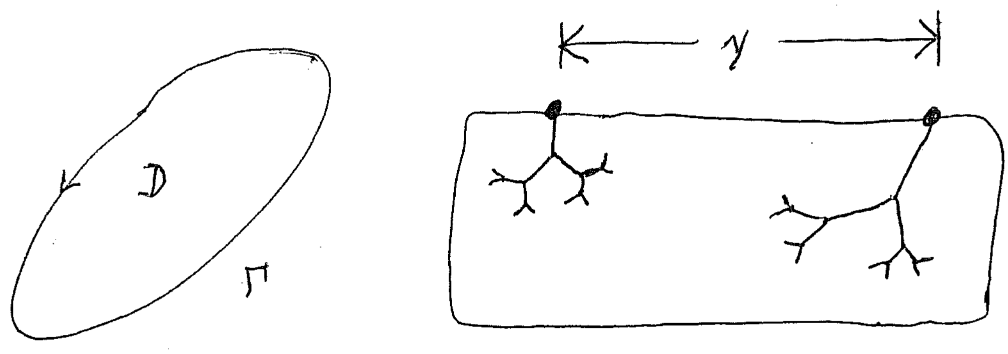
Lecture 20 Tues, 22 Nov 2011

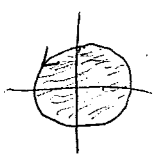
I am eager to move forward with some further aspects of conformal mappings (i.e., analytic + univalent maps) and harmonic functions.

I had some supplemental stuff to say about NORMAL FAMILIES — but I think it's best to skip that temporarily.

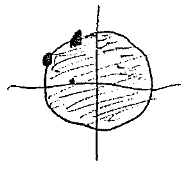
[[Please see the Supplement to Lec 20 for a bit more regarding the proof of the Riemann mapping theorem.]]

Ahlfors discusses the Riemann mapping theorem in chap 6, section 1. He also discusses there some aspects of the boundary behavior, especially when our given simply-connected $D$ is bounded by a Jordan curve $\Gamma$, or at least has PART of its boundary looking like a Jordan arc $\gamma$.



$f: D \to \mathcal{U}$ à la RMT



One of the things Ahlfors stresses is the importance of the

Schwarz Reflection Principle.

I agree with his highlighting this; it really <u>is</u> quite important.

He handles the Schwarz reflection principle in 2 phases: [3rd edition]

chap 4 section 6.5 ;

chap 6 section 1.3, also 1.4.

We'll proceed similarly.

To arrange things so we can work <u>efficiently</u>, it is wise to "back up" a bit — and first recall (OR develop) some basic properties of HARMONIC functions and elementary conformal mappings.

We begin with a very fundamental result about nonconstant analytic functions.

FACT

Let $H(z)$ be analytic on $|z - z_0| < R$. Suppose that $H(z) \not\equiv$ constant. Let $w_0 = H(z_0)$ and

$$m = \text{mult}\{\text{zero of } H(z) - w_0 \text{ at } z = z_0\}.$$

For $\delta$ very small, the function $H(z)$ can be expressed on $\{|z - z_0| < \delta\}$ as

$$w_0 + t^m,$$

where $t = \Psi(z)$ is analytic + univalent on $\{|z - z_0| < \delta\}$ and $\Psi(z_0) = 0$.

Proof

Write

$$H(z) - w_0 = (z - z_0)^m \varphi(z) \quad , \quad |z - z_0| < R,$$

where

$$\varphi(z) = \sum_{k=0}^{\infty} b_k (z-z_0)^k \, , \quad b_0 \neq 0 .$$ (yes!)

