Lecture 25

Fri, 9 Dec 2011

Today's lecture will be half some new math, then half some review (of topics for the exam).

Since time is limited, in part of the new things, I will be a bit sketchy.

Topic A:

Recall HFT for an entire fcn of order $\rho < \infty$.

$$f(z) = z^m \exp[Q(z)] \prod_{\text{zeros}} E\left(\frac{z}{a_n}; p\right)$$

$p = [\![\rho]\!]$, Q = polynomial, degree $\leq \rho$

know $\sum \frac{1}{|a_n|^{p+1}} < \infty$ since $n(r) = O(r^{\rho+\varepsilon})$

Recall too "Ahlfors' key estimate"

$$\ln|E(u;p)| \leq (2p+1)|u|^{p+1}, \quad u \in \mathbb{C}.$$

Ahlfors p. 209 or Lec 10, p. 2.

Thm 1 (from before; Lec 9, pp. 19, 21)

Suppose f is entire, order $\rho < \infty$.
Suppose $f(z) \neq 0$ and $f(z) \not\equiv$ constant.
For every $A \neq 0$, the equation $f(z) = A$ has at least one root.

Proof

Apply HFT. No zeros. Get

$$f(z) = e^{Q(z)} .$$

Since $f \not\equiv$ const, Q is a polynomial of degree ≥ 1.
Just solve $Q(z) = \log A$ (any branch) by fund. thm. of algebra. ∎

If I had about 3 more lectures, I would use Schwarz reflection (repeatedly) and conformal mapping to develop a famous theorem of Picard. Compare Ahlfors p. 307.

I would stress that this can be done with a strong geometric flavor as, for instance, in Rudin's book, *Real and Complex Analysis*. [Ahlfors uses the elliptic modular function. I prefer a more geometric approach. ∎

See also: Lang's book on <u>Complex Analysis</u>. Less rigorous!

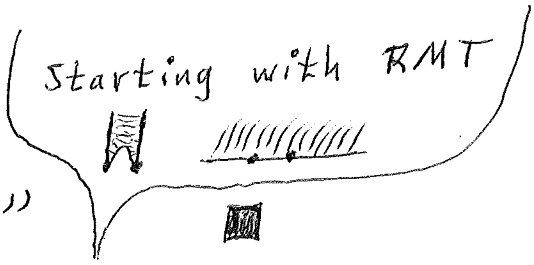
<u>Theorem 2</u> (the little Picard theorem).

Suppose $f(z)$ is entire, no assumption about order. Suppose $f(z) \neq 0$. Suppose $f(z) \neq b$ for some complex number $b \neq 0$.

THEN:

$$f(z) \equiv \text{constant}.$$

<u>Proof</u>



"Give me 3 more lectures!!"

THIS IS A VERY FAMOUS THEOREM THAT <u>ALL</u> STUDENTS OF COMPLEX ANALYSIS SHOULD KNOW.

$$\boxed{\begin{array}{l} f \text{ entire} \\ f(z) \neq a, b \end{array}} \Longrightarrow \boxed{f \equiv \text{constant}}$$

HFT proves this theorem very easily <u>provided</u> $\rho(f) < \infty$.

Corollary

Thm 1 is true even if $\rho = +\infty$.

Proof

$f(z) \neq 0$. Suppose $f(z) \sim A \neq 0$. Just apply Thm 2.
Get $f \equiv$ const. Contrad! ▮

Topic B:

A bit more on *identical vanishing*.

Recall Lec 22, p. 15 for bounded analytic fcns $A(z)$. The proof with

$$B(z) = \prod_{j=0}^{N-1} A(\omega^j z)$$

was very nice. Even "cute".

One wonders if there exists an alternate, more fundamental, "less slick", proof.

To attack this, we recall Schwarz reflection for harmonic and analytic functions. {Lec 21, pp. 16, 23, 25.}

Schwarz Refl. = VERY IMPORTANT!!

