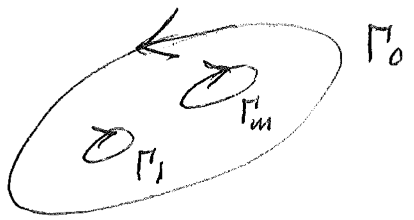
Lecture 21 Fri 25 Nov

More today on harmonic functions.

We had Green's 1^{st} identity

$$\iint_D \left[\vec{\nabla} f \cdot \vec{\nabla} g + f(\Delta g) \right] dx\,dy = \int_{\partial D} f \frac{\partial g}{\partial \vec{n}} \, ds$$

For D bounded by several (piecewise C^1) Jordan curves, $\vec{n}$ = outer unit normal, ∂D taken with positive orientation.



$$\left\{ \begin{array}{c} \partial D = \Gamma_0 + \Gamma_1 + \cdots + \Gamma_m \\ \text{say} \end{array} \right\}$$

This was a key fact.

If u was harmonic on $\overline{D}$, we immediately got

$$\int_{\partial D} \frac{\partial u}{\partial \vec{n}} \, ds = 0 .$$

But what is the significance of the individual integrals

$$\int_{\Gamma_j} \frac{\partial u}{\partial \vec{n}}\, ds \quad ?$$

To appreciate this, first work locally, on a little disk $|z - z_0| < h$. Let $u + iv$ and $u + iv_2$ both be analytic here. By C-R, we have:

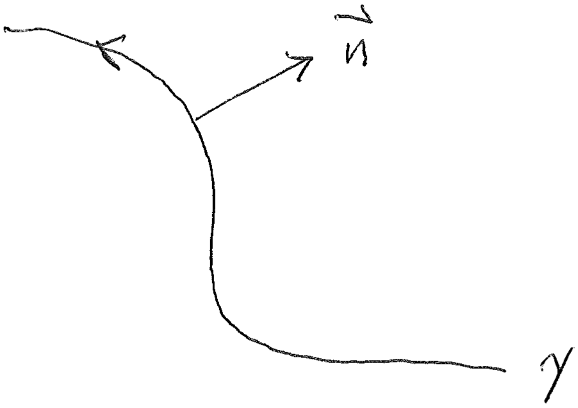
$$dv = -u_y\, dx + u_x\, dy = dv_2 .$$

Hence:

$$d(v - v_2) = 0 , \quad \text{i.e.} \quad v_2 \equiv v + \text{constant}.$$

Thus two harmonic conjugates of u can only differ by a (real) constant.

Now let γ be a small <u>directed</u> Jordan arc inside $\{|z - z_0| < h\}$. Assume unit normal $\vec{n}$ points off <u>to</u> <u>the</u> <u>right</u>.



Clearly:

$$\vec{n} = \frac{dy}{ds}\vec{i} - \frac{dx}{ds}\vec{j} \, .$$

Thus, along arc γ,

$$\frac{\partial v}{\partial s} = v_x \frac{dx}{ds} + v_y \frac{dy}{ds}$$

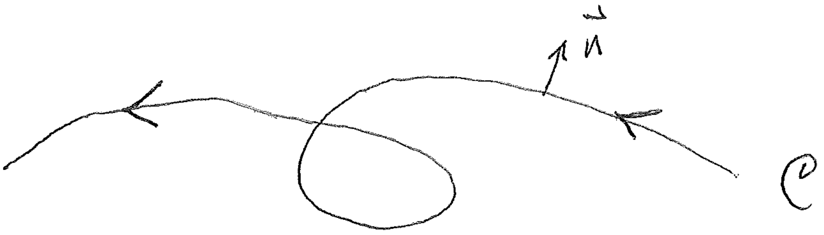
$$= -u_y \frac{dx}{ds} + u_x \frac{dy}{ds} \qquad (C-R)$$

$$= \langle u_x, u_y \rangle \cdot \left\langle \frac{dy}{ds}, -\frac{dx}{ds} \right\rangle$$

