Fri 16 Sep 2011

Lecture 2

Today we will begin discussing some explicit functions. Before I do that, I want to note 3 very general facts that will come in handy. (For Now / later.)

We saw in lecture 1 that $f(z)$ analytic on $D \Rightarrow$ Re f and Im f are C^∞ functions of (x, y) on D.

This means that all basic theorems of undergraduate complex analysis based on Green's theorem are true.

$\boxed{\text{since } f \text{ analytic}, \ f = u + iv \Rightarrow u_x = v_y, \ u_y = -v_x}$

E.G.

Let D be a domain bounded by $m \geq 1$ piecewise smooth (C^1) simple closed curves.

{connectivity m}

Let $f(z)$ be analytic on $D \cup \partial D$, i.e. on a slightly bigger domain.

We then have:

$$0 = \int_{\partial D} f\,dz = \oint_{C_0} f\,dz + \oint_{C_1} f\,dz + \cdots + \oint_{C_{m-1}} f\,dz$$

(C.I.T.) $\boxed{\text{Directly by Green's thm}}$

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0}\,dz \quad \text{for} \quad z_0 \in D$$

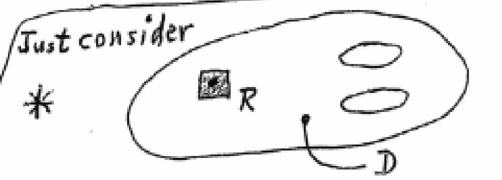
(C.I.F.) *

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_{\partial D} \frac{f(z)}{(z - z_0)^{k+1}}\,dz \,, \quad z_0 \in D$$

(C.I.F. for $f^{(k)}$) *

Ahlfors has more fancy versions, but we do <u>not</u> need them!

Just consider



with $g \equiv \dfrac{f(z)}{z - z_0}$ say

That's the 1st general fact!

The 2nd concerns infinite series.

Let me first RECALL for you —

Weierstrass M-test

Given $E \subseteq \mathbb{C}$, $\sum_{n=1}^{\infty} u_n(z)$, $z \in E$.

$u_n(z)$ complex-valued.

Assume $|u_n(z)| \leq M_n$ for $z \in E$

and

$$\sum_{n=1}^{\infty} M_n < \infty.$$

Then:

$\sum_{n=1}^{\infty} u_n(z)$ conv. uniformly on E.

I.E.

$$\boxed{\left| \sum_{n>N}^{\infty} u_n(z) \right| < \varepsilon \text{ for } \underline{\text{all}}\ z \in E, \text{ whenever } N \geq N_\varepsilon}$$

NOTE HERE THAT E MIGHT BE AN INTERVAL ALONG $\mathbb{R}$, AND THE u_n REAL-VALUED.

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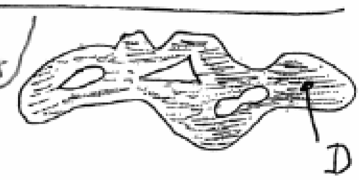
for analytic functions



Weierstrass Conv. Thm.

let D be a domain in $\mathbb{C}$.

let $u_n(z)$ be ANALYTIC on D.

let $\sum_{n=1}^{\infty} u_n(z)$ converge uniformly on each compact subset of D.

let $T(z) = \sum_{n=1}^{\infty} u_n(z)$.

THEN:

(a) $T(z)$ is automatically analytic on D;

(b) $T'(z) = \sum_{n=1}^{\infty} u_n'(z)$, with unif conv on compact subsets of D;

(c) likewise for
$$T^{(k)}(z) = \sum_{n=1}^{\infty} u_n^{(k)}(z).$$

Sketch of Proof

It suffices to prove (a) and (b).

It suffices to treat the case where D is a disk. Say

$$D = \{ |z| < R \}.$$

Consider $0 < \varepsilon < h$ SMALL.

