Lecture 17 Fri, 11 Nov

We want to continue a little further with our study of the Riemann zeta function $\zeta(z)$ before we change topic.

We saw on Tues, Lec 16, using E-M, that $\zeta(z)$ is analytic on $\mathbb{C} \sim \{1\}$, simple pole at $z=1$ residue 1.

In fact, Lec 16, p. 24

$$|\zeta(x+iy)| \leq C \frac{|y|^{2R+1}}{\varepsilon} \qquad \begin{array}{c} -2R+\varepsilon \leq x \leq 2 \\ |y| \geq 1000 \end{array}$$

for any $R \geq 0$. For $x > 1$, we always have (for ANY y)

$$|\zeta(z)| \leq \sum_{n=1}^{\infty} \frac{1}{n^x} \leq 1 + \int_1^{\infty} t^{-x}\,dt = 1 + \frac{1}{x-1}.$$

Riemann proved that

$$\xi(z) = \xi(1-z) \qquad (\star\star)$$

for the modified function

$$\xi(z) = \pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z).$$

Let's assume (**) for now. Thus,

$$\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{\frac{z-1}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z).$$

Exploit the relation

$$|\Gamma(x+iy)| \sim \sqrt{2\pi}\,|y|^{x-\frac{1}{2}}e^{-\frac{\pi}{2}|y|}$$

which holds uniformly for $x_1 \le x \le x_2$, $|y| \to \infty$.
(This was a HW problem done via Stirling.)

Get:

$$|\zeta(z)| = \left|\frac{\pi^{\frac{z-1}{2}}\Gamma\left(\frac{1-z}{2}\right)}{\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)}\right| |\zeta(1-z)|$$

$$|\zeta(x+iy)| = \pi^{x-\frac{1}{2}}\left|\frac{\Gamma\left(\frac{1-x}{2}+i\frac{y}{2}\right)}{\Gamma\left(\frac{x}{2}+i\frac{y}{2}\right)}\right| |\zeta(1-x+iy)|$$

asymptotic to $\dfrac{|y/2|^{-x/2}}{|y/2|^{\frac{x}{2}-\frac{1}{2}}} = \left|\dfrac{y}{2}\right|^{\frac{1}{2}-x}$

uniformly $a_1 \le x \le a_2$

$$\Downarrow$$

$$|\zeta(x+iy)| \sim \left|\frac{y}{2\pi}\right|^{\frac{1}{2}-x} |\zeta(1-x+iy)|$$

uniformly for $a_1 \leqq x \leqq a_2$ as $|y| \to \infty$. IE

$$\boxed{|\zeta(x+iy)| = \left|\frac{y}{2\pi}\right|^{\frac{1}{2}-x} |\zeta(1-x+iy)| \left[1+o(1)\right] \qquad (\ast\ast\ast)}.$$

For each $x \in \mathbb{R}$, let's define

$$\mu(x) = \limsup_{|y| \to \infty} \frac{\ln|\zeta(x+iy)|}{\ln|y|}.$$

Equivalently,

$$\mu(x) = \inf\left\{\beta : |\zeta(x+iy)| = O(y^{\beta}) \text{ for } |y| \geqq 1000\right\}.$$

By p. 1 middle, $\mu(x) < +\infty$.

By $(\ast\ast\ast)$, clearly

$$\mu(x) = \frac{1}{2} - x + \mu(1-x)$$

for all $x \in \mathbb{R}$.

FACT

$$\mu(x) = 0 \quad \text{for } x \geq 2.$$

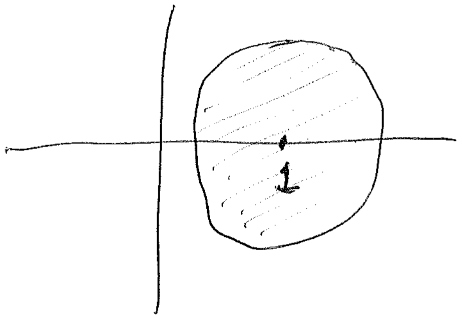
Proof

This is very easy.