Keep $\delta$ small enough so that $\varphi(z) \neq 0$ on $\{|z-z_0| < 2\delta\}$ (say). Form $\operatorname{Log} \varphi(z)$ here; then form

$$p(z) = \sqrt[m]{\varphi(z)} = \exp\left\{ \frac{1}{m} \operatorname{Log} \varphi(z) \right\} .$$

Clearly

$$H(z) - w_0 = (z-z_0)^m p(z)^m .$$

Let $\psi(z) = (z-z_0)p(z)$. Clearly $\psi$ is analytic on $|z-z_0| < \delta$ and $\psi'(z_0) = p(z_0) \neq 0$. By the standard inverse fcn thm of multi-variable calculus, $\psi(z)$ is univalent provided we again keep $\delta$ sufficiently small. Clearly:

$$H(z) = w_0 + \psi(z)^m .$$

[See Ahlfors, chap 3, section 2.3, third paragraph following eq (4).] ∎

↖ p. 74

The fact on page (3) is very fundamental and very geometric. I have found that most students have never seen it before!

Notice that Ahlfors discusses it in chap 4 section 3.3 after theorem 11. [p. 133]

## Corollary of the Fact ← on (3)

Let $N$ be a domain. Let $f(z)$ be analytic on $N$. Suppose $f(z) \not\equiv$ constant. Then $f(N)$ is open.

**Proof**

Trivial. ▮

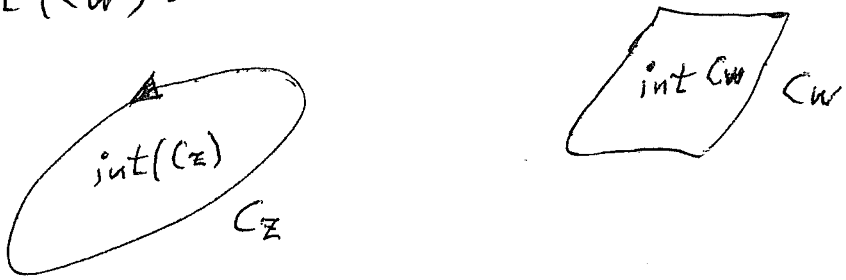
(Take $N$ = tiny disk on p. (3).)

Nonconstant analytic functions thus act as open mappings (open ↪ open).

Compare Ahlfors chap 4 section 3.3 corollary 1. [p. 132]

**FACT** (conformal mapping via "edges")

Let $C_z$ and $C_w$ be (piecewise $C^1$) Jordan curves. Orient $C_z$ counterclockwise. Let $\text{int}(C_z)$ be the "interior" of $C_z$; similarly for $\text{int}(C_w)$.




Let $w = f(z)$ be continuous on $C_z + \text{int}(C_z)$. Let $f(z)$ be analytic on $\text{int}(C_z)$. Let $f$ map $C_z$ in a 1-1 way onto $C_w$. Then:

(A) $f$ maps $C_z$ onto $C_w$ counterclockwise;

(B) $f$ is univalent on $\text{int}(C_z)$;

(C) $f[\text{int}(C_z)] = \text{int}(C_w)$.

**Proof**

Clearly $f \not\equiv$ const on $\text{int}(C_z)$.

Recall the argument principle.

Let $C_{z\varepsilon}$ be a nested family of (piecewise $C^1$) Jordan curves that approximate $C_z$ from the inside, closer and closer as $\varepsilon \to 0$.

Let $w_1$ be any point of $\mathrm{int}(C_w)$.

Since $f$ maps $C_z$ onto $C_w$ 1-1 and $f$ is uniformly continuous on $C_z + \mathrm{int}(C_z)$, the sets $f(C_{z\varepsilon})$ remain "far away" from $w_1$ when $\varepsilon \to 0$.