Let D be either

$$\{ |z| < h, \ y > 0 \} \qquad \text{half-disk}$$

or

$$\{ -h < x < h, \ 0 < y < h \} \qquad \text{rectangle.}$$

THM (identical vanishing result)

Let $f(z)$ be analytic on D. Assume that

$$\lim_{z \to \xi} f(z) = 0 \qquad (z \in D)$$

for each $\xi \in (-h, h)$. Then $f(z) \equiv 0$.

Proof

Let D_0 be the corresponding rectangle with $h \hookrightarrow \frac{h}{2}$.

By defining $f(x) = 0$ for $-h < x < h$, we clearly get a "bigger" function $f(z)$ that is continuous on $\overline{D_0}$. Since $\overline{D_0}$ is compact, this "bigger" f is uniformly continuous on $\overline{D_0}$.

Look at $\operatorname{Im} f(z)$ on D_0 as in the Schwarz refl principle for analytic fcns. The function $f(z)$ thus continues to an analytic function $F(z)$ on

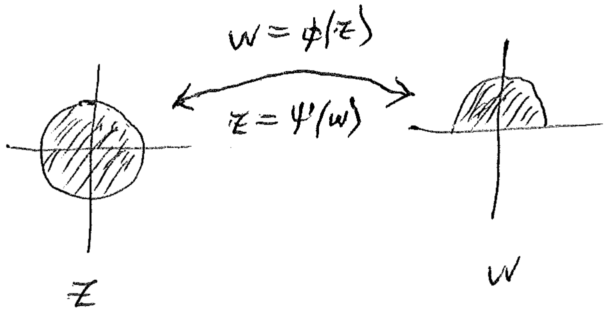
$$D_0 \cup \left(-\frac{h}{2}, \frac{h}{2}\right) \cup \text{conjugate}(D_0).$$

$\square$

But, $F(x) = 0$ on $(-\frac{h}{2}, \frac{h}{2})$. Get nonisolated zeros. Hence $F(z) \equiv 0$. Hence $f(z) \equiv 0$. ▩

Clearly the foregoing theorem can then be modified under <u>lots</u> of auxiliary conformal mappings (e.g. LF maps).

It is a standard fact in elem complex analysis that $|z| < 1$ can be mapped to $\{|w| < 1, \; Im(w) > 0\}$ by a relatively simple-looking conformal map $w = \phi(z)$, $z = \Psi(w)$.



⟨ Details = exercise. ⟩

By spinning $|z| < 1$ first, one can assume that a little arc on $\{|z| = 1\}$ gets mapped to a little line segment centered at $w = 0$.

Our old theorem in Lec 22, p. 15, is thus <u>re</u>-established by looking at $A[\Psi(w)]$ in the theorem on p. 5 above.

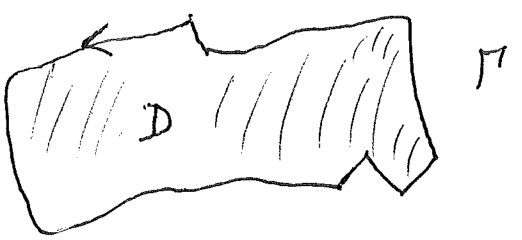
NOTICE THAT WE DID <u>NOT</u> USE "A" bounded!

The identical vanishing theorem on p. 5 is VERY important. Note it.
It is often applied together with Caratheodory's thm (Lec 22, p. 14).

Topic C :

I will now show how the Perron process, more specifically the Green's fcn $g(z;z_0)$, leads to a proof of a good portion of the Riemann mapping theorem. YES!

In a certain sense, the "main case" of RMT is when D is bounded by a piecewise C^1 Jordan curve Γ.



Recall that, by forming little exterior line segments, each pt of ∂D has a barrier.
Lec 24, p. 15, 17.