$$|\zeta(z) - 1| = \left| \sum_{n=2}^{\infty} n^{-z} \right| \leq \sum_{n=2}^{\infty} n^{-x}$$

$$|\zeta(z) - 1| \leq \sum_{n=2}^{\infty} n^{-2} < 2^{-2} + \int_2^{\infty} t^{-2}\,dt$$

$$|\zeta(z) - 1| < \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$



[$w = \zeta(z)$ lies in this open disk of radius $\frac{3}{4}$]

$$\frac{1}{4} < |\zeta(z)| < \frac{7}{4}$$

$$\ln\left(\frac{1}{4}\right) < \ln|\zeta(z)| < \ln\left(\frac{7}{4}\right)$$

$\mu(x) = 0$ is immediate. ∎

FACT

We have $\mu(x) \geqq 0$ for all $x \in \mathbb{R}$.

Proof

Suppose that $\mu(x_0) < 0$ for some x_0.
Clearly $x_0 < 2$. See 4.

Recall p. 1 middle with a giant R.

$$\boxed{|y|^{2R+1} \lessapprox e^{G|y|}}$$

Apply the convexity corollary of Lec 16 to the half-strip $[x_0, 3] \times [1000, +\infty)$. See Lec 16, p. 14. Use A = negative (slightly bigger than $\mu(x_0)$); $B = 0$. Get

$$\mu(x) \lessapprox A\left(\frac{3-x}{3-x_0}\right) + 0\left(\frac{x-x_0}{3-x_0}\right) = A\left(\frac{3-x}{3-x_0}\right)$$

for $x_0 \leqq x \leqq 3$. Since A is negative, we deduce that

$$\mu(x) \lessapprox A\frac{(1/2)}{3-x_0} < 0$$

for $x_0 \leqq x \leqq 2.5$. This contradicts 4. ∎

FACT

$$\mu(x) = 0 \quad \text{for} \quad x \geq 1.$$

Proof

By 5, we already know $\mu(x) \geq 0$.

For $x > 1$, use page 1 line 12. This gives $\mu(x) \leq 0$. Hence $\mu(x) = 0$. OK

For $x = 1$, recall that $\psi(1+iy) = O(\ln y)$ for $y \geq 1000$. See Lec 16, p. 24. This gives $\mu(1) \leq 0$. Hence $\mu(1) = 0$. OK

∎

By p. 3 (bottom), ie

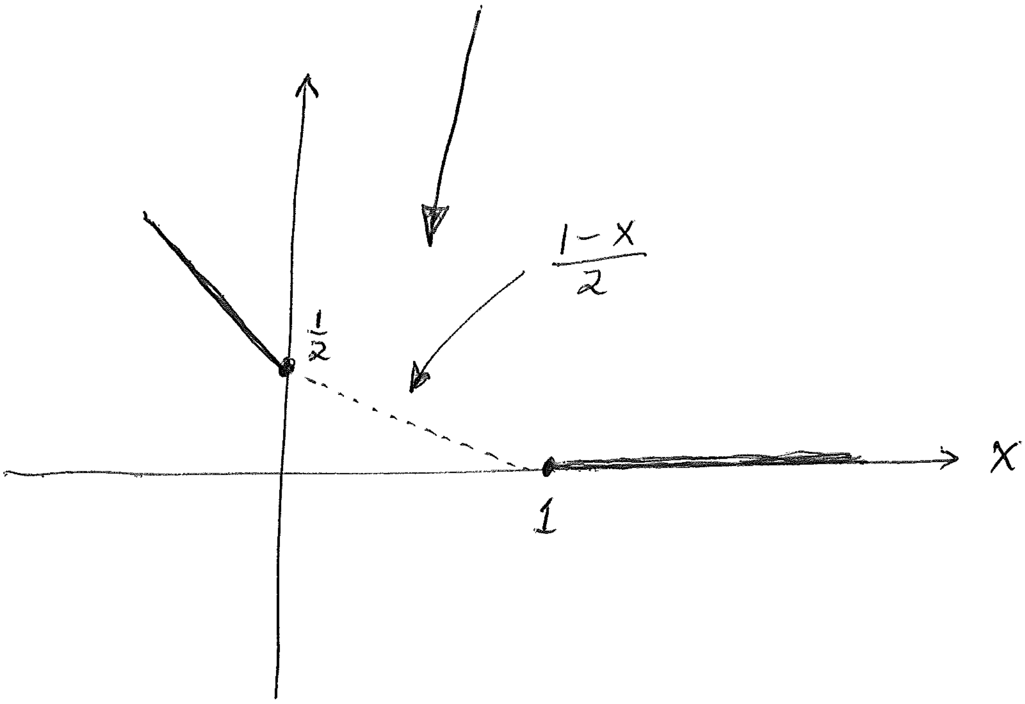
$$\mu(x) = \tfrac{1}{2} - x + \mu(1-x),$$

