Supplement for Lecture 13

Know:

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k \,, \quad 0 < |t| < 2\pi \qquad (B_k \in \mathbb{Q})$$

$$\zeta(2k) = \frac{(-1)^{k+1}}{2} \frac{B_{2k}}{(2k)!} (2\pi)^{2k} \,, \quad k \geq 1 \quad \leftarrow \text{Euler } 1755 \; ⊛$$

E-M version 2

$$\sum_{j=0}^{N} f(j) = \frac{1}{2} f(0) + \frac{1}{2} f(N) + \int_0^N f(x)\,dx$$
$$+ \sum_{k=1}^{R} \frac{B_{2k}}{(2k)!} \left[f^{(2k-1)}(N) - f^{(2k-1)}(0) \right]$$

+ Remainder, wherein

$$\text{Remainder} = \int_0^N f^{(2R+1)}(x) \left[2(-1)^{R+1} \sum_1^{\infty} \frac{\sin(2\pi n x)}{(2\pi n)^{2R+1}} \right] dx$$

$$= -\int_0^N f^{(2R)}(x) \left[2(-1)^{R+1} \sum_1^{\infty} \frac{\cos(2\pi n x)}{(2\pi n)^{2R}} \right] dx$$

$$= \int_0^N f^{(2R+2)}(x) \left[2(-1)^{R} \sum_1^{\infty} \frac{1 - \cos 2\pi n x}{(2\pi n)^{2R+2}} \right] dx$$

(whichever is most convenient).

NOTE:

$$\frac{\tilde{B}_{2R+1}(x)}{(2R+1)!} = 2(-1)^{R+1} \sum_1^{\infty} \frac{\sin 2\pi n x}{(2\pi n)^{2R+1}} \;;$$

$$\frac{\tilde{B}_{2R}(x)}{(2R)!} = 2(-1)^{R+1} \sum_1^{\infty} \frac{\cos 2\pi n x}{(2\pi n)^{2R}} \,.$$

⊛ Hence $(-1)^{k+1} B_{2k} = |B_{2k}|$.

$B_n(x)$ = Bernoulli polynomial $\quad (0 \leqq x < 1)$

$= x^n + \cdots + B_n$

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n \qquad 0 < |t| < 2\pi$$

$\tilde{B}_n(x)$ = periodic 1 extension of $B_n(x)$

Our plan is to discuss the Remainder term on page ① more carefully, ASSUMING THAT f IS REAL.

For complex f, write $f = f_1 + if_2$ and use linearity.

What we need to do is impose various conditions and see what happens.

$\boxed{A}$

First, let's proceed quite generally.
Look at the REMAINDER TERM:

Alternate Form
$$2(-1)^R \int_0^N \left[\sum_{n=1}^{\infty} \frac{1-\cos(2\pi n x)}{(2\pi n)^{2R+2}} \right] f^{(2R+2)}(x)\, dx .$$

The bracket is definitely <u>nonnegative</u>!!!
Let $m = \min f^{(2R+2)}$, $M = \max f^{(2R+2)}$
on $0 \leqq x \leqq N$. Therefore:

$$m \int_0^N [\cdots]\, dx \leqq \int_0^N [\cdots] f^{(2R+2)} dx$$
$$\leqq M \int_0^N [\cdots]\, dx$$

$$\Downarrow$$

$$\int_0^N [\cdots] f^{(2R+2)}(x)\,dx = C\int_0^N [\cdots]\,dx$$

for a certain C in the interval $[m, M]$.
Apply the intermediate value theorem to $f^{(2R+2)}(x)$. So, we can find $\xi \in [0, N]$ such that

$$f^{(2R+2)}(\xi) = C.$$

IE

$$\text{Remainder} = 2(-1)^R \int_0^N \left[\sum_{n=1}^{\infty} \frac{1-\cos(2\pi n x)}{(2\pi n)^{2R+2}}\right] f^{(2R+2)}\,dx$$

$$= 2(-1)^R C \int_0^N \left[\sum_{n=1}^{\infty} \frac{1-\cos(2\pi n x)}{(2\pi n)^{2R+2}}\right] dx$$

