Analytic Functions

Analytiska Funktioner

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Lecture 1

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I begin with a few words about the layout/structure of the course!

This course is intended as a "follow-up" to the standard undergrad complex variable course. A kind of "second course" in complex analysis.

I've given the course before in a number of different formats.

As you will see, there is a mixture of <u>general</u> and <u>explicit</u>.

The "official contents" are these:

Infinite series and products. Partial fractions and factorisation. The gamma and the beta functions, Stirling's formula, the method of steepest descent. Riemann's zeta function. Normal families, the Riemann mapping theorem. Harmonic functions, Poisson's formula, Jensen's formula, the distribution of zeros of entire functions. Analytic continuation: continuation along arcs, the monodromy theorem, the modular function and Picard's theorem. Subharmonic functions. Dirichlet's problem.

The "official learning outcomes" are these:

In order to pass the course (grade 3) the student should

- know the Weierstrass factorization theorem for entire functions and the Mittag-Leffler theorem for meromorphic functions;
- be able to give an account of the concept of normal family;
- be able to outline a proof of the Riemann mapping theorem;
- be able to give an account of the basic properties of harmonic functions, Poisson´s formula and the principle of Harnack;
- know Jensen's formula and how to apply it;
- be familiar with the concept of analytic continuation and the monodromy theorem;
- be able to outline the construction of a modular function and the proof of the "little Picard theorem";
- be familiar with the basic properties of subharmonic functions, and know how to use subharmonicity for the study of Dirichlet´s problem;
- be able to solve problems within the area of the course and to give proofs of central theorems.

In several older versions of this class, there was always time to do a couple of special topics, too.

E.G.

introduction to Riemann zeta fcn. and Riemann Hypothesis;

Euler-Maclaurin summation;

Hadamard factorization theorem (with some examples);

Stirling's formula on $|\mathrm{Arg}(z)| \leq \pi - \delta$;

Additional material on conformal mapping.

(Total number of lectures ≈ 25.)

I should also say that, in general, this will be a traditional type math class.

So,

1 final exam
1 oral final exam
Regular Homework
(starred, unstarred).

hand in — Do not hand in

Textbook (used as our basic "guide"):
L. Ahlfors, Complex Analysis,
3rd edition.
(2nd also OK – basically)

I think I will also make a copy of my lecture notes available for your Xeroxing.

Borrow from Library.

Today, I want to **discuss** a topic that often gets "bypassed" in undergrad complex variable.

I will build essentially on the old Cauchy – Goursat proof of the Cauchy integral theorem for rectangles.

$\boxed{\text{See Ahlfors, chap 4, §1.4.}}$

TO GET STARTED —

$\boxed{\text{open connected sets in } \mathbb{R}^N \text{ are called domains !!}}$

Recall:

Def: Let $f(z)$ be given on an open + connected set $D \subseteq \mathbb{R}^2$.
We say $f(z)$ is analytic on D when $f'(z_0)$ **exists** at every $z_0 \in D$.

$$\left\{ f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right\}$$

BIG THM.

Let $f(z) = u(x,y) + iv(x,y)$ be analytic on D. Then:

(i) $u(x,y)$ & $v(x,y)$ are $\overbrace{C^{\infty}}^{\text{automatically}}$ on D;

(ii) $f'(z)$ is analytic on D.

$\boxed{\text{hence } f^{(k)}(z) \text{ is analytic on } D \text{ for ANY } k \geq 1}$

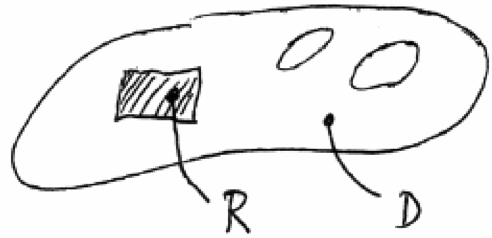
This theorem often gets short shrift. We want to be perfectly clear why this result is true!

IT IS A KEY THEOREM.

We approach it by way of rectangles and Cauchy integral theorem.

Theorem (Cauchy Integral Thm)

Let $f(z)$ be analytic on a domain D that includes rectangle R.



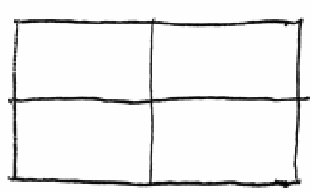
Then:

$$\oint_{\partial R} f(z)\,dz = 0.$$

Proof (Goursat) ← FAMOUS PROOF

Use contradiction.

Suppose $c = \oint_{\partial R} f(z)\,dz \neq 0$.



Partition R as shown.