$$= \frac{\partial u}{\partial \vec{n}} \, .$$

Any sensible "big" curve $\mathcal{C}$ can be subdivided into a bunch of tiny arcs γ. We will still have, step by step,

$$\frac{\partial v}{\partial s} = \frac{\partial u}{\partial \vec{n}} .$$



Thus, in loose notation,

$$v(P_2) - v(P_1) = \int_{\mathcal{C}} dv = \int_{\mathcal{C}} \frac{\partial v}{\partial s}\, ds$$

$$= \int_{\mathcal{C}} \frac{\partial u}{\partial \vec{n}}\, ds ,$$

a formula well worth remembering.

Line integrals $\int_{\mathcal{C}} \frac{\partial u}{\partial \vec{n}}\, ds \equiv \int_{\mathcal{C}} -u_y\, dx + u_x\, dy$ thus tell the net change in v as one travels along $\mathcal{C}$ (as a <u>directed</u> path).

If $\mathcal{C}$ is closed  , v does NOT necessarily "come back to itself". It comes back to $v_2 \equiv v +$ constant.

Cf. page 2.

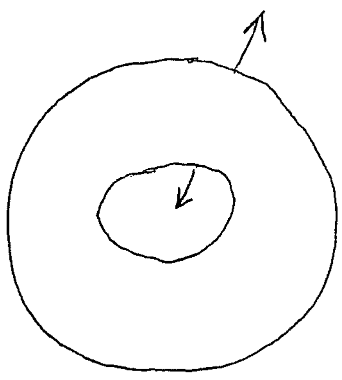
Green's 2^{nd} identity says

$$\iint_D [f(\Delta g) - g(\Delta f)]\,dx\,dy = \int_{\partial D} \left[f \frac{\partial g}{\partial \vec{n}} - g \frac{\partial f}{\partial \vec{n}}\right] ds.$$

Change notation slightly. Let u be harmonic on $\{R_1 \leq |z| \leq R_2\}$, $0 < R_1 < R_2 < \infty$. Take

Take $D = \{R_1 < |z| < R\}$, where $R_1 < R \leq R_2$. Take $f = \ln|z| = \ln r$, $g = u$. Get:

$$0 = \int_{\partial D} \left[(\ln r)\frac{\partial u}{\partial \vec{n}} - u \frac{\partial}{\partial \vec{n}}(\ln r)\right] ds$$



on C_R, $\vec{n}$ = radial

on C_{R_1}, $\vec{n}$ = $-$radial

$$0 = (\ln R)\int_{C_R} \frac{\partial u}{\partial r}\,ds - \frac{1}{R}\int_{C_R} u\,ds$$

$$+ (\ln R_1)\int_{C_{R_1}} \left(-\frac{\partial u}{\partial r}\right)_{R_1} ds + \frac{1}{R_1}\int_{C_{R_1}} u\,ds$$

$$\{\text{introduce } 0 \leqq \theta \leqq 2\pi\}$$

$$0 = (\ln R)\int_{C_R} \frac{\partial u}{\partial r}\,ds - \int_0^{2\pi} u(Re^{i\theta})\,d\theta$$

$$-(\ln R_1)\int_{C_{R_1}} \left(\frac{\partial u}{\partial r}\right)_{R_1} ds + \int_0^{2\pi} u(R_1e^{i\theta})\,d\theta$$

$$\Downarrow$$

$$\int_0^{2\pi} u(Re^{i\theta})\,d\theta = (\ln R)\int_{C_R} \left(\frac{\partial u}{\partial r}\right) ds$$

$$+ \int_0^{2\pi} u(R_1e^{i\theta})\,d\theta$$

$$-(\ln R_1)\int_{C_{R_1}} \left(\frac{\partial u}{\partial r}\right)_{R_1} ds$$

Recall p. (1) bottom. We have:

$$\int_{C_R} \frac{\partial u}{\partial \vec{n}}\, ds + \int_{C_{R_1}} \frac{\partial u}{\partial \vec{n}}\, ds = 0$$

$$\Downarrow$$

$$\int_{C_R} \left(\frac{\partial u}{\partial r}\right)_R ds = \int_{C_{R_1}} \left(\frac{\partial u}{\partial r}\right)_{R_1} ds .$$

Think of R_1 as frozen, R as <u>movable</u>.