we thus get

$$\mu(x) = \begin{cases} \tfrac{1}{2} - x, & x \leq 0 \\ 0, & x \geq 1 \end{cases}.$$

By Lec 16 (the convexity corollary, p. 14) we know that the graph of $\mu(x)$ must be <u>convex</u>.

Because of this, $\mu(x)$ must lie on or below the dotted line for $0 \leq x \leq 1$.



IN PARTICULAR,

$$\boxed{\mu\left(\tfrac{1}{2}\right) \leq \tfrac{1}{4}}$$

i.e.

$$|\zeta(\tfrac{1}{2}+iy)| = O(y^{\frac{1}{4}+\delta})$$

for every $\delta > 0$.

> We got this "for free" from (★★) and the convexity corollary.

At present, no one knows the exact value of $\mu(\frac{1}{2})$.

It is conjectured to be $\underline{\underline{0}}$. This is Lindelöf's conjecture. from about 1900

$\boxed{= \text{A FAMOUS } \underline{\text{OPEN}} \text{ PROBLEM.}}$

Notice that $\mu(\frac{1}{2}) = 0 \Rightarrow$

$$\mu(\xi) = 0 \quad \text{for } \tfrac{1}{2} < \xi < 1 \quad \left\{ \begin{array}{l} \text{use convexity} \\ \text{and } \mu(2) = 0 \end{array} \right\}$$

$$\Downarrow$$

$$\mu(\eta) = \tfrac{1}{2} - \eta \quad \text{for } 0 < \eta < \tfrac{1}{2}$$

$$\left\{ \text{since } \mu(x) = \tfrac{1}{2} - x + \mu(1-x) \right\}$$

$$\Downarrow$$

$$\mu(x) = \left\{ \begin{array}{ll} 0, & x \geq \frac{1}{2} \\ \frac{1}{2} - x, & x \leq \frac{1}{2} \end{array} \right\}.$$

Nice!

In line with p. (7), we can improve things very slightly. The method is interesting.

FACT

$$|\zeta(\tfrac{1}{2}+iy)| = O(y^{\frac{1}{4}} \ln y), \quad y \geqq 1000.$$

Proof

Take $0 < \varepsilon < \frac{1}{10}$.

Recall that

$$|\zeta(1+\varepsilon+iy)| \leqq \zeta(1+\varepsilon) \leqq 1+\frac{1}{\varepsilon} \leqq \frac{2}{\varepsilon};$$

see page (1).

Apply (★★★) on page (3), which holds uniformly for $-1 \leqq x \leqq 2$. Also see (1) line 9.

Get

$$|\zeta(-\varepsilon+iy)| = \left|\frac{y}{2\pi}\right|^{\frac{1}{2}+\varepsilon} |\zeta(1+\varepsilon+iy)| \left[1+o(1)\right]$$

for large y

$$\Downarrow$$

$$\left\{\begin{array}{l} |\zeta(-\varepsilon+iy)| \leqq C y^{\frac{1}{2}+\varepsilon}\left(\frac{2}{\varepsilon}\right) \\ |\zeta(1+\varepsilon+iy)| \leqq \frac{2}{\varepsilon} \end{array} \quad \text{for all large } y \right\}.$$

We propose to apply the convexity corollary to the function $\varepsilon\,\mathcal{I}(z)$ on the half-strip $[-\varepsilon, 1+\varepsilon] \times [y_0, +\infty)$, where y_0 = very large.

