Supplement to Lec 17

We wish to give a "slicker" proof of the relation

$$\xi(z) = \xi(1-z),$$

$$\xi(z) \equiv \pi^{-z/2}\Gamma(\tfrac{z}{2})\zeta(z).$$

We use the Poisson summation formula.

Consider functions $f \in C^{\infty}(\mathbb{R})$ such that

$$|f^{(j)}(x)| \leq A_{j\Delta}(1+|x|)^{-\Delta}$$

for every $j \geq 0$ and $\Delta \geq 1$. This is the so-called Schwartz class of functions on $\mathbb{R}$.

Let

$$\hat{f}(u) = \int_{\mathbb{R}} f(x) e^{-2\pi i u x}\, dx.$$

the Fourier transform of f

By repeatedly integrating by parts,

$$|\hat{f}(u)| \leq B_{\Delta}(1+|u|)^{-\Delta}, \quad u \in \mathbb{R}$$

for every $\Delta \geq 1$.

A similar inequality holds for any derivative $\hat{f}^{(m)}(u)$ after using Leibnitz's rule for improper integrals.

Thus, $\hat{f}$ is <u>also</u> in the Schwartz class of functions on $\mathbb{R}$.

The function

$$g(x) \equiv \sum_{n=-\infty}^{\infty} f(x+n), \quad x \in \mathbb{R}$$

is very interesting.*

* Remember that an infinite series $\sum_{k=1}^{\infty} u_k(x)$ can be differentiated term-by-term when $u_k \in C^1(\mathbb{R})$, $\sum_1^{\infty} u_k'(x)$ conv <u>unif</u> on $\mathbb{R}$ compacta, and $\sum_1^{\infty} u_k(x)$ conv pointwise.

Clearly RHS converges unif + absolutely on $\mathbb{R}$~compacta. We also have:

$$g(x+1) = g(x) \; ;$$

$$g^{(j)}(x) = \sum_{n=-\infty}^{\infty} f^{(j)}(x+n) .$$

Here $j \geq 1$ and the RHS again converges unif + absolutely on $\mathbb{R}$ compacta.

The function g is thus $\underline{C^\infty}$ and periodic. Expand it as a <u>Fourier series</u> !!!

à la Lec. 12 p. 18

$$g(x) = \sum_{k=-\infty}^{\infty} C_k e^{2\pi i k x}$$

$$C_k = \int_0^1 g(x) e^{-2\pi i k x} dx$$

But,

$$C_k = \lim_{N\to\infty} \int_0^1 \left(\sum_{-N}^{N} f(x+n) \right) e^{-2\pi i k x} dx$$

$$= \lim_{N\to\infty} \sum_{-N}^{N} \int_n^{n+1} f(y) e^{-2\pi i k y} dy$$

$$= \lim_{N\to\infty} \int_{-N}^{N+1} f(y)\, e^{-2\pi i k y}\, dy$$

$$= \int_{-\infty}^{\infty} f(y)\, e^{-2\pi i k y}\, dy = \hat{f}(k).$$

In other words: $(x \in \mathbb{R})$

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\, e^{2\pi i k x}.$$

Both sides are unif + abs convergent on $\mathbb{R}$ – compacta.

Put $x = 0$. Get:

$$\boxed{\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k)}.$$

This is the famous Poisson summation formula. $\{$NB: The conditions on $f(x)$ can be relaxed significantly.$\}$

One knows that

$$\int_{-\infty}^{\infty} e^{-q x^2}\, dx = \sqrt{\frac{\pi}{q}} \quad (q > 0).$$

By baby complex analysis,

$$f(x) = e^{-qx^2}$$

$$\Rightarrow \hat{f}(u) = \sqrt{\frac{\pi}{q}}\, e^{-\frac{\pi^2 u^2}{q}}.$$

Thus,

$$\sum_{n=-\infty}^{\infty} e^{-qn^2} = \sqrt{\frac{\pi}{q}} \sum_{k=-\infty}^{\infty} e^{-\frac{\pi^2 k^2}{q}}.$$

Put $q = \pi\beta$ to rephrase this as:

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 \beta} = \sqrt{\frac{1}{\beta}} \sum_{k=-\infty}^{\infty} e^{-\frac{\pi}{\beta}k^2}, \quad \beta > 0.$$

(This is the so-called THETA IDENTITY.)

