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Dugga
Komplex Analys
TF
070917

Exam Time: $13^{15} - 14^{00}$. Closed book, closed notes, no calculators. Solutions must be accompanied by explanatory text. Svenska är bra! Each problem is worth 5 points. To pass you must get at least 8 points.

1. Find <u>all</u> solutions to the equation

$$(1+i)(\sin z + i\cos z) = e^{4iz} .$$

Express answer in form $a+bi$, a and b real.

2. Find <u>all</u> solutions to the equation $\sin z = 4i$. Express answer in form $a+bi$, a and b real.

3. Find <u>all</u> analytic functions $f(z) = u+iv$ on $\mathbb{C}$ such that

$$3u + v = x^3 - 3xy^2 - y$$

and $f(0) = 2-6i$. Your answer should be expressed in terms of z $(= x+iy)$, not x and y.

Solutions

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$\boxed{1}$

$$(1+i)(\sin z + i\cos z) = e^{4iz}$$
$$(1+i)\,i(\cos z - i\sin z) = e^{4iz}$$
$$(1+i)\,i\,(e^{-iz}) = e^{4iz} \quad \Rightarrow \quad e^{5iz} = -1+i$$

So, $5iz = \log(-1+i) = \ln\sqrt{2} + i\arg(-1+i)$

$$5iz = \ln\sqrt{2} + i\left(\tfrac{3}{4}\pi + 2n\pi\right)$$
$$iz = \frac{1}{10}\ln 2 + i\left(\frac{3\pi}{20} + \frac{2n\pi}{5}\right) \qquad (n \in \mathbb{Z})$$
$$z = -\frac{i}{10}\ln 2 + \frac{3\pi}{20} + \frac{2n\pi}{5}.$$

$\boxed{2}$

$$\sin z = 4i \iff \frac{e^{iz} - e^{-iz}}{2i} = 4i \iff e^{iz} - e^{-iz} = -8.$$

Let $u = e^{iz}$. Get $u^2 - 1 = -8u$ or $u^2 + 8u - 1 = 0$.

So:
$$u = \frac{-8 \pm \sqrt{64+4}}{2} = \frac{-8 \pm \sqrt{68}}{2} = -4 \pm \sqrt{17}.$$

(Students are expected to know quadratic formula.)

So,
$$iz = \log(-4 \pm \sqrt{17}), \text{ all values.}$$

Get:

$$iz = \ln(-4+\sqrt{17}) + i\arg(-4+\sqrt{17}) \qquad \sqrt{17} = 4.1^+$$

OR

$$iz = \ln|-4-\sqrt{17}| + i\arg(-4-\sqrt{17}).$$

Ie,

$$iz = \ln(\sqrt{17}-4) + i(0 + 2n\pi);$$
$$iz = \ln(\sqrt{17}+4) + i(\pi + 2n\pi).$$

So,

$$z = -i\ln(\sqrt{17}-4) + 2n\pi \qquad (n \in \mathbb{Z})$$
$$z = -i\ln(\sqrt{17}+4) + \pi + 2n\pi \quad .$$

$\boxed{3}$ $f(z) = u + iv$ analytic on $\mathbb{C}$, $f(0) = 2-6i$,
$3u + v = x^3 - 3xy^2 - y$.

<u>Sol #1</u> (preferred)

What is $3u+v$?? $\quad (3-i)(u+iv) = (3u+v) + i(3v-u)$

Want:
$$\mathrm{Re}\left[(3-i)f(z)\right] = \mathrm{Re}(z^3) + \mathrm{Re}(iz),$$

$$\text{since} \left\{ \begin{array}{c} (x+iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 \\ + (iy)^3 \\ \hline \text{and} \quad iz = ix - y \end{array} \right\}.$$

So, we need:

$$\mathrm{Re}\left[(3-i)f(z) - z^3 - iz\right] = 0 \ .$$

<u>But</u>, $(3-i)f(z) - z^3 - iz$ is <u>analytic</u> on $\mathbb{C}$.
By C-R, we get

$$(3-i)f(z) - z^3 - iz = iA \ , \quad A \in \mathbb{R} \ .$$

So,

$$f(z) = \frac{z^3 + iz}{3-i} + \frac{iA}{3-i} \ .$$

Need $\quad 2-6i = \dfrac{iA}{3-i}$.

We must calculate to see if A is <u>real</u>.
If it isn't, then no such f exists.

Fortunately,

$$2-6i = \frac{iA}{3-i} = A\,\frac{3i-1}{10} \quad \Rightarrow \quad A = -20.$$

ANSWER:

$$f(z) = \frac{z^3 + iz}{3-i} + (2-6i).$$

<u>Sol #2</u> (<u>too</u> long!!)

$$3u + v = x^3 - 3xy^2 - y \quad \Rightarrow$$

$$3u_x + v_x = 3x^2 - 3y^2 \qquad \text{but} \qquad u_x = v_y \quad \text{by C-R}$$
$$3u_y + v_y = -6xy - 1 \qquad\qquad u_y = -v_x$$

so need

$$3u_x - u_y = 3x^2 - 3y^2$$
$$u_x + 3u_y = -6xy - 1.$$

Get: (by some algebra)

$$u_x = \frac{9x^2 - 9y^2 - 6xy - 1}{10}, \qquad u_y = \frac{-18xy - 3 - 3x^2 + 3y^2}{10}.$$

These yield:

$$u = \frac{3x^3 - 9xy^2 - 3x^2y - x}{10} + A(y)$$
$$u = \frac{-9xy^2 - 3y - 3x^2y + y^3}{10} + B(x).$$

Must find a <u>consistent</u> choice of A & B.

This implies that

$$A(y) = \frac{-3y + y^3}{10} + C \qquad (C = \underline{\text{real}})$$

$$B(x) = \frac{3x^3 - x}{10} + C .$$

So,

$$u = \frac{3x^3 - 9xy^2 - 3x^2y - x - 3y + y^3}{10} + C .$$

But $v = x^3 - 3xy^2 - y - \underline{3}u$.

Get:

$$v = \frac{10x^3 - 30xy^2 - 10y}{10} - \frac{1}{10}(9x^3 - 27xy^2 - 9x^2y - 3x - 9y + 3y^3) - 3C$$

$$\Rightarrow v = \frac{1}{10}(x^3 - 3xy^2 - y + 9x^2y + 3x - 3y^3) - 3C .$$

So:

$$f = \frac{3x^3 - 9xy^2 - 3x^2y - x - 3y + y^3}{10} + \frac{i}{10}(x^3 - 3xy^2 - y + 9x^2y + 3x - 3y^3) + C - 3iC$$

(Rigged to be ANALYTIC)

$$\Rightarrow f = \frac{3+i}{10}(x^3 - 3xy^2 + 3ix^2y - iy^3) + \frac{3+i}{10}(-y + ix) + C(1 - 3i)$$

(look at terms one-by-one)

$$\Rightarrow f = \frac{3+i}{10}(z^3 + iz) + C(1 - 3i) .$$

To get $f(0) = 2 - 6i$, take $C = 2$.