If one is worried, one can FIRST make a change of variable

$$z = -\varepsilon + (1+2\varepsilon)\mathfrak{z} \qquad (\mathfrak{z} = u + iv)$$

and keep $\mathfrak{z} \in [0,1] \times [y_0^*, \infty)$. Here

$$x = -\varepsilon + (1+2\varepsilon)u$$
$$y = (1+2\varepsilon)v \quad .$$

Compare Lec 16, p. 22 bottom.

In any event, we then get: $\boxed{\text{Lec 16, p. 14}}$

$$|g| = O(1)\, y^{(\frac{1}{2}+\varepsilon)(1-u) + 0u}$$
$$= O(1)\, y^{\frac{1+\varepsilon-x}{1+2\varepsilon}(\frac{1}{2}+\varepsilon)+0}$$
$$= O(1)\, y^{\frac{1}{2}(1+\varepsilon-x)}$$

$$\boxed{\begin{array}{c} -\varepsilon \lessapprox x \lessapprox 1+\varepsilon \\ y \text{ big} \end{array}}$$

$$\Downarrow$$

$$|\zeta(x+iy)| = \frac{O(1)}{\varepsilon}\, y^{\frac{1}{2}(1+\varepsilon-x)}$$

for $-\varepsilon \lessapprox x \lessapprox 1+\varepsilon$, $y \gtrapprox y_0$ (big).

We put $x = \frac{1}{2}$ and take $\varepsilon = \frac{1}{10 \ln y}$. Get:

$$\frac{1}{\varepsilon}\, y^{\frac{\varepsilon}{2}} = (\text{const}) \ln y$$

$$|\zeta(\tfrac{1}{2}+iy)| = O(1)\, y^{\frac{1}{4}} \ln y. \quad \blacksquare$$

In the above, we clearly also get

$$\zeta(x_0+iy) = O(1)\, y^{\frac{1-x_0}{2}} \ln y$$

for every $x_0 \in [0,1]$.

Nice!

When $\operatorname{Re}(z)>1$, Euler found a very important alternate expression for $\zeta(z)$.

<u>Theorem</u> (Euler.)

For $\operatorname{Re}(z)>1$,

$$\zeta(z) = \prod_p \frac{1}{1-p^{-z}}$$

where p ranges over the prime numbers.
The RHS is uniformly and abs conv on compacta.

<u>Proof</u>

Recall that every $n \geqq 2$ can be factored uniquely into a product $p_1^{e_1} \cdots p_r^{e_r}$ of prime powers.

Keep $\operatorname{Re}(z) \geqq 1+\varepsilon$. Note $|p^{-z}| = p^{-x} \leqq 2^{-x} \leqq \frac{1}{2}$.

$$\frac{1}{1-p^{-z}} = 1 + p^{-z} + p^{-2z} + \cdots$$

$$|p^{-z} + p^{-2z} + \cdots| \leqq p^{-x} + p^{-2x} + \cdots = \frac{p^{-x}}{1-p^{-x}} = \frac{1}{p^x - 1}$$

$$\leqq \frac{1}{2^{1+\varepsilon}-1} < 1$$

Therefore, by Weierstrass M-test for products,

$$\prod_p \frac{1}{1-p^{-z}} \quad \text{conv unif + abs on } \{x \geq 1+\varepsilon\}.$$

Notice that

$$\prod_{p \leq N} \frac{1}{1-p^{-z}} = \prod_{p \leq N} \left\{ 1 + p^{-z} + (p^2)^{-z} + \cdots \right\}$$

$$= \sum_{n \in \mathcal{C}_N} \frac{1}{n^z}$$

where

$$\mathcal{C}_N = \{ n \geq 1 : \text{no prime factor of } n \text{ exceeds } N \}.$$

Clearly

$$\mathcal{C}_N = \{1, 2, \ldots, N\} \cup \mathcal{D}_N$$

for some set $\mathcal{D}_N \subseteq \{N+1, \ldots\}$. Thus,

$$\prod_{p \leq N} \frac{1}{1-p^{-z}} = \sum_{n=1}^{N} n^{-z} + \sum_{n \in \mathcal{D}_N} n^{-z}.$$

But,

$$\left| \sum_{\mathcal{D}_N} n^{-z} \right| \leq \sum_{N+1}^{\infty} n^{-x}.$$

Let $N \to \infty$ to get

$$\prod_p \frac{1}{1-p^{-z}} = \zeta(z) + 0 .$$ ■

<u>Corollary</u> (of the proof).

