Supplement to Lecture 20

There are NO miracles! More precisely, none in the proof of RMT.

Was the miracle really a miracle?

NO!

Recall: $f(D) \subseteq \mathcal{U}$, $f(z_0) = 0$, etc.

$$e(z) = \frac{f(z) - w_0}{1 - \bar{w}_0 f(z)} \qquad f(z) \neq w_0$$

$$F(z) = \sqrt{e(z)}$$

$$G(z) = \frac{|F'(z_0)|}{F'(z_0)} \frac{F(z) - F(z_0)}{1 - \overline{F(z_0)} F(z)} .$$

Let "self-map" mean <u>Möbius</u> <u>transformation</u> $\mathcal{U}$ onto $\mathcal{U}$.

L.F.

Let $\mathcal{W} = G(z)$. We want to go from $\mathcal{W}$ back to $f(z)$. EASY! Baby Algebra!

$$\mathcal{W} \xrightarrow[\text{map}]{\text{self}} F(z) \xrightarrow{\text{square}} e(z) \xrightarrow[\text{map}]{\text{self}} f(z)$$

So,

$$f(z) = A(\mathcal{W}) \qquad \text{i.e.} \quad \boxed{f(z) = A[G(z)]}$$

where A is analytic on $\mathcal{U}$ and sends $\underline{\mathcal{U}}$ onto $\underline{\mathcal{U}}$. (NOT SCHLICHT.) univalent

Note too that

$$(\mathcal{W}=0) \xrightarrow{\text{self}} (F(z_0)) \xrightarrow{\text{sq.}} (F(z_0)^2 = e(z_0) = -W_0)$$
$$\downarrow \text{self}$$
$$(0).$$

So: $A(0) = 0$. $\leftarrow$ $\boxed{\text{or just take } z = z_0 \text{ in } f(z) = A[G(z)].}$

By Schwarz lemma, since A is NOT schlicht,

$$|A'(0)| < 1.$$

But,

$$f(z) = A(\mathcal{W}) = A[G(z)]$$
$$\Downarrow$$
$$f'(z_0) = A'[G(z_0)]\,G'(z_0) = A'(0)\,G'(z_0)$$

$$\Rightarrow \quad |G'(z_0)| > |f'(z_0)|. \quad *$$

[This was a contradiction, since $G \in \mathcal{F}$ and $f'(z_0) = \max$.]

* by pure thought

Thus, we got:

Riemann Mapping Theorem.

Let D be a simply-connected domain, $D \neq \mathbb{C}$. Let $z_0 \in D$.
There exists a <u>unique</u> f such that:

(a) f is analytic and univalent on D;

(b) $f[D] = \{|w|<1\}$;

(c) $f(z_0) = 0$;

(d) $f'(z_0) > 0$.

> f is a 1-1 conformal mapping of D onto $\mathcal{U}$. We say D and $\mathcal{U}$ are conformally equivalent!!

Corollary!

Let D be <u>any</u> simply-connected domain $\subsetneqq \mathbb{C}$. Then, D is homeomorphic to the open unit disk $\mathcal{U}$.

<u>Proof</u>

If $D \neq \mathbb{C}$, apply the RMT.
If $D = \mathbb{C}$, look at the map Φ:

$$x+iy \longrightarrow x+ie^y \longrightarrow \frac{(x+ie^y)-i}{(x+ie^y)+i}.$$

Each "chunk" is a C^∞ homeomorphism. Hence Φ is a C^∞ homeomorphism $\mathbb{C} \leftrightarrow \mathcal{U}$. Clearly $\Phi(0)=0$. ∎

NOTE

The complex plane $\mathbb{C}$ cannot be mapped with a univalent analytic function f onto $\mathcal{U}$. If it could, f would violate Liouville's theorem.

Univalent + C^∞ is OK.