Form Green's fcn $g(z;z_0)$. (Lec 24, p. 21.)

$$g(z;z_0) = \ln\frac{1}{|z-z_0|} + \text{HARMONIC} \quad \text{near } z=z_0$$

g harmonic on $D \sim \{z_0\}$

$$\lim_{z\to\xi} g(z;z_0) = 0, \quad \text{each } \xi\in\Gamma$$

$$g(z;z_0) > 0 \quad \text{on } D$$

$\boxed{\text{WRITE } r = |z-z_0|}$

Locally, we can form analytic function

$$g + i g^* = g + ih$$

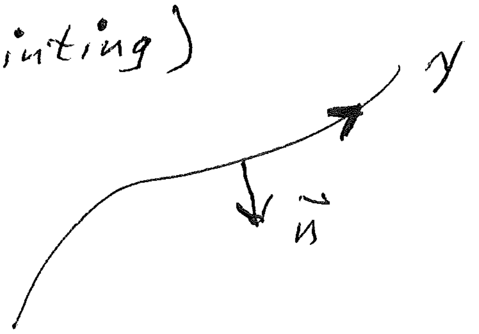
on $D\sim\{z_0\}$.

Recall the business about "local" $u+iv$ and

$$\frac{\partial u}{\partial \vec{n}} = \frac{\partial v}{\partial s} \quad \text{on smooth arc } \gamma,$$

as discussed Lec 21, pp. 1 ~ 4.

($\vec{n}$ = right pointing)



Recall too that grad $u = 0$ for $P_0 \in \gamma$

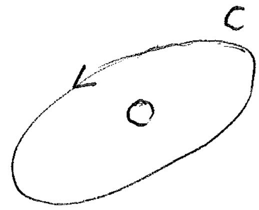
$$\Leftrightarrow \quad \frac{\partial u}{\partial \vec{n}} = \frac{\partial u}{\partial s} = 0 \quad (\text{two } \perp \text{ directions}).$$

The Cauchy – Riemann equations (locally) just say

$$\frac{\partial u}{\partial \vec{n}} = \frac{\partial v}{\partial s} \qquad \text{(Easy)}$$

$$\frac{\partial v}{\partial \vec{n}} = -\frac{\partial u}{\partial s} .$$

Take any nice Jordan curve C close to Γ. Let N_δ be the circle $|z-z_0|=\delta$.



We know (Lec 21, p. 1)

$$\oint_C \frac{\partial g}{\partial \vec{n}} ds + \oint_{N_\delta} \frac{\partial g}{\partial \vec{n}} ds = 0 .$$

outward $\qquad$ $\vec{n} = -$ radial

Keep δ small.

$$g = -\ln r + \text{harm} \qquad \text{near } z_0$$

$$\frac{\partial g}{\partial r} = -\frac{1}{r} + \text{continuous}$$

$$\frac{\partial g}{\partial (-r)} = \frac{1}{r} + \text{continuous}$$

$$\int_{N_\delta} \frac{\partial g}{\partial(-r)}\, ds = \int_{N_\delta} \left[\frac{1}{r} + \text{cont.} \right] ds$$

$$\simeq \int_{N_\delta} \frac{ds}{\delta} + O(\delta)$$

$$\simeq 2\pi + O(\delta) \,.$$

By letting $\delta \to 0^+$, we get

$$\oint_C \frac{\partial g}{\partial \vec{n}}\, ds = -2\pi \,.$$

Nice!

Thus:

$$\oint_C dg^* = \int_C \frac{\partial h}{\partial s}\, ds = -2\pi \,.$$

Ie g^* always comes back to a value of 2π "less" when C is traversed!

Yes, but then locally analytic function

$$F(z) = e^{-g-ih}$$

is single-valued on $D - \{z_0\}$!

YES!