$\zeta(z) \neq 0$ on $\{Re(z) > 1\}$; in fact,

$$\log \zeta(z) = \sum_p \left\{ p^{-z} + \tfrac{1}{2} p^{-2z} + \tfrac{1}{3} p^{-3z} + \cdots \right\} .$$

The RHS is unif + abs conv on compacta.

<u>Proof</u>

$$\text{Log} \frac{1}{1-w} = w + \frac{w^2}{2} + \frac{w^3}{3} + \cdots \quad \text{with} \begin{Bmatrix} \text{abs + unif conv} \\ \text{on } |w| \leq 1-\delta \end{Bmatrix} .$$

On page 12, we $\overset{\text{(can)}}{\text{now}}$ simply plug in $w = p^{-z}$ for each p and remember that

$$|p^{-z}| = p^{-x} \leq 2^{-1-\varepsilon} \leq \frac{1}{2} \quad \text{for } Re(z) \geq 1+\varepsilon .$$

Recall, too, our old theorems from Lec 6 and 7 about

$$\prod_{k=1}^{\infty} (1+b_k(z)) \quad \text{versus} \quad \sum_{k=1}^{\infty} \text{Log}(1+b_k(z)) . !$$ ■

Define

$$\Lambda(n) = \begin{Bmatrix} \ln p, & n = p^r, \ r \geq 1 \\ 0, & n = \text{otherwise} \end{Bmatrix}.$$

Recall that:

$$\frac{d}{dz}(A^{-z}) = (-\ln A) A^{-z}.$$

By (14) and Weierstrass conv thm,

$$\frac{\zeta'(z)}{\zeta(z)} = \sum_p \left\{ -\frac{\ln p}{p^z} - \frac{\ln p}{p^{2z}} - \dots \right\}$$

$$= -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z}. \qquad \boxed{\mathrm{Re}(z) > 1}$$

The <u>product</u> formula for $\zeta(z)$ and the <u>sum</u> formulae for $\log \zeta(z)$, $\frac{\zeta'(z)}{\zeta(z)}$ are very basic in what is called analytic number theory.

Theorem

The function $\xi_0(z) \equiv z(z-1)\pi^{-\frac{z}{2}}\Gamma(\frac{z}{2})\zeta(z)$ is entire. It satisfies $\xi_0(1)=1$ and

$$\xi_0(z) = \xi_0(1-z).$$

The order of ξ_0 is 1; the type is $+\infty$.

The function ξ_0 is nonzero on

$$\{Re(z)>1\} \cup \{Re(z)<0\}.$$

But, in the strip $\{0 \leqq Re(z) \leqq 1\}$, it has infinitely many zeros. None of these zeros lie on the line segment $\{0 \leqq x \leqq 1, y=0\}$.

Proof

$\xi_0(z) = z(z-1)\xi(z)$. By ① (✱✱), $\xi_0(z)=\xi_0(1-z)$.

Clearly $\xi_0(z)$ is analytic on $Re(z)>0$ except possibly at $z=1$. By letting $z \to 1$, we see $\xi_0(1)=1$ since $\Gamma(\frac{1}{2})=\sqrt{\pi}$.

$$\left\{ \Gamma(w)\Gamma(1-w) = \frac{\pi}{\sin \pi w}, \quad w=\frac{1}{2} \Rightarrow \Gamma(\tfrac{1}{2})=\sqrt{\pi} \right\}$$

It follows that ξ_0 is entire.

To study the size of $|\xi_0(z)|$, it suffices to look at $Re(z) \geqq \frac{1}{2}$. We divide into 2 cases:

$$\tfrac{1}{2} \leqq Re(z) \leqq 2 \quad \text{AND} \quad Re(z) > 2.$$

In case #1, we use 1 line 9 and 2 line 4. Clearly $|\xi_0(x+iy)| \rightrightarrows 0$ as $|y| \to \infty$.