With

$$\Psi(x) \equiv \sum_{n=1}^{\infty} e^{-\pi n^2 x} \qquad (x > 0),$$

we have

$$\boxed{1 + 2\Psi(x) = \sqrt{\frac{1}{x}}\left[1 + 2\Psi\left(\frac{1}{x}\right)\right]}.$$

Riemann found a nice formula for $\zeta(s)$ using $\Psi(x)$. It gives $\xi(s) = \xi(1-s)$ immediately.

$$\boxed{\text{We follow Riemann and put } s \equiv \sigma + it.}$$

Let $\mathcal{R} = \{ Re(s) > 1 \}$. Let K be any closed disk in $\mathcal{R}$. Keep $s \in K$.

Note that: p. 5 $\boxed{\Psi \in C[(0,\infty)]}$

$$\Psi(x) = O(e^{-\pi x}) \ , \quad x \to +\infty$$

$$\Psi(x) = O\left(\frac{1}{\sqrt{x}}\right) \ , \quad x \to 0^+ .$$

Look at: principal value

$$\int_0^\infty \Psi(x)\, x^{\frac{s}{2}-1}\, dx \qquad (s \in K).$$

This integral satisfies a nice Weierstrass M-test for $s \in K$. We propose to integrate term-by-term.

At once:

$$\text{integral} = \sum_{n=1}^{\infty} \left(\int_0^{\infty} e^{-\pi n^2 x} x^{s/2} \frac{dx}{x} \right) \qquad \boxed{x = \frac{y}{\pi n^2}}$$

$$= \sum_{n=1}^{\infty} \left(\int_0^{\infty} e^{-y} \frac{y^{s/2}}{\pi^{s/2} n^s} \frac{dy}{y} \right)$$

$$= \pi^{-s/2} \sum_{n=1}^{\infty} n^{-s} \left(\int_0^{\infty} y^{\frac{s}{2}-1} e^{-y} \, dy \right)$$

$$= \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

NICE!

On the other hand,

$$\text{integral} = \int_0^1 \Psi(x) x^{\frac{s}{2}-1} dx + \int_1^{\infty} \Psi(x) x^{\frac{s}{2}-1} dx \qquad \underline{s \in K}$$

$$\left\{ \text{but} \quad \Psi(x) = -\frac{1}{2} + \frac{1}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \Psi\left(\frac{1}{x}\right) \right\}$$

$$= \int_0^1 \left[-\frac{1}{2} + \frac{1}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \Psi\left(\frac{1}{x}\right) \right] x^{\frac{s}{2}-1} dx + \int_1^{\infty} \Psi(x) x^{\frac{s}{2}-1} dx$$

$$= -\frac{1}{2}\int_0^1 x^{\frac{s}{2}-1}\,dx + \frac{1}{2}\int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\,dx$$

$$+ \int_0^1 \psi\left(\frac{1}{x}\right) x^{\frac{s}{2}-\frac{3}{2}}\,dx + \int_1^\infty \psi(x)\, x^{\frac{s}{2}-1}\,dx$$

$$= -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty \psi(y)\, y^{\frac{3}{2}-\frac{s}{2}}\,\frac{dy}{y^2}$$

$$+ \int_1^\infty \psi(x)\, x^{\frac{s}{2}-1}\,dx$$

$$= -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty \psi(x)\left[x^{\frac{s}{2}-1} + x^{-\frac{1}{2}-\frac{s}{2}}\right]dx$$

$$= \frac{1}{s(s-1)} + \int_1^\infty \psi(x)\left[x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right]\frac{dx}{x}.$$

$\boxed{\text{continuous and } O(e^{-\pi x}) \text{ for } x \text{ large}}$

In other words,

$$(\star)\quad \overbrace{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)}^{\xi(s)} = \frac{1}{s(s-1)} + \int_1^\infty \psi(x)\left[x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right]\frac{dx}{x}$$

for every $s \in K$. NOTICE THAT the RHS is a perfectly good analytic function of s on $\mathbb{C} - \{0,1\}$; the integral is actually an <u>entire</u> function.

$\boxed{\text{THINK ANALYTIC !! CONTINUATION}}$

Moreover, ~~RHS~~ RHS remains the same when

$$s \longleftrightarrow 1-s.$$

QED ■