Keep $\varepsilon$ very small. The number of roots of $f(z) - w_1 = 0$ (counted with mult.) for $z \in \mathrm{int}(C_{z\varepsilon})$ is

$$\frac{1}{2\pi i} \oint_{C_{z\varepsilon}} \frac{f'(z)}{f(z) - w_1}\, dz .$$

Arg Principle

But this expression reduces to

$$= \frac{1}{2\pi} \Delta_{C_{z\varepsilon}} \arg\left[f(z) - w_1\right],$$

in the sense of "net change". Letting $\varepsilon \to 0$, we get:

$$\frac{1}{2\pi} \Delta_{\underline{\underline{C_z}}} \arg\left[f(z) - w_1\right].$$

It follows that:

| number of roots of $f(z) - w_1 = 0$ for $z \in int(C_z)$ | $= \frac{1}{2\pi} \Delta_{C_z} \arg [f(z) - w_1]$. |
|---|---|

If $f$ maps $C_z$ onto $C_w$ counterclockwise to counterclockwise, the RHS is $+1$.

If $f$ maps $C_z$ onto $C_w$ counterclockwise to clockwise, the RHS is $-1$.

The LHS is $\geq 0$. Hence, we must have

counterclo. $\rightarrow$ countercl.

YES!

i.e. (A) is OK.

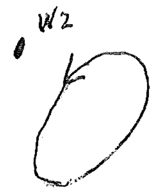
From the $+1$, we also see:

$$(\star) \qquad int(C_w) \subseteq f[int(C_z)].$$

Replace $w_1$ in the above by any $w_2 \in ext(C_w)$.

Note

$$\frac{1}{2\pi} \Delta_{C_z} \arg [f(z) - w_2] = 0.$$

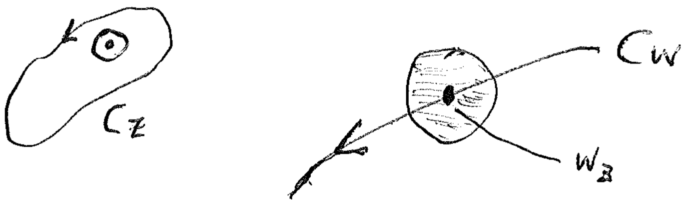


Therefore

$$(\star\star) \quad ext(C_w) \cap f[int(C_z)] = \phi .$$

Take any $w_3 \in C_w$. Can it happen that $f(z_3) = w_3$ for some $z_3 \in int(C_z)$ ??

ANSWER: NO! Indeed, since $f$ is an open mapping, a neighborhood of $z_3$ would go to a neighborhood of $w_3$.




See pages 3 + 5. This violates $(\star\star)$.

HENCE:

$$f[int(C_z)] = int(C_w) .$$

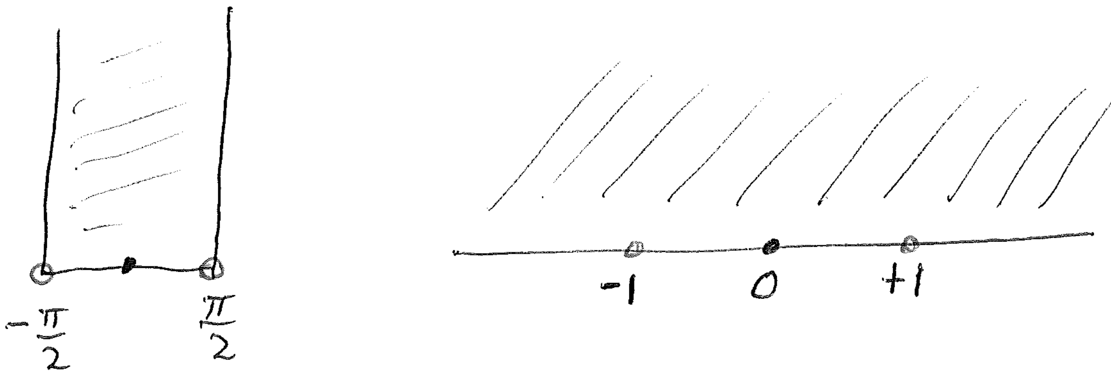
⇒ (C) OK

And, by applying the +1 again, we deduce that $f$ is univalent on $int(C_z)$. Hence, (B) is OK too.