$\boxed{\text{Bisect} \times \text{Bisect}}$

Traverse each small rectangle. Get:

$$\text{“} I_1 + I_2 + I_3 + I_4 = c \text{”}.$$

[since equal/opposite arcs cancel]

Select one, call it $R^{(1)}$, so that

$$\left| \oint_{\partial R^{(1)}} f(z)\,dz \right| \geqq \frac{|c|}{4} .$$

Repeat. Get $R^{(2)}$ <u>inside</u> $R^{(1)}$ ~~so~~

$$\left| \oint_{\partial R^{(2)}} f(z)\,dz \right| \geqq \frac{1}{4}\frac{|c|}{4} .$$

etc etc

$$R \supseteq R^{(1)} \supseteq R^{(2)} \supseteq \cdots$$

NESTED RECTANGLES

$$\left| \oint_{\partial R^{(n)}} f(z)\,dz \right| \geqq \frac{|c|}{4^n} .$$

Clearly, by Bolzano-Weierstrass (etc),

$$\bigcap_{n=1}^{\infty} R^{(n)} = \text{one point } \{\xi\} .$$

see Ahlfors, p. 63, prob. 4

3rd edition

Know

$$f'(\xi) = \lim_{\substack{z \to \xi \\ (z \neq \xi)}} \frac{f(z) - f(\xi)}{z - \xi} \quad \underline{\text{exists}}.$$

Write

$$\frac{f(z) - f(\xi)}{z - \xi} = f'(\xi) + c(z)$$

for $z \neq \xi$ and $c(\xi) = 0$. Notice that $c(z)$ __is__ __continuous__ near the point ξ.

{ NOTE: f' exists at $z_0 \Rightarrow$ f is <u>continuous</u> at z_0, as in elementary calculus. }

Get:

$$f(z) = f(\xi) + f'(\xi)(z - \xi) + c(z)(z - \xi)$$

for z near ξ.

By elementary line integrals and anti-derivatives

$$\left[\int_A^B g'(z)\,dz = g(B) - g(A) \quad \text{"baby stuff"} \right]$$

$$\oint_\gamma (z - z_1)^n\, dz = 0 \quad , \quad n \geq 0 \,. \quad (z_1 \in \mathbb{C})$$

Accordingly:

$$\int_{\partial R^{(n)}} f\,dz = 0 + 0 + \int_{\partial R^{(n)}} c(z)(z-\xi)\,dz \,.$$

Hence:

$$\left| \int_{\partial R^{(n)}} f\,dz \right| \leq \int_{\partial R^{(n)}} |c(z)|\,|z-\xi|\,\underbrace{|dz|}_{\text{arc length}} \,.$$

Watch carefully now!!

Let d_n = diameter of $R^{(n)}$.

IE diagonal

Clearly,

$$d_n = \frac{\alpha}{2^n} \quad \text{(say)}.$$

Self-Similar Aspect

α = diagonal of R

We have:

$$|z - \xi| \leq d_n \quad \text{for} \quad z \in \partial R^{(n)};$$

$$\text{length}(\partial R^{(n)}) \leq 4 d_n.$$

Choose *any* tiny $\varepsilon > 0$. Since $c(z)$ is continuous and $c(\xi) = 0$,

$$|c(z)| < \varepsilon \quad \text{for} \quad z \in \partial R^{(n)}$$

whenever $n \geq N_\varepsilon$.

For $n \geq N_\varepsilon$, get:

$$\left| \int_{\partial R^{(n)}} f\,dz \right| \leq \int_{\partial R^{(n)}} \varepsilon \cdot d_n \cdot |dz|$$

$$\leq \varepsilon \cdot d_n \cdot 4 d_n$$

$$= 4\varepsilon \frac{d^2}{4^n} .$$

But, this then gives

$$\frac{|c|}{\cancel{4^n}} \leq \left| \int_{\partial R^{(n)}} f\,dz \right| \leq \frac{4\varepsilon d^2}{\cancel{4^n}} .$$

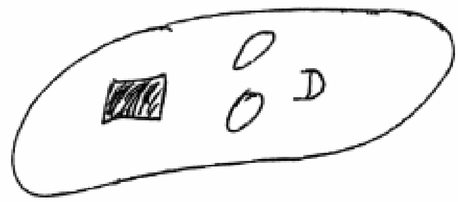
For sufficiently small ε, this is a contradiction.

We conclude that:

$$\oint_{\partial R} f(z)\,dz = 0 . \blacksquare$$

"TRICK" COROLLARY.

Let R be a rectangle contained in domain D.