Put

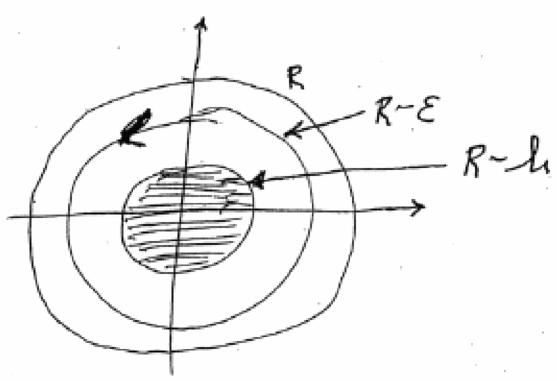
$$S_N(z) \equiv \sum_{n=1}^{N} u_n(z).$$

$\boxed{\text{standard def}}$

For $|z| \leq R - h$, we have:

$$(\alpha) \quad S_N(z) = \frac{1}{2\pi i} \oint_{|\xi| = R - \varepsilon} \frac{S_N(\xi)}{\xi - z}\, d\xi$$

$$(\beta) \quad S_N'(z) = \frac{1}{2\pi i} \oint_{|\xi| = R - \varepsilon} \frac{S_N(\xi)}{(\xi - z)^2}\, d\xi.$$



Cauchy integral formulae

We know that $S_N(w) \rightrightarrows T(w)$ on compact subsets of D.

$\boxed{\rightrightarrows \text{ means UNIFORM}}$

Hence, $T(w)$ is continuous on D.

In ⊛, $\dfrac{S_N(\xi)}{\xi - z} \rightrightarrows \dfrac{T(\xi)}{\xi - z}$ for each z.

So, we get [by ⊛]

$$T(z) = \frac{1}{2\pi i} \oint_{|\xi| = R - \varepsilon} \frac{T(\xi)}{\xi - z} \, d\xi$$

for each $|z| \leq R - h$. Let $h \to \varepsilon^+$.

Get:

$$T(z) = \frac{1}{2\pi i} \oint_{|\xi| = R - \varepsilon} \frac{T(\xi)}{\xi - z} \, d\xi$$

for all $|z| < R - \varepsilon$. OK!

One checks with a straightforward difference quotient that the RHS is analytic for $|z| < R - \varepsilon$; in fact,

$$(\text{RHS})' = \frac{1}{2\pi i} \oint_{|\xi| = R - \varepsilon} \frac{T(\xi)}{(\xi - z)^2} \, d\xi .$$

We conclude that $T(z)$ is <u>analytic</u> on each $\{|z| < R - \varepsilon\}$, hence on all of D.

This proves (a)!

To prove (b), we shall use β. ← p. 5
We must show that

$$S_N'(z) \rightrightarrows T'(z)$$

on <u>compact</u> subsets of D.

It suffices to consider a compact set $\{|z| \leq R - h\}$ and to utilize β with some tiny $\varepsilon < h$. We do so.

One easily checks

$$\frac{S_N(\xi)}{(\xi - z)^2} \rightrightarrows \frac{T(\xi)}{(\xi - z)^2} \quad \text{as } N \to \infty$$

with uniformity in both z and ξ.

We get:

$$\frac{1}{2\pi i} \oint_{|\xi| = R - \varepsilon} \frac{S_N(\xi)}{(\xi - z)^2} d\xi \rightrightarrows \frac{1}{2\pi i} \oint_{|\xi| = R - \varepsilon} \frac{T(\xi)}{(\xi - z)^2} d\xi$$

THIS EQUALS $S_N'(z)$

for $|z| \leq R-h$. By β, thus,

$$S_N'(z) \rightrightarrows \frac{1}{2\pi i} \oint_{|\xi|=R-\varepsilon} \frac{T(\xi)}{(\xi-z)^2}\, d\xi .$$

But, $T(z)$ is now <u>known</u> to be analytic on D. By the Cauchy integral formula (for T) we thus have

$$S_N'(z) \rightrightarrows T'(z),$$

as required. $\blacksquare$

Now for the 3^{rd} very general fact.

LEMMA (VERY useful in its own right).

Let $f(t,z)$ be complex-valued and CONTINUOUS for $a \leq t \leq b$, $z \in D$, D = a domain. Suppose that, for each fixed t, $f(t,z)$ is analytic in z.