∎

It is very important to notice that when working with unbounded $z$-regions (simply-connected) or unbounded $w$-regions (simply-connected), one can often use an AUXILIARY LINEAR FRACTIONAL TRANSFORMATION to reduce things to the "bounded" situation of p. (6).

A good example is studying $w = \sin(z)$ :




(details = exercise).

Let $D$ be a domain. Let $u \in C^2(D)$ and real. We say $u$ is harmonic on $D$ if and only if

$$u_{xx} + u_{yy} = 0 \quad \text{on } D.$$

We often write $\Delta \varphi \equiv \varphi_{xx} + \varphi_{yy}$. | The Laplacian

FACT

Let $f(z)$ be analytic on domain $D$. Then $\mathrm{Re}[f(z)]$ is harmonic on $D$.

Proof

Write $f(z) = u(x,y) + iv(x,y)$. We proved earlier that $u$ and $v \in C^\infty(D)$. (Lec 1 pp. 21, 22)

Apply C-R eqs:

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned} \implies \begin{aligned} u_{xx} &= v_{yx} \\ u_{yy} &= -v_{xy} \end{aligned}.$$

Get $u_{xx} + u_{yy} = 0$. ∎

Similarly for $\mathrm{Im}[f(z)]$.

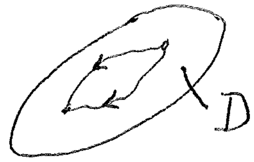
## FACT

Let $D$ be simply-connected. Let $u$ be harmonic on $D$. Then $u = \operatorname{Re}(F)$ for some $F$ which is analytic on $D$.

## Proof

Use multi-variable calculus for line integrals and Green's theorem. Since $D$ is simply-connected, the line integrals

$$\int_\gamma -u_y\,dx + u_x\,dy$$




are indep of the path (for $\gamma \subseteq D$).

Indeed,

$$(-u_y)_y = (u_x)_x .$$

We can therefore find a potential function $\phi \in C^1(D)$ such that

$$\operatorname{grad}\phi = \langle -u_y, u_x \rangle .$$

Hence

$$\phi_x = -u_y, \quad \phi_y = u_x .$$

Apply C-R to $u + i\phi$. This is analytic!

Just take $F = u + i\phi$. ∎

If $u$ is $C^2$, notice that harmonicity is a LOCAL property. (Just need to study $u_{xx}$, $u_{yy}$.)

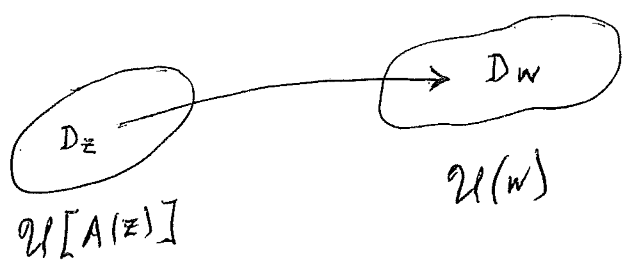
Cor 1

Every harmonic fcn is $C^\infty$.

Cor 2

Let $\mathcal{U}(w)$ be harmonic on domain $D_w$.
Let $A(z)$ be analytic on domain $D_z$.
Suppose that $A(D_z) \subseteq D_w$. Then
$\mathcal{U}[A(z)]$ is harmonic on $D_z$.

In our applications, often $A$ is univalent and $A(D_z) = D_w$.



**Cor 3**

Let $u$ be harmonic on domain $D$.

Let $u \equiv K$ on $\{|z-z_0|<\varepsilon\} \subseteq D$.

Then $u \equiv K$ on $D$.

**Proof**

Near each $z_1 \in D$, (write) $u = \mathrm{Re}(F)$, $F$ analytic.
But, then, $u_x - iu_y = F'(z)$. It follows that $u_x - iu_y$ is analytic on $D$.