<u>Theorem</u> $\boxed{u \text{ harmonic on } R_1 \leq |z| \leq R_2}$

$$\int_0^{2\pi} u(Re^{i\theta})\,d\theta = \alpha(\ln R) + \beta ,$$

where

$$\alpha = \int_{C_{R_1}} \left(\frac{\partial u}{\partial r}\right)_{R_1} ds = \int_{C_R} \left(\frac{\partial u}{\partial r}\right)_R ds$$

β = some constant.

See Ahlfors p. 165 eq (61).

It is important to note the <u>significance</u> of coefficient

$$q = \int_{C_{R_1}} \left(\frac{\partial u}{\partial r}\right)_{R_1} ds = \int_{C_R} \frac{\partial u}{\partial \vec{n}} ds .$$

<u>Cor</u>

Let u be harmonic on $\{|z|<A\}$ and continuous on $\{|z|\le A\}$. Then

$$\int_0^{2\pi} u(Re^{i\theta})\,d\theta = 2\pi u(0) , \quad 0<R\le A .$$

<u>Proof</u>

By continuity, it suffices to treat $0<R<A$. Take R_1 very close to 0, R_2 very close to A, and $R_1 < R < R_2$.

Apply ① bottom to $\{|z|\le R\}$. Get $q = 0$.

Hence

$$\int_0^{2\pi} u(Re^{i\theta})\,d\theta = \beta .$$

Let $R_1 \to 0$ and $R \to 0$. Clearly $\beta = 2\pi u(0)$.

∎

Needless to say, this proof of the mean value property is different than what we originally used.

Lec 7, p. 20

Original Fact (Mean Value Property)

Let $u(z)$ be harmonic on $\{|z|<R\}$ and continuous on $\{|z|\leq R\}$. Then

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})\,d\theta .$$

Proof

Write $u(z) = \mathrm{Re}\,[f(z)]$, f analytic on $\{|z|<R\}$.

By the CIF,

$$f(0) = \frac{1}{2\pi i}\oint_{|z|=r} \frac{f(z)}{z}\,dz .$$

Write $z = re^{it}$. Get

$$f(0) = \frac{1}{2\pi i}\int_0^{2\pi} f(re^{it})\frac{1}{re^{it}}\, rie^{it}\,dt$$

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{it})\,dt .$$

Take real part of both sides.

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{it})\,dt .$$

By uniform continuity of $u(z)$ on $\{|z| \leq R\}$,

$$u(re^{it}) \rightrightarrows u(Re^{it})$$

on $[0, 2\pi]$ as $r \to R$. Therefore,

$$\lim_{r \to R} \int_0^{2\pi} u(re^{it})\,dt = \int_0^{2\pi} u(Re^{it})\,dt .$$

Hence, after multiplying thru by $1/2\pi$,

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{it})\,dt . \quad \blacksquare$$

<u>Fact</u> (Poisson integral <u>representation</u>)

Let $u(z)$ be harmonic on $\{|z| < R\}$ and <u>continuous</u> on $\{|z| \leq R\}$. Let $|\xi| < R$.

Then:

$$u(\xi) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\left[\frac{Re^{i\theta} + \xi}{Re^{i\theta} - \xi}\right] u(Re^{i\theta})\,d\theta .$$

<u>Proof.</u>

Please note that we are assuming $u(z)$ harmonic on $\{|z| < R\}$ <u>and</u> <u>continuous</u> on $\{|z| \leq R\}$.

The situation is very close to our earlier Poisson integral formula of Lecture 8, pp. 25 + 26!