Then:

(A) $\frac{\partial f}{\partial z}(t,z)$ has continuity and analyticity properties akin to $f(t,z)$;

(B) $$\lim_{h \to 0} \frac{f(t,z+h) - f(t,z)}{h} = \frac{\partial f}{\partial z}(t,z)$$

uniformly on compact subsets of $[a,b] \times D$.

Sketch of Proof

We want to work with some suitable compact subsets of $[a,b] \times D$ that can easily be pieced together. For this purpose, we can use:

$[a,b] \times$ closed <u>disk</u>

$\uparrow$ say $\{|z| \leqq R\} \subseteq D$.

OK, then! Notice that (CIF)

$$f(t,z) = \frac{1}{2\pi i}\oint_{|\xi|=R} \frac{f(t,\xi)}{\xi - z}\, d\xi \quad \left\{\begin{matrix} \text{each } t \\ |z| \leqq R - \delta \end{matrix}\right\}$$

~~so~~

$$\frac{f(t,z+h) - f(t,z)}{h} = \frac{1}{2\pi i \cancel{h}}\oint_{|\xi|=R} f(t,\xi)\frac{\cancel{h}}{(\xi - z)(\xi - z - h)}\, d\xi$$

$$\delta = \text{tiny} \quad \left\{\begin{matrix} \text{for} & |z| \leqq R - \underline{3}\delta \\ & 0 < |h| \leqq \delta \\ & a \leqq t \leqq b \end{matrix}\right\}.$$

Clearly:

$$\text{R.H.S.} \Longrightarrow \frac{1}{2\pi i}\oint_{|\xi|=R} \frac{f(t,\xi)}{(\xi - z)^2}\, d\xi \quad \text{as } h \to 0$$

In other words, by CIF,

KEY POINT

$$\frac{f(t, z+h) - f(t, z)}{h} \rightrightarrows \frac{\partial f}{\partial z}(t, z)$$

as $h \to 0$ over $\{a \le t \le b,\ |z| \le R - \underline{3\delta}\}$.

Keep in mind here that $\delta > 0$ is arbitrary and $\{|z| \le R\}$ depicts an arbitrary closed disk within D. (B) follows!

Since the LHS is continuous wrt (t, z), the limit function $\frac{\partial f}{\partial z}(t, z)$ is necessarily continuous wrt (t, z) too. This fact suffices to prove (A).

$\blacksquare$

VERY USEFUL

CORollary (Leibnitz's rule).

If we write (in the LEMMA)

$$G(z) = \int_a^b f(t, z)\,dt,$$

then $G(z)$ is analytic on D and

$$G'(z) = \int_a^b \frac{\partial f}{\partial z}(t, z)\,dt.$$

Proof

Just work on D compacta and form the difference quotient (noting p. (9) B)

$$\frac{G(z+h) - G(z)}{h}. \quad \blacksquare$$

Compare Ahlfors §4.2.3 (exercise 6)

WE NOW GO TO EXPLICIT FCNS...

FACT

Let $D = \mathbb{C} \sim \mathbb{Z}$. Then:

$$\pi \operatorname{ctn} \pi z = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right)$$

for $z \in D$.

$\boxed{\text{We shall give } \underline{2} \text{ proofs of this fact.}}$

For the first proof, we begin with a simple lemma (almost an observation).

Lemma

Let

$$F(z) = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right), \quad z \in D.$$

Then:

(a) $F(z)$ converges uniformly on D compacta;

(b) $F(z+1) = F(z)$;

(c) $|F(x+iy)| \leq 10$ for $|y| \geq 1000$ (say);

(d) at each integer, $F(z)$ has a simple pole of residue 1.

THINK BIG!