Since $u_x - iu_y \equiv 0$ on $\{|z-z_0|<\varepsilon\}$, by analyticity we get $u_x - iu_y \equiv 0$ on $D$. (Lec 19 p.1)

Hence $\mathrm{grad}\, u = \vec{0}$ on $D$. Therefore $u \equiv$ constant. Plug in $z_0$ to get $u \equiv K$.

∎

Recall Poisson integral formula

$$u(re^{i\omega}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\omega - \theta)}\, u(Re^{i\theta})\, d\theta$$

denominator $= |Re^{i\theta} - z|^2$

$$\mathrm{Re}\left\{\frac{Re^{i\theta} + z}{Re^{i\theta} - z}\right\}$$

$z = re^{i\omega}$

when $u$ is harmonic on $|z| < R$, continuous on $|z| \leq R$.

[Lec 8, p. 25 (also Ahlfors 168) and Lec 15, p. 18 Note]

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$r = 0 \Rightarrow$

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})\, d\theta$$

(mean-value property).

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For later use, note $u = 1 \Rightarrow$

$$1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\omega - \theta)}\, d\theta .$$

<u>Theorem</u> (Max Principle for Harmonic Fcns)

Let $D$ be a bounded domain. Let $u$ be harmonic on $D$. Assume that

$$(\star) \quad \limsup_{z \to \xi} u(z) \leq M, \text{ all } \xi \in \partial D.$$

Then

$$u(z) \leq M, \text{ all } z \in D.$$

If equality ever holds, $u(z) \equiv M$.

<u>Proof</u>

Very similar to Lec 19, pages 2–4.

Let $\mathcal{M} = \sup_D u$.

Sequential compactness of $\bar{D} \Rightarrow \mathcal{M} < \infty$.

(Use $\star$.)

Assume $\mathcal{M} > M$. Get some $\zeta \in D$ with $u(\zeta) = \mathcal{M}$. Apply mean value property as in Lec 19 page 3. Get $u \equiv \mathcal{M}$ near $\zeta$.

Use Cor 3 ABOVE. Get $u \equiv \mathcal{M}$ on $D$.

Contradicts $\star$.

Thus $\mathfrak{M} \leq M$.

But, then, clearly $u(z) \leq M$ on $D$. (OK)

If $u(z_2) = M$ for some $z_2 \in D$, repeat the mean-value trick at $z_2$. Get $u \equiv M$ via Cor 3 above. ∎

On p. (16), by switching to $-u$, we clearly get a corresponding <u>minimum</u> principle for $u$ on $D$.

These max/min relations are also discussed in Ahlfors, chap 4, section 6.2, theorem 21 (3rd edition).
<u>p. 166</u>

**Theorem** (P–L form of the max principle)

Let $D$ be a bounded domain. Let $u$ be harmonic on $D$. Let $u$ be bounded from above by some giant $G$. Let

$$(\star) \quad \limsup_{z \to \xi} u(z) \leq M$$

for all $\xi \in \partial D$ except $\xi = \xi_1, \ldots, \xi_q$ ($q < \infty$). Then

$$u(z) \leq M, \quad z \in D.$$

If equality ever holds, $u(z) \equiv M$.

**Proof** Compare Lec 15, p. 21, 24.

Choose $B > \text{diam}(D)$. Look at the harmonic function

$$u_\varepsilon(z) = u(z) - \varepsilon \sum_{j=1}^{q} \ln \frac{B}{|z - \xi_j|}$$

for $z \in D$, $\varepsilon > 0$. Notice that

$$\frac{B}{|z - \xi_j|} > 1, \quad z \in D.$$

Clearly

$$\limsup_{z\to\xi} u_\varepsilon(z) \leqq M, \quad \xi \neq \xi_1, \dots, \xi_q$$

$$\limsup_{z\to\xi_k} u_\varepsilon(z) = -\infty, \quad \text{since } u \leqq G.$$

Apply thm on p. 16. Get

$$u_\varepsilon(z) \leqq M, \quad \text{all } z \in D.$$

Let $\varepsilon \to 0$. Get

$$u_\varepsilon(z_1) \leqq M, \quad \text{each } z_1 \in D.$$

Thus $u \leqq M$ on $D$. If equality holds at $z_2 \in D$, apply the mean-value trick at $z_2$ to get $u \equiv M$ (via Cor 3 above).