For case #2, we use Stirling for $\Gamma(\frac{z}{2})$ and page 4 middle.

$$\left\{ \begin{array}{l} \log \Gamma(w) = (w-\frac{1}{2}) \operatorname{Log} w - w + \frac{1}{2}\ln(2\pi) \\ \qquad\qquad + O(\frac{1}{w}) \\ w = \frac{z}{2} \quad \text{etc} \end{array} \right\}$$

Clearly $\ln|\xi_0(z)|$ remains under

$$c_1 R \ln R + c_2 R + c_3$$

for $|z| \leqq R$ (R big), $2 < \operatorname{Re}(z) < \infty$.

This shows that $\xi_0(z)$ has order <u>at most</u> 1.

By specializing to the case $z = x > 2$, and recalling that $\zeta(x) \to 1$ as $x \to \infty$, and using Stirling for $\Gamma(\frac{x}{2}) > 0$, we see that the order is exactly 1 and that the type is $+\infty$ (due to the term $x \ln x$).

By 14, clearly $\xi_0(z) \neq 0$ on $\{\operatorname{Re}(z) > 1\}$. $\boxed{\Gamma \neq 0}$
Since $\xi_0(z) = \xi_0(1-z)$, get likewise on $\{\operatorname{Re}(z) < 0\}$.

Any zeros of ξ_0 therefore lie in the strip
$\{0 \leq \mathrm{Re}(z) \leq 1\}$. OK

Let $f(z) = \xi_0(\frac{1}{2}+z)$. This fcn has order 1, type ∞, and satisfies $f(z) = f(-z)$.

The fcn $g(z) = f(\sqrt{z})$ is therefore a well-defined entire fcn of order $\frac{1}{2}$ and type ∞.

Recall HFT in Lec 9. In particular, see the corollary to HFT on page 19. We conclude that g has infinitely many zeros! Hence so do f and $\xi_0(z)$.

It remains to show $\xi_0(x) \neq 0$ for $0 \leq x \leq 1$.
Since $\xi_0(1) = \xi_0(0) = 1$, wlog $0 < x < 1$.

The summation

$$F(z) \equiv \sum_{k=1}^{\infty} \left((2k-1)^{-z} - (2k)^{-z} \right)$$

is unif conv on compact subsets of $\mathrm{Re}(z) > 0$.
Indeed, simply write

$$(2k-1)^{-z} = (2k)^{-z}\left(1-\frac{1}{2k}\right)^{-z}$$

and use the binomial series for $(1+t)^{-z}$ on $|t|<1$. The original k-summation is thus

$$\sum_{k=1}^{\infty} O(1)\,(2k)^{-z}\,\frac{1}{k} \qquad \text{[on compacta]}$$

and the Weierstrass M-test applies. The function F is thus nicely <u>analytic</u> on $\{\mathrm{Re}(z)>0\}$. Notice that $F(x)>0$ for $x>0$.

For $\mathrm{Re}(z)>1$, however, observe that

$$(1-2^{1-z})\,\zeta(z) = \sum_{n=1}^{\infty} n^{-z} - 2\sum_{m=1}^{\infty}(2m)^{-z}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{z}}$$

$$= \sum_{k=1}^{\infty}\left((2k-1)^{-z}-(2k)^{-z}\right) = F(z).$$

By analyticity, we conclude that

$$F(z) = (1-2^{1-z})\,\zeta(z) \quad \text{for } \underline{\text{all}}\ \mathrm{Re}(z)>0.$$

Since $F(x)>0$ for $0<x<1$, we discover that $\zeta(x)<0$ for $0<x<1$. Hence $\xi_0(x)\neq 0$.

$\blacksquare$

Remarks.

Riemann conjectured that all zeros of $\xi_0(z)$ lie on the line $\{Re(z) = \frac{1}{2}\}$. This is the famous RIEMANN HYPOTHESIS (unproved since 1859).

Notice that $\xi_0(1+2l) > 0$, $l \geq 1$. Hence $\xi_0(-2l) > 0$. Since $\Gamma(\frac{z}{2})$ has a simple pole at $z=-2l$, the function $\zeta(z)$ must have a simple zero at $z=-2l$. This justifies Lec 15, p. 20 bottom.