Proof of lemma

(a) follows easily from the Weierstrass M-test.
In fact, we get:

$$(\star) \qquad F(z) = \frac{1}{z} + \lim_{N\to\infty} \sum_{1\le |n| \le N} \left(\frac{1}{z-n} + \frac{1}{n}\right)$$

$$= \lim_{N\to\infty} \sum_{|n|\le N} \frac{1}{z-n} .$$

This last formula shows that

$$F(z+1) - F(z) = \lim_{N\to\infty} \sum_{|n|\le N} \left(\frac{1}{z+1-n} - \frac{1}{z-n}\right)$$

$$= \lim_{N\to\infty} \left[\frac{1}{z+1+N} + \frac{1}{z+N} + \cdots + \frac{1}{z+1-N} - \frac{1}{z+N} - \cdots - \frac{1}{z+1-N} - \frac{1}{z-N} \right]$$

$$= \lim_{N\to\infty} \left(\frac{1}{z+N+1} - \frac{1}{z-N} \right) = 0 ,$$

proving (b). To prove (d), it suffices by (b) to look at the situation near $z=0$. But, there, we simply use the original formula for $F(z)$ and the Weierstrass M-test.

It remains to prove (c).

[Note: the "10" is not very important.]

By (b), it suffices to treat $\underline{\underline{0 \leq x \leq 1}}$, $|y| \geq 1000$.

Take $y > 0$; the case of negative y is similar.

Let $L = [\![y]\!]$; thus $L \leq y < L+1$.

$$F(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{z-k} + \frac{1}{z+k} \right)$$
$$= \frac{1}{z} + \sum_{k=1}^{L} \left(\frac{1}{z-k} + \frac{1}{z+k} \right) + \sum_{k=L+1}^{\infty} \left(\frac{1}{z-k} + \frac{1}{z+k} \right).$$

For $k \geq 1$, notice that $\quad z = x + iy$

$$|z-k| \leq |z+k| \longleftarrow \text{geom obvious!}$$
$$|z+k| \geq y, \quad |z-k| \geq y; \quad |z| \geq y, \quad |z| \leq 1+y.$$

Therefore,

$$|F(z)| \leq \frac{1}{y} + \sum_{k=1}^{L} \frac{2}{y} + \sum_{k=L+1}^{\infty} \left| \frac{2z}{z^2-k^2} \right|$$

$$|F(z)| \leq \frac{2L+1}{y} + 2(y+1) \sum_{k=L+1}^{\infty} \frac{1}{|z-k||z+k|}$$

$$|F(z)| \leq \frac{2L+1}{y} + 2(y+1) \sum_{k=L+1}^{\infty} \frac{1}{|z-k|^2}$$

$$|F(z)| \leq \frac{2y+1}{y} + 2(y+1) \sum_{k=L+1}^{\infty} \frac{1}{(k-x)^2}$$

$$|F(z)| \leq 3 + 2(y+1) \sum_{k=L+1}^{\infty} \frac{1}{(k-1)^2}$$

$$|F(z)| \leq 3 + 2(y+1) \sum_{m=L}^{\infty} \frac{1}{m^2}$$

$$\left\{ \text{but} \quad \sum_{m=L}^{\infty} \frac{1}{m^2} \leqq \int_{L-1}^{\infty} \frac{1}{t^2}\,dt \right\} \quad \text{by calculus}$$

$$|F(z)| \leqq 3 + 2(y+1)\frac{1}{L-1}$$

$$|F(z)| \leqq 3 + 2\frac{L+2}{L-1} \qquad (\text{but } L \geqq 1000)$$

$$|F(z)| \leqq 3 + 3 = 6 .$$

By p. 14 (★), we easily see that $F(z)$ is <u>odd</u> ; ie $F(-z) = -F(z)$.

Proof #1 of the FACT on p. 13

Let

$$g(z) = \pi \operatorname{ctn}(\pi z) - F(z) .$$

Notice that $g(z+1) = g(z)$. The only possible singularities of $g(z)$ are the points $\{0, \pm 1, \pm 2, \ldots\}$. Near $z = 0$, we have

$$\pi \operatorname{ctn}(\pi z) = \pi \frac{\cos(\pi z)}{\sin(\pi z)} = \frac{1}{z} + [\text{analytic}]$$

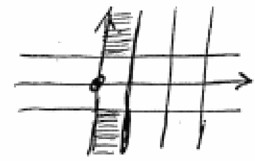
$$F(z) = \frac{1}{z} + [\text{analytic}]$$

$$\Rightarrow \quad g(z) = [\text{analytic}] .$$

By periodicity, we conclude that $g(z)$ is an <u>entire</u> function. (Entire fcn $\equiv$ analytic on $\mathbb{C}$.)

