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Dugga #2
Komplex Analys
TF
071010

Exam Time: $14^{15} - 15^{00}$. Closed book, closed notes, no calculators. Solutions must be accompanied by explanatory text in problems 2 and 3. Svenska är bra! Each problem is worth 5 points. To pass you must get ≥ 8 points.

1. State the type of singularity at the given point. Removable, Pole of order N (you must tell N), Essential.

(a) $\left(\dfrac{\cos z}{\sin z}\right) \cdot e^{\frac{3}{z+1}}$ at $z = 0$

(b) $\dfrac{z^8 e^{i/z}}{(z^2+4)^4}$ at $z = 2i$

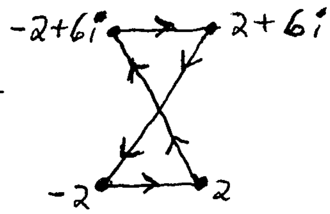
(c) $\dfrac{z^8 e^{i/z}}{(z^2+4)^4}$ at $z = \infty$

(d) $e^{-\left(z+\frac{1}{z}\right)^4}$ at $z = 0$

(e) $\dfrac{1}{z^2 \sin z} - \dfrac{1}{z^3 \cos z}$ at $z = 0$

2. Using the Cauchy integral theorem and/or Cauchy integral formula, calculate

$$\int_\Gamma \frac{1}{(z^2+4)^3}\, dz \qquad \text{where } \Gamma =$$



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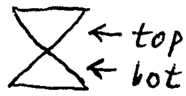
3. Consider the function $\dfrac{z}{(z^2+1)(z^2+4)}$. Assume it is expanded as a Laurent series $\sum_{n=-\infty}^{\infty} c_n z^n$ for $1 < |z| < 2$. Evaluate c_{-5} and c_5. Your final answers should be fractions $\frac{p}{q}$.

Solutions ~ Dugga #2

1. (a) pole, order 1
   (b) pole, order 4
   (c) removable
   (d) essential
   (e) pole, order $\underline{1}$

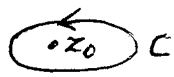
2. Let the path Γ be divided into 2 triangular pieces Γ_{bot} and Γ_{top}. $f(z) = (z-2i)^{-3}(z+2i)^{-3}$.



Γ_{top} encloses $\underline{\text{no}}$ singularities; so, by CIT,

$$\int_{\Gamma_{top}} f\,dz = 0.$$

Γ_{bot} encloses $2i$. So, we use a CI$\underline{\text{F}}$ set-up instead. We know (CIF):



$$g^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{g(z)}{(z-z_0)^{n+1}}\,dz, \quad n \geq 0.$$

Notice that

$$\left\{ \begin{array}{l} \displaystyle\oint_{\Gamma_{bot}} f\,dz = \oint_{\Gamma_{bot}} \frac{(z+2i)^{-3}}{(z-2i)^3}\,dz = 2\pi i \, \frac{g''(2i)}{2!} \\ \text{where } g = (z+2i)^{-3} \leftarrow \begin{array}{l}\text{ANALYTIC} \\ \text{INSIDE } \Gamma_{bot}\end{array} \end{array} \right\}.$$

But, $g''(z) = 3 \cdot 4 (z+2i)^{-5}$, so

$$\oint_{\Gamma_{bot}} f\,dz = 2\pi i \, \frac{12(4i)^{-5}}{2} = \pi i \, 12 (4)^{-5} i^{-5} = \frac{12\pi}{4^5}$$

$$= \frac{3\pi}{4^4} = \frac{3\pi}{256}.$$

Hence,

$$\int_{\Gamma} f(z)\,dz = 0 + \frac{3\pi}{256} = \frac{3\pi}{256}.$$

3. $$f = \frac{z}{(z^2+1)(z^2+4)} = \sum_{-\infty}^{\infty} c_n z^n, \quad 1 < |z| < 2.$$

One should seek to get the expansion in powers of z in the simplest way. This may not involve going "all the way" to partial fractions. Note:

$$f = z \cdot \frac{1}{3}\left(\frac{1}{z^2+1} - \frac{1}{z^2+4}\right) \quad \text{trivially.}$$

So, for $1 < |z| < 2$, we write

$$f = \frac{z}{3}\left(\frac{1}{z^2(1+z^{-2})} - \frac{1}{4(1+\frac{z^2}{4})}\right)$$

because

$$\frac{1}{1+u} = 1 - u + u^2 - u^3 \pm \cdots \quad \text{when } \underline{|u| < 1}.$$

We get:

$$f = \frac{z}{3}\left[z^{-2}(1 - z^{-2} + z^{-4} - z^{-6} \pm \cdots)\right]$$
$$- \frac{z}{12}\left[1 - \left(\frac{z^2}{4}\right) + \left(\frac{z^2}{4}\right)^2 - \left(\frac{z^2}{4}\right)^3 \pm \cdots\right].$$

By inspection,

$$c_{-5} = \frac{1}{3}$$

$$c_5 = -\frac{1}{12} \cdot \frac{1}{16} = -\frac{1}{192}.$$