$$= 2(-1)^R f^{(2R+2)}(\xi) \int_0^N \left[\sum_{n=1}^{\infty} \frac{1-\cos(2\pi n x)}{(2\pi n)^{2R+2}}\right] dx$$

$$= 2(-1)^R f^{(2R+2)}(\xi) \cdot N \sum_{n=1}^{\infty} \frac{1}{(2\pi n)^{2R+2}}$$

$$\approx 2(-1)^k f^{(2k+2)}(\xi)\cdot N \frac{1}{(2\pi)^{2k+2}}\, \zeta(2k+2)$$

use p. [redacted] 1

$$= (-1)^k f^{(2k+2)}(\xi)\cdot N \cdot \frac{|B_{2k+2}|}{(2k+2)!} .$$

Now we must review the table of Bernoulli numbers again. We recall [redacted] that

$$(-1)^k |B_{2k+2}| = B_{2k+2} .$$

See page [redacted] 1.

Accordingly:

$$\text{Remainder} = f^{(2k+2)}(\xi)\cdot N \cdot \frac{B_{2k+2}}{(2k+2)!}$$

with $0 \le \xi \le N$,

Hildebrand 200 bottom.
Jordan 261 (1)
Steffensen 133 (11)
Abramowitz–Stegun 886

$\boxed{B}$ There is a very <u>slight</u> modification that can also be verified. Sometimes useful!

$$\text{Remainder} = 2(-1)^R \int_0^N \left[\sum_{n=1}^{\infty} \frac{1-\cos(2\pi n x)}{(2\pi n)^{2R+2}} \right] f^{(2R+2)}(x)\,dx$$

$$\equiv 2(-1)^R \int_0^1 \left[\sum_{n=1}^{\infty} \frac{1-\cos(2\pi n x)}{(2\pi n)^{2R+2}} \right] g^{(2R+2)}(x)\,dx$$

where $g(x) \equiv f(x) + f(x+1) + \cdots + f(x+N-1)$

But, just like before,

$$\int_0^1 \left[\sum_{n=1}^{\infty} \frac{1-\cos(2\pi n x)}{(2\pi n)^{2R+2}} \right] g^{(2R+2)}(x)\,dx$$

$$= g^{(2R+2)}(\theta) \int_0^1 \left[\sum_{n=1}^{\infty} \frac{1-\cos(2\pi n x)}{(2\pi n)^{2R+2}} \right] dx$$

$$\{ 0 \leq \theta \leq 1 \}$$

$$= g^{(2R+2)}(\theta)\,(2\pi)^{-2R-2}\,\zeta(2R+2)$$

$$\Downarrow$$

$$\text{remainder} = 2(-1)^R g^{(2R+2)}(\theta)\,(2\pi)^{-2R-2}\,\zeta(2R+2)$$

$$= (-1)^R g^{(2R+2)}(\theta) \frac{|B_{2R+2}|}{(2R+2)!}$$

↑ p. ①

$$\text{remainder} = \frac{B_{2k+2}}{(2k+2)!}\, g^{(2k+2)}(\theta)$$

$$\text{remainder} = \frac{B_{2k+2}}{(2k+2)!}\left[\sum_{j=0}^{N-1} f^{(2k+2)}(\theta+j)\right]$$

for some $0 \leqq \theta \leqq 1$.

This form is slightly more <u>precise</u> than form A.

This form is found in:

Jordan, page 261 (2)

Abramowitz - Stegun, page 806 (23.1.30).

$\boxed{C}$ One of the most __useful__ versions (for purposes of computation) is the following.

SUPPOSE THAT $f^{(2k+2)}(x)$ DOES __NOT__ CHANGE SIGN ON $[0,N]$.