Let $f(z)$ be analytic on D **except** **for** a point z_1 interior to D where

$$\lim_{z \to z_1} (z - z_1) f(z) = 0 .$$

Then, we still have

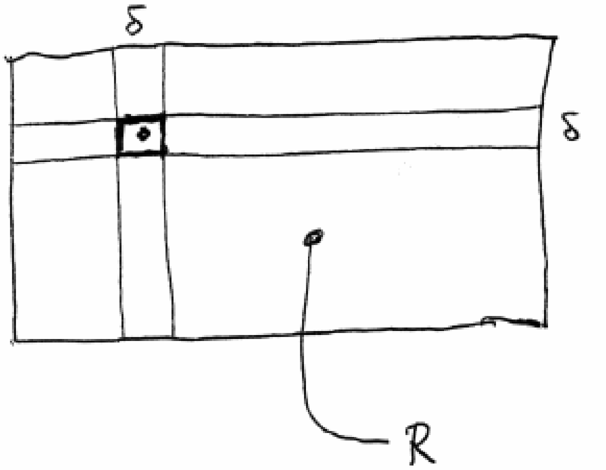
$$\oint_{\partial R} f(z)\, dz = 0 .$$

Similarly for m interior points $z_1, \ldots, z_m$.

Proof

Remember that we have just proved Cauchy – Goursat for any rectangle.

Look at



Let R_δ be the δ-sided square centered at z_1. Notice that:

$$\oint_{\partial R} f\,dz \approx \text{sum over 9 baby integrals}$$

$$= \oint_{\partial R_\delta} f\,dz \quad \text{by Cauchy - Goursat THM.}$$

Choose any $\varepsilon > 0$. By taking δ appropriately small,

$$|f(z)| < \frac{\varepsilon}{|z - z_1|}, \quad z \in \partial R_\delta .$$

KEY POINT!

Hence,

$$\left| \oint_{\partial R_\delta} f(z)\,dz \right| \leq \oint_{\partial R_\delta} |f(z)|\,|dz|$$

$$< \oint_{\partial R_\delta} \frac{\varepsilon}{|z - z_1|}\,|dz|$$

$$\left\{ \text{but } |z - z_1| \geq \tfrac{1}{2}\delta \right\}$$

$$\leq \oint_{\partial R_\delta} \frac{\varepsilon}{\frac{1}{2}\delta}\,|dz|$$

$$= \frac{\varepsilon}{\frac{1}{2}\delta}(4\delta) = 8\varepsilon .$$

Thus:

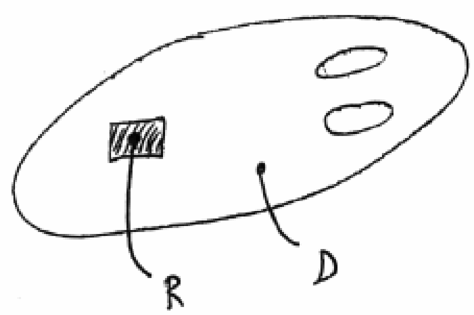
$$\left| \oint_{\partial R} f\,dz \right| \leq 8\varepsilon$$

$$\Rightarrow \oint_{\partial R} f\,dz = 0 . \quad \blacksquare$$

THM (Cauchy integral formula)

Let R be a rectangle contained in domain D. Let $g(z)$ be analytic on D. Let z_0 be any interior point of R. Then:

$$g(z_0) = \frac{1}{2\pi i} \oint_{\partial R} \frac{g(z)}{z - z_0}\, dz .$$



Proof

Let

$$f(z) = \begin{cases} \dfrac{g(z) - g(z_0)}{z - z_0}, & z \neq z_0 \\ g'(z_0), & z = z_0 \end{cases} .$$

This f works in the "TRICK" COROLLARY. One takes $z_1 \leftrightarrow z_0$, of course.

Accordingly:

$$\oint_{\partial R} \frac{g(z) - g(z_0)}{z - z_0}\, dz = 0 .$$

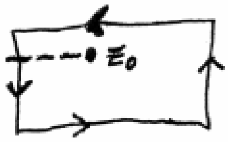
Hence:

$$\oint_{\partial R} \frac{g(z)}{z - z_0}\, dz = \oint_{\partial R} \frac{g(z_0)}{z - z_0}\, dz$$

$$= g(z_0) \oint_{\partial R} \frac{1}{z - z_0}\, dz .$$

To calculate the last integral, we recall that $\mathrm{Log}(z - z_0)$ is an anti-derivative of $(z - z_0)^{-1}$ except when $z - z_0 =$ negative. At once,

$$\oint_{\partial R} \frac{1}{z - z_0}\, dz =$$



$$= 2\pi i .$$

$$\left\{ \begin{array}{l} \Delta_{\partial R} \mathrm{Log}(z - z_0) \\ \text{STYLE} \end{array} \right\}$$

Thus:

$$\oint_{\partial R} \frac{g(z)}{z-z_0}\,dz = 2\pi i\, g(z_0)$$

and we are done. ∎

Once we have

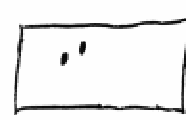
$$g(z_1) = \frac{1}{2\pi i}\oint_{\partial R} \frac{g(z)}{z-z_1}\,dz$$

for (any) $\forall z_1$ inside R, we can now start forming difference quotients!