From the fact that $\xi_0(x) \neq 0$ for $x > 0$, we conclude that $\xi_0(x) > 0$ for all $x \in \mathbb{R}$. This fact (and knowledge of the poles of Γ) enables one to deduce that $\zeta(x) \neq 0$ along the real axis except at $\{-2, -4, \dots\}$.

The zeros of $\xi_0(z)$ are thus synonymous with the nonreal zeros of $\zeta(z)$.

It is not too hard (via a trick!) to show that all zeros of ξ_0 satisfy $0 < Re(z) < 1$.

As a last remark, we mention that it can be shown [too complicated to do here] that

truth of Riemann Hypothesis $\Longrightarrow$ truth of Lindelöf's conjecture.

Currently, the best that is known is:

(a) over 40% of the zeros of $\xi_0(z)$ lie on $\operatorname{Re}(z) = \frac{1}{2}$;

(b) $\mu(\frac{1}{2}) \leqq \frac{1}{6} - \lambda$, where λ is a tiny fraction like $\frac{1}{200}$.

Proof of either RH or Lindelöf's Conj would make you famous.

Check "google.com", ETC.

Let E be a subset of, say, a metric space (X, d). We say E is <u>sequentially compact</u> if and only if

for every sequence $\{e_n\}_{n=1}^{\infty} \subseteq E$,
we can find a subsequence e_{n_j}
with
$$n_1 < n_2 < n_3 < \cdots$$
such that $\{e_{n_j}\}$ converges to a point <u>of E</u>.

By the Bolzano–Weierstrass theorem of elementary real (or mathematical) analysis,

$E \subseteq \mathbb{R}^N$ is sequentially compact if and only if E is closed + bounded.

We want to study a kind of sequential compactness in a setting where X is a space of <u>functions</u> [not "points"].

By the way, the "official" Bolzano-Weierstrass thm states that every bounded sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ has a subsequence $\{a_{n_j}\}$ that converges to some <u>real number</u> c.

B-W for $\mathbb{R}^N$ follows from B-W for $\mathbb{R}^1$.

I have stressed that <u>analytic functions</u> are very special objects. They have many properties other types of functions do not have.

Analytic functions also satisfy important <u>structural</u> properties; eg

expandability into power series
integral representations
etc etc.

The following nice theorem picks up on this.

Theorem ☆

Let $D = \{ |z - z_0| < R \}$. Suppose that $f_n(z)$ is analytic on D for each $n \geq 1$ and that

$$|f_n(z)| \leq M, \quad \text{all } n.$$

We can then find a subsequence $\{n_j\}$ so that $f_{n_j}(z)$ converges uniformly on D ~~compacta~~.

Proof

WLOG $z_0 = 0$. We know

$$f_n(z) = \sum_{k=0}^{\infty} c_{kn} (z)^n, \quad \leftarrow \text{power series}$$

$$c_{kn} = \frac{1}{2\pi i} \oint_{|z| = R - \varepsilon} \frac{f_n(z)}{z^{k+1}} \, dz .$$

Clearly, for each k,

$$|c_{kn}| \leq \frac{1}{2\pi} \frac{M}{(R-\varepsilon)^{k+1}} 2\pi (R - \varepsilon) = \frac{M}{(R-\varepsilon)^k}$$

Let $\varepsilon \to 0$. Get:

$$|c_{kn}| \leq \frac{M}{R^k} \quad \text{for each } k \geq 0, \text{ all } n.$$

The sequence of complex numbers $\{c_{kn}\}_{n=1}^{\infty}$ is bounded for **each** k.

$k=0$ Extract a convergent subsequence
$$\{c_{0n} : n \in S_0\}.$$

$k=1$ Look at $n \in S_0$ and $\{c_{1n}\}$. Get
$S_1 \subseteq S_0$; $\{c_{1n} : n \in S_1\}$ convergent.

