2x) + x$

$u_y = -v_x = 2e^{2y}\sin(2x) - y$.

Get $u = e^{2y}\sin(2x) + \frac{x^2}{2} + \underline{b(y)}$, then

$$2e^{2y}\sin(2x) + 0 + b'(y) = 2e^{2y}\sin(2x) - y$$

$$\Rightarrow b(y) = -\frac{y^2}{2} + A \Rightarrow u = e^{2y}\sin(2x) + \frac{x^2-y^2}{2} + A.$$

$\boxed{2}$

So

$$u + iv = e^{2y}\sin(2x) + \frac{x^2 - y^2}{2} + i\left(e^{2y}\cos 2x + xy\right)$$
$$+ A$$

$f(z) =$ MUST BE PLACED IN TERMS OF $\underline{z}$.

Must now either repeat first few lines of method I, or say

$f(z)$ analytic on $\mathbb{C}$

$$f(\underline{x}) = \sin(2x) + \frac{x^2}{2} + i\cos(2x) + A \qquad \boxed{y = 0}$$

$$f(x) = ie^{-2ix} + \frac{x^2}{2} + A$$

$$\Downarrow$$

$$f(z) = ie^{-2iz} + \frac{z^2}{2} + A \qquad \left[\begin{array}{l}\text{by a THM} \\ \text{in book}\end{array}\right].$$

Plug in $z = 0$.

$$4 + i = i + A, \qquad A = 4.$$

$$f = 4 + ie^{-2iz} + \frac{z^2}{2}. \quad ////$$