We know that:

$$\text{remainder} = 2(-1)^k \int_0^N \left[\sum_{n=1}^{\infty} \frac{1-\cos(2\pi n x)}{(2\pi n)^{2k+2}} \right] f^{(2k+2)}(x)\,dx.$$

Let's suppose $f^{(2k+2)}(x) \geq 0$; similarly for the other case. Then, clearly,

$$|\text{remainder}| = 2 \int_0^N \left[\sum_{n=1}^{\infty} \frac{1-\cos(2\pi n x)}{(2\pi n)^{2k+2}} \right] f^{(2k+2)}(x)\,dx$$

$$|\text{remainder}| \leq 2 \int_0^N \sum_{n=1}^{\infty} \frac{2}{(2\pi n)^{2k+2}} f^{(2k+2)}(x)\,dx$$

$$|\text{remainder}| \leq 2 \cdot \frac{2}{(2\pi)^{2k+2}} \zeta(2k+2) \int_0^N f^{(2k+2)}(x)\,dx$$

page (1)

$$|\text{remainder}| \leq 2 \cdot \frac{2}{(2\pi)^{2k+2}} \left[2^{2k+1} \pi^{2k+2} \frac{|B_{2k+2}|}{(2k+2)!} \right] \cdot \int_0^N f^{(2k+2)}(x)\,dx$$

$$|\text{remainder}| \leq 2 \frac{|B_{2k+2}|}{(2k+2)!} \left[f^{(2k+1)}(N) - f^{(2k+1)}(0) \right]$$

NOTE THE EXTRA 2 !

In other words, FAMOUS RESULT

$$|\text{remainder}| \leq 2 \left| \frac{B_{2k+2}}{(2k+2)!} \langle f^{(2k+1)} \rangle \right|$$

when $f^{(2k+2)}(x)$ does not change sign. The RHS is **twice** the absolute value of the "next term up" in the expansion in page 1.

[Hildebrand 201
Steffensen 133
Abramowitz–Stegun 886]

attached at end of these notes

$\boxed{D}$ Suppose next that $f^{(2R+2)}(x)$ and $f^{(2R+4)}(x)$ both have constant sign on $[0,N]$, and that this sign is THE SAME for both functions.

Apply version $\boxed{A}$ first [redacted] (1) with R, then with $R+1$. In view of page ✓, we get:

$$\sum_{j=0}^{N} f(j) = \int_0^N f(x)\,dx + \frac{1}{2}\left[f(0)+f(N)\right] + \sum_{k=1}^{R} \frac{B_{2k}}{(2k)!}\langle f^{(2k-1)}\rangle + N f^{(2R+2)}(\xi)\,\frac{B_{2R+2}}{(2R+2)!}$$

(5) [redacted] above

$$\sum_{j=0}^{N} f(j) = \int_0^N f(x)\,dx + \frac{1}{2}\left[f(0)+f(N)\right] + \sum_{k=1}^{R} \frac{B_{2k}}{(2k)!}\langle f^{(2k-1)}\rangle + \frac{B_{2R+2}}{(2R+2)!}\langle f^{(2R+1)}\rangle + N f^{(2R+4)}(\eta)\,\frac{B_{2R+4}}{(2R+4)!}$$

← (5) again

$$\Downarrow$$

So,

$$N f^{(2\ell+2)}(\xi)\,\frac{B_{2\ell+2}}{(2\ell+2)!} = \frac{B_{2\ell+2}}{(2\ell+2)!}\,\langle f^{(2\ell+1)}\rangle + N f^{(2\ell+4)}(\eta)\,\frac{B_{2\ell+4}}{(2\ell+4)!}$$

We can now do something very cheap!!

Remember that $B_{2\ell+2}\,B_{2\ell+4} < 0$.

$\boxed{\text{page } 1}$

Write the foregoing equality as

$$A = B + C .$$

By hypothesis, $AC < 0$. Hence

$$A^2 = AB + AC$$

$$\Downarrow$$

$$0 \leq A^2 < AB .$$

But, then;

$$0 \leqq \frac{A^2}{AB} < 1$$

or

$$0 \leqq \frac{A}{B} < 1 .$$

We can write

$$A = \lambda B ,$$

with $0 \leqq \lambda < 1$.

In other words:

$$\boxed{[\text{remainder}]_{\text{with } R} = \lambda \frac{B_{2R+2}}{(2R+2)!} \langle f^{(2R+1)} \rangle}$$

with $0 \leqq \lambda < 1$. Here the remainder has the ~~same~~ sign as the "next term up" and is numerically smaller in absolute value.

Just like the famous alternating series test

This version agrees with

Jordan page 261(4), 255(7)
Steffensen page 133 (middle).