$k=2$ Look at $n \in S_1$ and $\{c_{2n}\}$. Get
$S_2 \subseteq S_1$, $\{c_{2n} : n \in S_2\}$ convergent.

etc

Do the "Cantor diagonal" on array

$S_0 : \quad n_1^{(0)} < n_2^{(0)} < n_3^{(0)} < \cdots$

$S_1 : \quad n_1^{(1)} < n_2^{(1)} < n_3^{(1)} < \cdots$

$S_2 : \quad n_1^{(2)} < n_2^{(2)} < n_3^{(2)} < \cdots$

etc.

Get sequence $n_j^{(j-1)}$ of positive integers.

These are strictly increasing; eg

$$n_3^{(2)} > n_2^{(2)} \geq \{2^{\text{nd}} \text{ entry of sequence } S_1\}$$
$$= n_2^{(1)}.$$

By construction, the sequence $\{n_j^{(j-1)}\}_{j=1}^{\infty}$ is eventually contained in each $\mathcal{S}_K$.

Look at the original array.

Conclude that:

$$\lim_{\substack{n \to \infty \\ n \in \{\text{Cantor diagonal}\}}} c_{kn} \quad \text{exists} \quad \left(\text{call it } c_k\right)$$

for each $k \geq 0$. yes

Since $|c_{kn}| \leq \frac{M}{R^k}$, each limit value c_k satisfies the same estimate.

Write $\mathcal{S} = \{n_j^{(j-1)}\}$. I claim that $\{f_n : n \in \mathcal{S}\}$ works in the theorem.

It suffices to prove the uniform conv on each $\{|z| \leq R-\delta\}$. The limit function is $\sum_{k=0}^{\infty} c_k z^k$.

freeze it!!

Observe that $(n \in \mathcal{S})$:

$$\left|\sum_{k=0}^{\infty} c_{kn} z^k - \sum_{k=0}^{\infty} c_k z^k\right| \leq \left|\sum_{k=0}^{G} (c_{kn} - c_k) z^k\right| + \sum_{k=G+1}^{\infty} |c_{kn} - c_k| |z|^k.$$

$\langle$any $G\rangle$

Choose any $\varepsilon > 0$.

The last sum is no bigger than

$$\sum_{k=G+1}^{\infty} 2\,\frac{M}{R^k}(R-\delta)^k$$

$$= 2M \sum_{k=G+1}^{\infty} \left(1-\frac{\delta}{R}\right)^k .$$

$\boxed{\delta \text{ was given } > 0}$

Choose G so big that

$$2M \sum_{k=G+1}^{\infty} \left(1-\frac{\delta}{R}\right)^k < \frac{\varepsilon}{2} .$$

G is now fixed! But, then,

$$\left| \sum_{k=0}^{G} (c_{kn}-c_k) z^k \right| \leq \sum_{k=0}^{G} |c_{kn}-c_k| (R-\delta)^k$$

$$\leq \sum_{k=0}^{G} |c_{kn}-c_k| R^k .$$

For $n \in \mathcal{S}$ large enough, this last sum will clearly become $< \varepsilon/2$. In other words, for such n, say $n \geq N_0$ in $\mathcal{S}$, we have

$$\left| f_n(z) - \sum_{k=0}^{\infty} c_k z^k \right| < \varepsilon$$

uniformly for $|z| \leq R-\delta$. ∎

Note that the limit fcn $g(z) = \sum_{0}^{\infty} c_k z^k$ that we obtain is again analytic on $\{|z|<R\}$ and, by pointwise convergence, we have $|g(z)| \leq M$.

The set $\mathcal{E}_M$ of analytic functions f on $\{|z|<R\}$ having absolute value $\leqq M$ is thus <u>sequentially</u> <u>compact</u> for each $M>0$.

One natural question is what happens if you replace $\{|z|<R\}$ by a general domain $D \subsetneqq \mathbb{C}$?

We'll start to discuss this next time.

Before closing for today, I have suggested looking in Ahlfors for one proof of $\xi(z) = \xi(1-z)$, $\xi(z) = \pi^{-z/2}\Gamma(\frac{z}{2})\zeta(z)$. To save class time, I will give my "slick" proof in a supplement. It will rest partly on Fourier series. Cf. Lec 12, pages 18 + 23 + 24 (top).