▨

Analogously to p. 10, when our initial $D$ on pages 18 or 16 is <u>unbounded</u>, it is a standard trick to consider $u[A(z)]$ instead where $A$ is linear fractional. See Cor 2 on p. 13. (Details = exercise)

We now go "backwards" in the Poisson integral formula.

Take $R = 1$, without loss of generality.

E.G. piecewise continuous

**THM**

Let $h(e^{i\phi})$ be any REASONABLE bounded, real-valued function on $\partial\mathcal{U}$. Define

$$u(z) \equiv \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{e^{i\phi}+z}{e^{i\phi}-z}\right) h(e^{i\phi})\,d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\phi-\theta)}\, h(e^{i\phi})\,d\phi$$

for $|z|<1$. Then:

(a) $u(z)$ is harmonic on $\{|z|<1\}$

(b) $A \leqq h(e^{i\phi}) \leqq B \Rightarrow A \leqq u(z) \leqq B$

(c) $\lim_{z\to\eta} u(z) = h(\eta)$ whenever $h(e^{i\phi})$ is <u>continuous</u> at $\eta$.

See Ahlfors, 3rd ed., page 169.

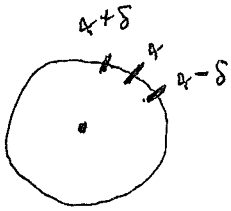
## Proof

(a) Simply stay on $\mathcal{U}$-compacta.
Notice that $\operatorname{Re}\left(\frac{e^{i\phi}+z}{e^{i\phi}-z}\right)$ is harmonic in $z$.
Apply Leibnitz's rule to calculate $u_{xx}+u_{yy}$.

(b) Trivial by (15) [redacted] (bottom).

(c) Choose any small $\varepsilon > 0$. Suppose $h(e^{i\phi})$ is continuous at $\eta = e^{i\alpha}$. Choose tiny $\delta > 0$ so that

$$|h(e^{i\phi}) - h(e^{i\alpha})| \leqq \varepsilon$$
for $|\phi - \alpha| \leqq \delta$.




Notice that: $\boxed{z = re^{i\theta}}$

$$u(z) = \frac{1}{2\pi}\int_{\partial\mathcal{U}} \frac{1-r^2}{|e^{i\phi}-z|^2}\, h(e^{i\phi})\, d\phi$$

$$u(z) - h(e^{i\alpha}) = \frac{1}{2\pi}\int_{\partial\mathcal{U}} \frac{1-r^2}{|e^{i\phi}-z|^2}\left[h(e^{i\phi}) - h(e^{i\alpha})\right] d\phi$$

$$= \frac{1}{2\pi}\int_{[\alpha-\delta,\alpha+\delta]} \boxed{*}\ (h(e^{i\phi}) - h(e^{i\alpha}))\, d\phi$$

$$+ \frac{1}{2\pi}\int_{\partial\mathcal{U}-[\alpha-\delta,\alpha+\delta]} \boxed{*}\ (h(e^{i\phi}) - h(e^{i\alpha}))\, d\phi .$$

slight abuse of $\phi \sim e^{i\phi}$ notation here

Clearly,

$$|u(z) - h(e^{i\theta})| \leq \frac{1}{2\pi} \int_{[\theta-\delta, \theta+\delta]} \frac{1-r^2}{|e^{i\phi} - z|^2} \varepsilon \, d\phi$$

$$+ \frac{1}{2\pi} \int_{\partial\mathcal{U} - [\theta-\delta, \theta+\delta]} \frac{1-r^2}{|e^{i\phi} - z|^2} |h(e^{i\phi}) - h(e^{i\theta})| \, d\phi .$$

Remember that $-M \leq h(e^{i\phi}) \leq M$ for some big $M$.