We just have to "touch up" one minor point. For this purpose, select any z in $\{|z|<R\}$. Take $\varepsilon>0$ very small. Let $R_1 = R-\varepsilon$. By Lecture 8, pp. 25+26,

$$u(z)=u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left[\frac{R_1e^{i\phi}+z}{R_1e^{i\phi}-z}\right]u(R_1e^{i\phi})\,d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{R_1^2-r^2}{R_1^2+r^2-2R_1r\cos(\theta-\phi)}\,u(R_1e^{i\phi})\,d\phi .$$

$\varepsilon \to 0$

Let $R_1 \to R$ in the last formula. Remember that $u(R_1e^{i\phi}) \rightrightarrows u(Re^{i\phi})$ by uniform continuity. AT ONCE,

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-\phi)}\,u(Re^{i\phi})\,d\phi$$

for EACH z in $\{|z|<R\}$. Hence, also,

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left[\frac{Re^{i\phi}+z}{Re^{i\phi}-z}\right]u(Re^{i\phi})\,d\phi .$$

This is equivalent to what we wanted to prove. ∎

In Tuesday's lecture (Lec 20), we turned the Poisson integral representation around, studying

$$u(z) \equiv \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{e^{i\phi}+z}{e^{i\phi}-z}\right) h(e^{i\phi})\, d\phi$$

for a general REASONABLE bounded (real) function $h(e^{i\phi})$ on $\partial\mathcal{U}$.

We found:

A) u harmonic on $\mathcal{U}$

B) u bounded

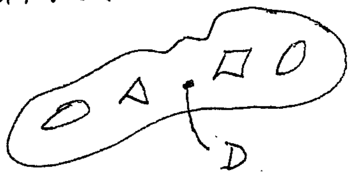
C) $\lim_{z\to\eta} u(z) = h(\eta)$ whenever h is <u>continuous</u> at η.

If h is, say, piecewise continuous on $\partial\mathcal{U}$, notice that u of the above sort must be <u>unique</u> BY THE P-L FORM OF THE MAX/MIN PRINCIPLE. Lec 20, p. 18.

As a preview of where we are headed, if time allows I will completely prove the following thm.

Theorem

Let $D \subsetneqq \mathbb{C}$ be a domain bounded by $N < \infty$ Jordan curves.



Let $h(\xi)$ be real, bounded, piecewise continuous on ∂D. The Dirichlet Problem can then be solved for h. I.e., there exists a unique bounded harmonic function u on D such that

$$\lim_{z \to \xi} u(z) = h(\xi)$$

at every point of continuity of h.

One possible approach will use the Perron method – developed in Ahlfors chap 6, sec 4. Another approach can be given via a TRICK + induction on N.

Important Example for pages 12 + 13:

Notice that $w = \frac{1+z}{1-z}$ maps $|z|<1$ onto $\{Re(w)>0\}$.
In fact,

$$w(0) = 1$$
$$w(e^{i\phi}) = i\left(\text{ctn}\,\tfrac{\phi}{2}\right), \quad 0<|\phi|\le\pi.$$

Form function $p(z) = Re\left(\frac{1+z}{1-z}\right)$ on $\{|z|<1\}$. The function $p(z)$ is obviously positive and harmonic. By construction,

$$\lim_{z\to\xi} p(z) = 0, \quad \xi \in \{|z|=1\} - \{1\}.$$

Let $u(z)$ be the Poisson integral (12) on $\{|z|<1\}$ for boundary function

$$h(e^{i\phi}) = \begin{cases} 1, & 0<\phi<\pi \\ -1, & -\pi<\phi<0 \end{cases} \quad \text{(say)}.$$

Clearly h is nicely piecewise continuous.

Though $u(z)+p(z)$ <u>appears</u> to solve the $\boxed{\text{same}}$ Dirichlet problem as $u(z)$, there is no contradiction to uniqueness because $p(z)$ is <u>UNBOUNDED</u>!

We now start some basic aspects of the famous

Schwarz Reflection Principle.

This will involve systematic use of Poisson-type integrals.