Note that both Hildebrand and Abramowitz–Stegun <u>screw</u> <u>it</u> <u>up</u> by forgetting the condition of SAME SIGN; see

Hildebrand 201 (para 1)
Abramowitz ~ Stegun 886.

derivative
(see p. 10)

Abramowitz - Stegun

$f_j \equiv f(x_j)$

(14)

Simpson's Rule

25.4.5

$$\int_{x_0}^{x_2} f(x)dx=\frac{h}{3}[f_0+4f_1+f_2]$$
$$+\frac{1}{6}\int_{x_0}^{x_1}(x_0-t)^2(x_1-t)f^{(3)}(t)dt$$
$$+\frac{1}{6}\int_{x_1}^{x_2}(x_2-t)^2(x_1-t)f^{(3)}(t)dt$$
$$=\frac{h}{3}[f_0+4f_1+f_2]-\frac{h^5}{90}f^{(4)}(\xi)$$

Extended Simpson's Rule

25.4.6

$$\int_{x_0}^{x_{2n}} f(x)dx=\frac{h}{3}[f_0+4(f_1+f_3+\ldots+f_{2n-1})$$
$$+2(f_2+f_4+\ldots+f_{2n-2})+f_{2n}]-\frac{nh^5}{90}f^{(4)}(\xi)$$

Euler-Maclaurin Summation Formula

25.4.7 eg $h=1$!

$$\int_{x_0}^{x_n} f(x)dx=h\left[\frac{f_0}{2}+f_1+f_2+\ldots+f_{n-1}+\frac{f_n}{2}\right]$$
$$-\frac{B_2}{2!}h^2(f'_n-f'_0)-\ldots-\frac{B_{2k}h^{2k}}{(2k)!}[f_n^{(2k-1)}-f_0^{(2k-1)}]+R_{2k}$$
$$R_{2k}=\frac{\theta n B_{2k+2}h^{2k+3}}{(2k+2)!}\max_{x_0\le x\le x_n}|f^{(2k+2)}(x)|,\qquad(-1\le\theta\le1)$$

(For B_{2k}, Bernoulli numbers, see chapter 23.)

X If $f^{(2k+2)}(x)$ and $f^{(2k+4)}(x)$ do not change sign for $x_0<x<x_n$ then $|R_{2k}|$ is less than the first neglected term. If $f^{(2k+2)}(x)$ does not change sign for $x_0<x<x_n$, $|R_{2k}|$ is less than twice the first neglected term.

Lagrange Formula

25.4.8

$$\int_a^b f(x)dx=\sum_{i=0}^{n}(L_i^{(n)}(b)-L_i^{(n)}(a))f_i+R_n$$

(See **25.2.1.**)

25.4.9

$$L_i^{(n)}(x)=\frac{1}{\pi'_n(x_i)}\int_{x_0}^{x}\frac{\pi_n(t)}{t-x_i}dt=\int_{x_0}^{x}l_i(t)dt$$

25.4.10 $$R_n=\frac{1}{(n+1)!}\int_a^b\pi_n(x)f^{(n+1)}(\xi(x))dx$$

Equally Spaced Abscissas

25.4.11

$$\int_{x_0}^{x_n} f(x)dx=\frac{1}{h^n}\sum_{i=0}^{n}f_i\frac{(-1)^{n-i}}{i!(n-i)!}\int_{x_0}^{x_n}\frac{\pi_n(x)}{x-x_i}dx+R_n$$

25.4.12 $$\int_{x_m}^{x_{m+1}}f(x)dx=h\sum_{i=-[\frac{n-1}{2}]}^{[\frac{n}{2}]}A_i(m)f_i+R_n \quad *$$

(See **Table 25.3** for $A_i(m)$.)