To prove $g(z)$ is <u>uniformly</u> <u>bounded</u>, it suffices to prove this for $0 \leq x \leq 1$. Or, for

$$\{0 \leq x \leq 1, \ |y| \geq 1000\}.$$



F is handled by Lemma, part (c). $\pi \operatorname{ctn}(\pi z)$ is handled by a trivial calculation. (write it using $e^{i\pi z}$)

By Liouville's thm, $g(z) \equiv$ constant.

But $g(z)$ is an <u>odd</u> function. Therefore, $g(z) \equiv 0$. ∎

Proof #2 of FACT

We use a Fourier series on $[-\pi,\pi]$! Take

$$f(x) = \cos(\alpha x) \,, \quad \alpha \neq \text{integer}$$

$$\boxed{\tfrac{1}{2}\left(e^{i\alpha x}+e^{-i\alpha x}\right)}$$

and write

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \,, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}\,dt \,.$$

Get:

$$c_n = \frac{1}{2\pi}\left[\frac{\sin(\pi n+\pi\alpha)}{n+\alpha} + \frac{\sin(\pi n-\pi\alpha)}{n-\alpha}\right]$$

$$c_n = \frac{(-1)^{n+1}}{\pi}\,\frac{\alpha}{n^2-\alpha^2}\,\sin(\pi\alpha) \,.$$

So,

$$\cos(\alpha x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}\alpha\sin(\pi\alpha)}{\pi(n^2-\alpha^2)}\, e^{inx}$$

for $-\pi \leq x \leq \pi$. Let $x=\pi$ to get

$$\cos(\pi\alpha) = \sum_n \frac{(-1)\alpha\sin(\pi\alpha)}{\pi(n^2-\alpha^2)}$$

$$= \sum_n \frac{\alpha\sin(\pi\alpha)}{\pi(\alpha^2-n^2)}$$

$$\Downarrow$$

$$\pi\,\mathrm{ctn}\,\pi\alpha = \sum_{n=-\infty}^{\infty} \frac{\alpha}{\alpha^2-n^2} = \frac{1}{\alpha} + 2\sum_{n=1}^{\infty}\frac{\alpha}{\alpha^2-n^2}$$

clearly

$\Downarrow$

$$\pi \operatorname{ctn} \pi \alpha = \frac{1}{\alpha} + \sum_{n=1}^{\infty} \left(\frac{1}{\alpha - n} + \frac{1}{\alpha + n} \right) .$$

In forming this Fourier series development, x is real, but α can be complex (there is no problem). ∎

By the way,

$$\operatorname{ctn}(z) \to \mp i$$

exponentially fast as $y \to \pm\infty$. This property is useful to remember.

Indeed, for $y > 0$ large, note that:

$$\operatorname{ctn}(z) + i = i\,\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} + i$$

$$= i\left(\frac{2e^{iz}}{e^{iz} - e^{-iz}}\right)$$

$$= \frac{2i}{1 - e^{-2iz}} = \frac{2i}{1 - e^{-2ix}e^{2y}}\,;$$

hence,

$$|\operatorname{ctn}(z) + i| \leq \frac{2}{e^{2y} - 1}\,.$$

OK

ADDENDUM

Extra Review Sheet for the Start of Komplex Analys D

Ahlfors, 3rd edition:

Read on your own —

Chapter 1. Note §1.2.4 (28) for later use.
CHAP 1 SECTION 2 .

Chapter 2. Pay careful attention to the PROOF in §2.2.4. You should omit Lucas' Thm in §2.1.3. You can also omit §2.2.5.

; then

Chapter 3. Section 1, on Elementary Point Set Topology, with which you should already have a general familiarity.*
Plus Section 2; §3.2.1; §3.2.2; and first 4 paragraphs of §3.2.3. STOP HERE.

* We do not use every tiny detail — just the main properties.