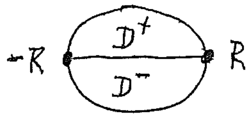
For the simplest example, let's take a disk

$$D = \{|z| < R\}.$$

Put

$$D^+ = D \cap \{Im(z) > 0\}$$
$$D^- = D \cap \{Im(z) < 0\}.$$

The Schwarz Reflection Principle will give us a NICE condition under which harmonic functions <u>on D^+</u> can be extended harmonically to all of <u>D</u>.



Theorem (Schwarz)

Let $u(z)$ be harmonic on D^+ and continuous on $D^+ \cup \partial D^+$. Suppose that

$$u(x) = 0 \quad \text{for } -R < x < R .$$

The function $u(z)$ then has exactly one harmonic extension to all of D; and this is

$$u_0(z) \equiv \left\{ \begin{array}{ll} u(z), & z \in D^+ \\ 0; & z = x \in (-R, R) \\ -u(\bar{z}), & z \in D^- \end{array} \right\} .$$

"reflection"

NOTE that this definition of u_0 makes perfectly good ~~sense~~ on $\bar{D}$ too.

$$\boxed{\text{u is understood to be REAL-VALUED}}$$

Proof

Define u_0 on $\bar{D}$ as above. Clearly u_0 is continuous on $\bar{D}$. Let $P(z)$ be the Poisson integral associated with boundary function $Q = u_0(Re^{i\theta})$. Write

$$P(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} \, u_0(Re^{i\theta}) \, d\theta .$$

$\boxed{\text{change θ to ϕ if you prefer}}$

When z is real and in $(-R, R)$, the integrand is clearly ODD with respect to θ. Hence,

$$P(x) = 0 \quad \text{for } -R < x < R .$$

By the THM on p. 20 (Lecture 20), we know that

$$\lim_{z \to \eta} P(z) = u_0(Re^{i\theta})$$

for every point $\eta = Re^{i\theta}$ along $|z| = R$.

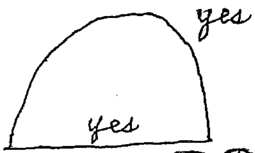
⟦ This uses the fact that $u(z)$ is continuous on $D^+ \cup \partial D^+$. Hence likewise for $u_0(z)$ on $D \cup \partial D$. ⟧

We know that $P(z)$ is harmonic on $\underline{\underline{D}}$.
⟨THM on p. 20 again⟩

Look at the harmonic function

$$u(z) - P(z)$$

on $\underline{\underline{D^+}}$. Notice that $u(z) - P(z) \to 0$ as $z \to \partial D^+$.



By the MAX/MIN principle on p. 16 18 ~~of~~ Lec 20, we get

$$u(z) - P(z) \equiv 0 \quad \text{on } \underline{\underline{D^+}},$$

i.e.

$$u(z) \equiv P(z) \quad \text{on } \underline{\underline{D^+}}.$$

$P(z)$ thus gives a harmonic function on D which extends u out of D^+.

By elementary calculus, one checks that

$$P(z) = -P(\bar{z}) \quad \text{for } z \in D^-.$$

Indeed,

$$P(\bar{z}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2 - |z|^2}{|Re^{i\theta} - \bar{z}|^2} u_0(Re^{i\theta})\,d\theta$$

$$\{\text{put } \theta = -\phi\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2 - |z|^2}{|Re^{-i\phi} - \bar{z}|^2} u_0(Re^{-i\phi})\,d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2 - |z|^2}{|Re^{i\phi} - z|^2} \left(-u_0(Re^{i\phi})\right)d\phi$$

by the def of u_0

$$= -P(z).$$

Putting things together, we now see that

$$P(z) = u_0(z) \quad \text{for each } z \in D.$$

Page 17 top takes care of points along ∂D.

The function $u_0(z)$ thus truly gives a harmonic extension as promised.

NICE!

Suppose that $\tilde{u}_0(z)$ is another harmonic function on D that works. Form the harmonic function

$$g(z) = u_0(z) - \tilde{u}_0(z) \quad \text{on } D.$$

Find analytic function $H = \underline{g} + ir$ on D. Notice that

$$H'(z) = g_x - i g_y .$$

Since $g \equiv 0$ on $\underline{D^+}$, we get $H'(z) \equiv 0$ on D^+, hence on $\underline{D}$. This implies $\text{grad}(g) \equiv 0$ on D, hence $g \equiv$ constant. We thus get $g \equiv 0$ on D. ∎

Compare Lec 20, corollary 3.