Newton-Cotes Formulas (Closed Type)

(For Trapezoidal and Simpson's Rules see **25.4.1–25.4.6.**)

25.4.13 **(Simpson's $\frac{3}{8}$ rule)**

$$\int_{x_0}^{x_3}f(x)dx=\frac{3h}{8}(f_0+3f_1+3f_2+f_3)-\frac{3f^{(4)}(\xi)h^5}{80}$$

25.4.14 **(Bode's rule)**

$$\int_{x_0}^{x_4}f(x)dx=\frac{2h}{45}(7f_0+32f_1+12f_2+32f_3+7f_4)-\frac{8f^{(6)}(\xi)h^7}{945}$$

25.4.15

$$\int_{x_0}^{x_5}f(x)dx=\frac{5h}{288}(19f_0+75f_1+50f_2+50f_3+75f_4+19f_5)-\frac{275f^{(6)}(\xi)h^7}{12096}$$

25.4.16

$$\int_{x_0}^{x_6}f(x)dx=\frac{h}{140}(41f_0+216f_1+27f_2+272f_3+27f_4+216f_5+41f_6)-\frac{9f^{(8)}(\xi)h^9}{1400}$$

25.4.17

$$\int_{x_0}^{x_7}f(x)dx=\frac{7h}{17280}(751f_0+3577f_1+1323f_2+2989f_3+2989f_4+1323f_5+3577f_6+751f_7)-\frac{8183f^{(8)}(\xi)h^9}{518400}$$

25.4.18

$$\int_{x_0}^{x_8}f(x)dx=\frac{4h}{14175}(989f_0+5888f_1-928f_2+10496f_3-4540f_4+10496f_5-928f_6+5888f_7+989f_8)-\frac{2368}{467775}f^{(10)}(\xi)h^{11}$$

25.4.19

$$\int_{x_0}^{x_9}f(x)dx=\frac{9h}{89600}\{2857(f_0+f_9)+15741(f_1+f_8)+1080(f_2+f_7)+19344(f_3+f_6)+5778(f_4+f_5)\}-\frac{173}{14620}f^{(10)}(\xi)h^{11}$$

*See page II.

X want SAME sign in both !!

Typical
<u>Problems</u>

Set up calculation of Euler constant $\gamma \equiv C$.

<u>Solution</u> (sketch)

$f(x) = (1+x)^{-1}$ in E-M, version 2. Page ① [redacted]

$f^{(j)}(x) = (-1)^j j!\,(1+x)^{-j-1}$ (yes)

$(R \geq 1)$

So:

$$1+\frac{1}{2}+\cdots+\frac{1}{N+1} = \frac{1}{2}+\frac{1}{2(N+1)} + \ln(N+1)$$

$$+ \sum_{k=1}^{R} \frac{B_{2k}}{(2k)!}(-1)\left[(2k-1)!\,(1+N)^{-2k} - (2k-1)!\,(1)\right]$$

$$+ \int_0^N \underbrace{(-1)(2R+1)!\,(1+x)^{-2R-2}}_{f^{(2R+1)}} \left[2(-1)^{R+1}\sum_1^\infty \frac{\sin(2\pi n x)}{(2\pi n)^{2R+1}}\right] dx.$$

Let $N \to \infty$. Clearly get:

$$\gamma = \frac{1}{2} + 0 + \sum_{k=1}^{R}\frac{B_{2k}}{2k}(1) + \int_0^\infty \underbrace{(-1)(2R+1)!\,(1+x)^{-2R-2}}\,[*]\,dx.$$

Now subtract <u>this</u> from original equation. Get:

$$1+\frac{1}{2}+\cdots+\frac{1}{N+1} - \ln(N+1) - \gamma = \frac{1}{2(N+1)} - \sum_{k=1}^{R}\frac{B_{2k}}{2k}(1+N)^{-2k}$$

$$- \int_N^\infty \underbrace{(-1)(2R+1)!\,(1+x)^{-2R-2}}\,[*]\,dx.$$

How big is remainder term?? Imitate p. ③ [redacted] line 6 to get

$$(-1)\int_N^\infty f^{(2R+2)}(x)\left[2(-1)^R \sum_1^\infty \frac{1-\cos 2\pi n x}{(2\pi n)^{2R+2}}\right]dx$$

$$= \int_N^\infty f^{(2R+2)}(x)\left[2(-1)^{R+1}\sum_1^\infty \frac{1-\cos 2\pi n x}{(2\pi n)^{2R+2}}\right]dx.$$

Can now mimic item $\boxed{C}$ on page 8 [redacted]. Or $\boxed{D}$.

[Must fill in all details, of course.]