WITH RESPECT TO z_1

Keep z_1 and z_1+h away from ∂R.

Get:



← ∂R

$$\frac{g(z_1+h)-g(z_1)}{h} = \frac{1}{2\pi i}\oint_{\partial R} g(z)\,\frac{1}{h}\left[\frac{1}{z-z_1-h} - \frac{1}{z-z_1}\right]dz$$

$$= \frac{1}{2\pi i} \oint_{\partial R} g(z) \frac{1}{(z-z_1-h)(z-z_1)} dz .$$

Write the last integral as a definite integral and ~~use~~ uniform convergence as $h \to 0$ (h complex). Get:

$$g'(z_1) = \frac{1}{2\pi i} \oint_{\partial R} \frac{g(z)}{(z-z_1)^2} dz .$$

We now repeat:

$$\frac{g'(z_1+h) - g'(z_1)}{h} = \frac{1}{2\pi i} \oint_{\partial R} \frac{g(z)}{h} \left[\frac{1}{(z-z_1-h)^2} - \frac{1}{(z-z_1)^2} \right] dz$$

{apply baby algebra}

$$= \frac{1}{2\pi i} \oint_{\partial R} \frac{g(z)}{h} \frac{h}{(z-z_1-h)(z-z_1)} \left[\frac{1}{z-z_1-h} + \frac{1}{z-z_1} \right] dz$$

$$\Downarrow$$

$$g''(z_1) = \frac{2}{2\pi i} \oint_{\partial R} \frac{g(z)}{(z-z_1)^3}\, dz .$$

To continue, recall that

$$x^m - y^m = (x-y)\left(\sum_{j=0}^{m-1} x^{m-1-j} y^j \right) .$$

By induction, we immediately deduce that

$$g^{(k)}(z_1) = \frac{k!}{2\pi i} \oint_{\partial R} \frac{g(z)}{(z-z_1)^{k+1}}\, dz$$

for all $k \geq 1$.

$\left\{ \begin{array}{l} \text{Cauchy integral} \\ \text{formula for } k^{th} \\ \text{derivatives} \end{array} \right\}$

Of course, this is a

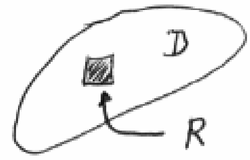
$\boxed{\text{BIG SURPRISE!}}$

Since $g^{(k+1)}(z_1)$ exists (as a nice integral) for <u>each</u> z_1 <u>inside</u> R, we conclude that $g^{(k)}(z)$ is <u>analytic</u> on the inside of R.

Since R is <u>any</u> rectangle inside D, we have thus verified that $g^{(k)}(z)$ is analytic on D.

With a change of letter, this proves the 2nd half of the BIG THM on page 6.

We have $f(z) = u(x,y) + iv(x,y)$. We must still prove u and v are C^∞ functions (in the sense of multivariable calculus) on each open rectangle R contained well inside D.



The trick is to recall Leibnitz's rule from advanced calculus.

Let $Q(t;x,y)$ and all of its partial derivatives with respect to (x,y) be continuous in (t,x,y) for

$$a \leq t \leq b, \quad \alpha_1 < x < \alpha_2, \quad \beta_1 < y < \beta_2.$$

Q complex OK!

Let

$$E(x,y) \stackrel{\text{def.}}{=} \int_a^b Q(t;x,y)\,dt \,.$$

{Think of x, y as auxiliary parameters}

Then:

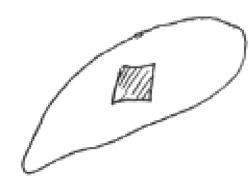
$$\frac{\partial E}{\partial x} = \int_a^b \frac{\partial Q}{\partial x}(t;x,y)\,dt\,, \quad \frac{\partial E}{\partial y} = \int_a^b \frac{\partial Q}{\partial y}(t;x,y)\,dt\,,$$

similarly for higher derivatives.

Moreover, each of these dt-integrals __is__ __continuous__ with respect to (x,y) for $\alpha_1 < x < \alpha_2$, $\beta_1 < y < \beta_2$.

On each open rectangle R, by CIF, p. 16,

$$g(x+iy) = \frac{1}{2\pi i}\oint_{\partial R} \frac{g(w)}{w-x-iy}\,dw \,.$$



Write the RHS as a sum of 4 "dt" integrals  using the parametric equation for ∂R. The __integrand__ function

$$\frac{g[w(t)]\,w'(t)}{w(t)-x-iy}$$

is a perfectly good $Q(t;x,y)$!

HENCE (by Leibnitz) $u(x,y)$ and $v(x,y)$ are C^∞ on R.