Very closely related to this is a very beautiful fact.

<u>Theorem</u>

Let D be any domain in $\mathbb{C}$.
Let $h(z)$ be any continuous function on D.
Suppose that, for <u>each</u> $z_0 \in D$, there is some $\delta_0 > 0$ so that

$$h(z_0) = \frac{1}{2\pi}\int_0^{2\pi} h(z_0 + re^{i\theta})\,d\theta$$

for <u>all</u> $0 < r < \delta_0$. Then $h(z)$ is harmonic.

Proof

Let A be any disk whose closure $\bar{A}$ lies in D. It suffices to prove $h(z)$ is harmonic on A. Let $P(z)$ be the Poisson integral on A with boundary values agreeing with $h(z)$ along ∂A. Look at the fcn

$$k(z) = h(z) - P(z).$$

Notice that $k(z)$ is continuous on $\bar{A}$ and 0 along ∂A.

$\boxed{\text{Also } k \text{ satisfies local mean-value property too on } A.}$

Claim: $k(z) \equiv 0$ on A.

Proof

Suppose, eg, that k has a POSITIVE max M on $\bar{A}$. It must be assumed somewhere in A. Let $k(z_0) = M$. Get δ_0 as in thm. WLOG $|z - z_0| \leqq \delta_0$ is contained in A. For $0 < r < \delta_0$, we thus get

$$M = k(z_0) = \frac{1}{2\pi}\int_0^{2\pi} k(z_0 + re^{i\theta})\, d\theta.$$

Of course, $k(z_0 + re^{i\theta}) \leqq M$. As in Lecture 20, we get $k(z_0 + re^{i\theta}) \equiv M$ for all θ and all $0 \leqq r < \delta_0$. In other words,

$$A_1 \equiv \{z \in A : k(z) = M\} \text{ is } \underline{\text{open}}.$$

By continuity,

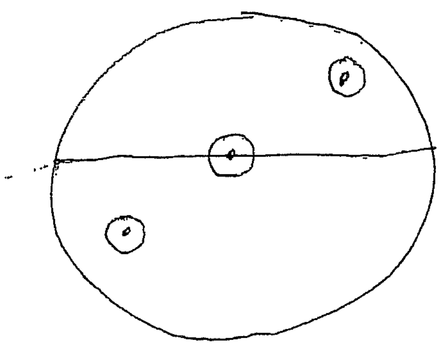
$$A_2 \equiv \{z \in A : k(z) < M\} \text{ is also open}.$$

Since $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \phi$, one of A_1 and A_2 must be <u>empty</u>. Since A_1 is nonempty, we get $A_2 = \phi$, which contradicts $h(z) = 0$ along ∂A. ■

We thus have $h(z) \equiv P(z)$ on A.
Since $P(z)$ is harmonic on A, we are done.

■

The fcn. u_0 in Schwarz's reflection theorem could have been proven harmonic in this way. Points $z_0 \in (-R, R)$ are handled by mere symmetry (in an obvious way).



"3 cases for z_0"

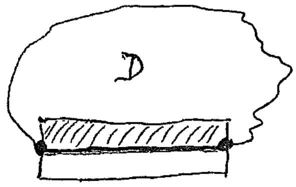
Since time is getting short, I'll now consider a *simplified* definition of what is called a "one-sided free boundary arc" of D (a domain).

Let D be a domain $\subseteq \mathbb{C}$.
Let $\gamma : \{a < x < b\}$ be part of ∂D. Note that I am using an open arc!

We say that γ is a *one-sided free boundary arc* of D when there exists a small $h > 0$ such that

$$(a,b) \times (0,h) \subseteq D$$
$$(a,b) \times (-h,0) \subseteq \mathbb{C} \sim D$$

OR the reverse.