As $z \to e^{i\theta}$ (in $\mathcal{U}$), clearly

$$\frac{1-r^2}{|e^{i\phi} - z|^2} \rightrightarrows 0$$

for $\phi$ corresponding to $\partial\mathcal{U} - [\theta-\delta, \theta+\delta]$. The 2nd integral above thus approaches 0 (trivially) as $z \to e^{i\theta}$.

The 1st integral is

$$\leq \frac{1}{2\pi} \int_{\partial\mathcal{U}} \frac{1-r^2}{|e^{i\phi} - z|^2} \varepsilon \, d\phi = \varepsilon .$$ ANY $z$

Hence,

$$\limsup_{z \to \eta} |u(z) - h(e^{i\theta})| \leq \varepsilon + 0 = \varepsilon .$$

Since $\varepsilon$ is arbitrary,

$$\lim_{z \to \eta} u(z) = h(e^{i\varphi}).$$

Qed ∎

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Let $D$ be bounded by several (piecewise $C^1$) Jordan curves.


Let $f$ and $g$ be $C^2$ on $D \cup \partial D$ (real).
Recall that, by Green's theorem,

$$\iint_D \left[ \vec{\nabla} f \cdot \vec{\nabla} g + f(\Delta g) \right] dx\,dy = \int_{\partial D} f \frac{\partial g}{\partial \vec{n}} \, ds$$

Green's 1st identity

where $\vec{n}$ = outer unit normal. [[ For positively oriented $\partial D$, thus

$$\vec{n} = \frac{dy}{ds}\vec{i} - \frac{dx}{ds}\vec{j} \quad . ]]$$

By switching $f \leftrightarrow g$ and subtracting, clearly

Green's $2^{nd}$ identity

$$\iint_D [f(\Delta g) - g(\Delta f)]\,dx\,dy = \int_{\partial D} \left[f \frac{\partial g}{\partial \vec{n}} - g \frac{\partial f}{\partial \vec{n}}\right] ds \,.$$

In particular, for harmonic $f$ and $g$, we automatically get:

$$\iint_D \vec{\nabla} f \cdot \vec{\nabla} g \,dx\,dy = \int_{\partial D} f \frac{\partial g}{\partial \vec{n}}\, ds$$

$$0 = \int_{\partial D} \left[f \frac{\partial g}{\partial \vec{n}} - g \frac{\partial f}{\partial \vec{n}}\right] ds \,.$$

For $f = 1$, $u =$ harmonic, we get

$$0 = \int_{\partial D} \frac{\partial u}{\partial \vec{n}}\, ds \,.$$

Notice that

$$\frac{\partial u}{\partial \vec{n}}\, ds = \left[ u_x \left( \frac{dy}{ds} \right) + u_y \left( -\frac{dx}{ds} \right) \right] ds$$

$$= -u_y\, dx + u_x\, dy$$

when $\partial D$ is positively oriented. Recall here the potential function $\phi$ used on p. 12. In other words, when $\underline{\underline{v}}$ is selected so that $u + iv$ is analytic locally, we simply have

$$\frac{\partial u}{\partial \vec{n}}\, ds = dv \quad \text{on } \partial D.$$

One sometimes writes

$$* du = -u_y\, dx + u_x\, dy = \left\{ \begin{array}{l} \text{the conjugate} \\ \text{differential} \\ \text{of } du \end{array} \right\}.$$

See Ahlfors, chap 4, section 6 around page 163 (3rd edition).

$\underline{\underline{v}}$ is called the local harmonic conjugate of $u$

By C-R,

$$dv = v_x\, dx + v_y\, dy = -u_y\, dx + u_x\, dy.$$