Similar definitions hold for line segments γ *in general position* and for circular arcs.

> Note: $D = \{|z|<1,\ y>0\}$.
> By our definitions $\{-\frac{n}{n+1} < x < \frac{n}{n+1}\}$ is a 1-sided free boundary arc for EACH $n \geq 1$. NOT $n = \infty$.

Let B be a general domain $\subseteq$ upper half-plane $\mathcal{H}$. Let (a,b) be a one-sided free boundary arc of B.

See figure on 22 with "B" in place of D!

Let

$$D = B \cup \{a<x<b,\ y=0\} \cup \{\text{conjugate of } B\}.$$

By using "h.", D is clearly a domain.

Theorem

Let $v(z)$ be harmonic on B. Let $\lim_{z\to(a,b)} v(z) = 0$. Then $v(z)$ automatically extends to a harmonic function $v_0(z)$ on D; in fact, one can take

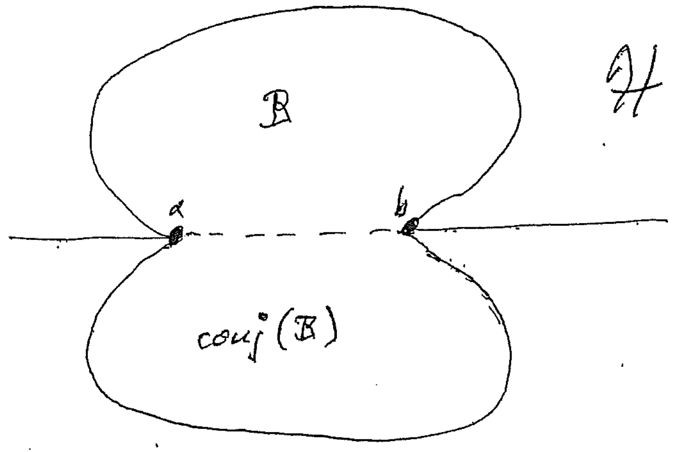
$$v_0(z) = \begin{cases} v(z), & z \in B \\ 0, & a<x<b,\ y=0 \\ -v(z), & z \in \text{conjugate of } B \end{cases}.$$

Proof

Use "h" and either the thm on p. 19 or p. 16 ▨

Let $\mathbb{B}$ be a (general) domain $\subseteq$ upper half-plane $\mathcal{H}$.
Let (a,b) be a 1-sided free boundary arc of $\mathbb{B}$.

Let
$$D = \mathbb{B} \cup \{a<x<b,\ y=0\} \cup \{\text{conjugate of } \mathbb{B}\}.$$



$\{D$ = the big domain$\}$

Remember the
$$(a,b) \times (0,h) \subseteq B$$
$$(a,b) \times (-h,0) \subseteq \mathbb{C} \sim B$$
idea from before.

BABY FACT (baby reflection principle)

Let $A(z)$ be analytic on D.
Let $A(x)$ = real for $a<x<b$.
Then $A(z) \equiv \overline{A(\bar{z})}$ on D.

Pf

$g(z) = A(z) - \overline{A(\bar{z})}$ analytic on D. But $g(x) = 0$, $a<x<b$. Nonisolated zeros. So $g \equiv 0$ on D. ∎

Theorem (Schwarz reflection for analytic functions).

Let $f(z)$ be analytic on B. Let $\lim_{z \to (a,b)} \mathrm{Im}[f(z)] = 0$. Assume nothing about $\mathrm{Re}[f(z)]$. Then, $f(z)$ automatically continues to an analytic function $f_0(z)$ on D which satisfies

$$f_0(\bar{z}) = \overline{f_0(z)} .$$

Proof

Let $\mathcal{B}$ be the disconnected open set

$$B \cup \text{conjugate}(B) .$$

Define $F(z)$ on $\mathcal{B}$ by writing

$$F(z) = \left\{ \begin{array}{ll} f(z), & z \in B \\ \overline{f(\bar{z})}, & z \in \text{conjugate}(B) \end{array} \right\} .$$

$F(z)$ is clearly analytic on $\mathcal{B}$.

To finish the proof, we must verify that
(☆) $F(z)$ agrees with an analytic function on ~~the~~ box $(a, b) \times (-h, h)$.

Write $f(z) = u(z) + iv(z)$ on B.

Recall p. 23 [redacted] THM. (or 16) By using that theorem over and over, we clearly get a harmonic extension $v_0(z)$ of $v(z)$ on the box $(a,b)\times(-h,h)$.

$$\left\{ \begin{array}{l} v_0(x-iy) = -v(x+iy) \\ \text{for } y>0 \end{array} \right\}$$

Since the box $(a,b)\times(-h,h)$ is simply-connected, we can write

$$v_0(z) = \operatorname{Im}[G(z)]$$

for <u>some</u> analytic function G on $(a,b)\times(-h,h)$.

We suspect that $F(z) = G(z) + \{\text{a real constant}\}$ on $(a,b)\times\{0<|y|<h\}$. Recall the simple format of $v_0(z)$. Clearly:

$$\operatorname{Im}[F(z)-G(z)] = v(z)-v(z) = 0 \quad \text{on } (a,b)\times(0,h)$$
$$\operatorname{Im}[F(z)-G(z)] = v_0(z)-v_0(z) = 0 \quad \text{on } (a,b)\times(-h,0).$$

We thus get

$$F(z) = G(z) + A_1 \quad \text{on } (a,b)\times(0,h)$$
$$F(z) = G(z) + A_2 \quad \text{on } (a,b)\times(-h,0)$$

with $A_1, A_2 \in \mathbb{R}$. But <u>why</u> is $A_1 = A_2$?

Notice that:

$$F(x+i\varepsilon) - F(x-i\varepsilon) = F(x+i\varepsilon) - \overline{F(x+i\varepsilon)} \quad \text{(by def)}$$
$$= 2i \operatorname{Im} F(x+i\varepsilon) = 2i\,v(x+i\varepsilon).$$

This gives

$$G(x+i\varepsilon) - G(x-i\varepsilon) + (A_1-A_2) = 2i\,v(x+i\varepsilon).$$

key

By letting $\varepsilon \to 0$, we discover that
$$A_1 = A_2 = A \text{ (say)}. \quad \text{real}$$

We thus have:
$$F(z) = G(z) + A \quad \text{on} \quad (a,b) \times \{0 < |y| < h\}.$$

The function $G(z) + A$ is quite interesting.
First of all, it gives an analytic continuation of our original $f(z)$ to all of $(a,b) \times (-h,h)$.

$F = f$ on B

Secondly, $G(z) + A$ is analytic on $(a,b) \times (-h,h)$. For $y \neq 0$, it agrees with $F(z)$. We have thus fulfilled 25 ☆. Done! ∎

Here is a slightly <u>different</u> way of concluding.

First get $v_0(z) = \mathrm{Im}[G(z)]$ on $(a,b) \times (-h,h)$ as before. Look at analytic fcn

as on 24 bottom
$$g(z) = G(z) - \overline{G(\bar{z})} \quad \text{on} \quad (a,b) \times (-h,h).$$

Clearly $g(x) = 0$ since $v_0(x) = 0$ by construction. Nonisolated zeros $\Rightarrow g(z) \equiv 0$. Hence:

$$(\star\star) \qquad G(\bar{z}) = \overline{G(z)}.$$

On $(a,b) \times (0,h)$, we have:
$$\mathrm{Im}[F(z) - G(z)] = v(z) - v(z) = 0.$$

Hence $F(z) = G(z) + A_1$ on $(a,b) \times (0,h)$, $A_1 \in \mathbb{R}$.
Thanks to $(\star\star)$, $\overline{F(z)} = \overline{G(z)} + A_1 \Longrightarrow$
$$F(\bar{z}) = G(\bar{z}) + A_1 \quad \text{on} \quad (a,b) \times (-h,0).$$

The function $G(z) + A_1$ thus fulfills 25 ☆.
Done! ∎