Clearly,

$$|F| = e^{-g} < 1 \quad \text{on} \quad D - \{z_0\}$$

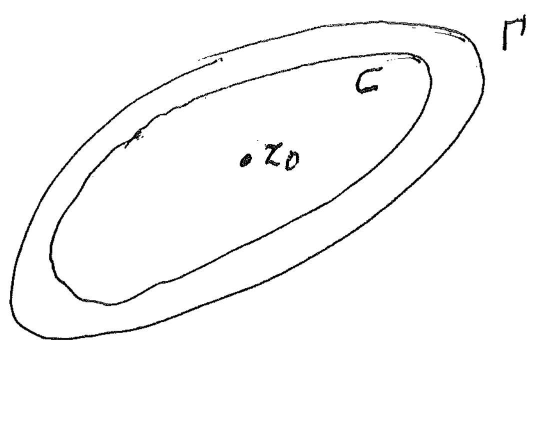
$$|F| = e^{\ln r + \text{harm}} = r e^{\text{harm.}} \quad \text{near} \quad z_0 .$$

At once, F has a removable singularity at $z = z_0$, F is analytic on D, z_0 = simple zero, $F[D] \subseteq \mathcal{U}$. Also $\lim_{z \to \Gamma} |F(z)| = 1$.

We suspect F is univalent and $F(D) = \mathcal{U}$.

To get this, we make systematic and appropriate use of 3 observations.

$\boxed{1}$ Take Jordan curve C fairly close to Γ. Let $B_1 = \min_C g$, $B_2 = \max_C g$.



By max-min principles, clearly $g < B_2$ in the $[C, \Gamma]$ "ring".

Also, similarly, $g > B_1$ inside C.

Of course, for $C \approx \Gamma$, the numbers B_1 and B_2 will be very small.

$\boxed{2}$ Recall our old local result about analytic functions $Q(z)$ near z_1. We had

$$w = Q(z) = w_1 + t^m,$$

$$t = \psi(z)$$

where $w_1 = Q(z_1)$, ψ is analytic + univalent near z_1, $m \geq 1$. This was discussed in Lec 20, p. 3 (Ahlfors 133). Because of this result, it is always easy to visualize the set $\operatorname{Re} g(z) = 0$ locally for any analytic g.

The simplest cases just correspond to the implicit function theorem of multi-variable calculus:

$$H(x,y) = 0 \quad \text{near } (x_2, y_2) \qquad \left\{ \begin{matrix} \text{think} \\ H = \text{harm.} \end{matrix} \right\}$$

$$\frac{dy}{dx} = -\frac{H_x}{H_y}$$

want $\operatorname{grad} H \neq 0$ at (x_2, y_2).

If $|\operatorname{grad} H|$ stays uniformly away from 0, the shape of $H = 0$ changes rather boringly as you travel along $\{H = 0\}$.

$\boxed{3}$ Let $\mathcal{R}$ be a Jordan region (including boundary) that lies in D and does not contain z_0.
Then $g(z;z_0)$ <u>cannot</u> reduce to a constant $\mathcal{L}$ along $\partial\mathcal{R}$.

Indeed, by max-min principle, get $g \equiv \mathcal{L}$ inside $\mathcal{R}$, hence $g \equiv \mathcal{L}$ on D. Contrad!

We now put 1, 2, 3 together.

The graph of $\{g = \varepsilon\}$ is very interesting when ε = small.

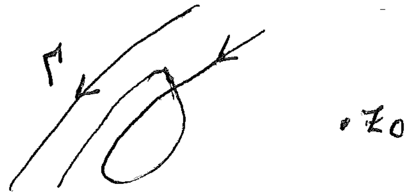
It is certainly a compact set. By the properties of g, it lies close to Γ. We can use "2" to visualize its shape LOCALLY.

For simplicity, we can assume that ε is always taken such that the analytic function $g_x - i g_y$ has <u>no</u> zeros on $\{g = \varepsilon\}$.

Remember: the zeros of $g_x - i g_y$ are isolated; they can only accumulate along Γ (AT WORST). "Just evaluate g at each such zero and avoid these ε-values."

By property 1, the locus $g=\varepsilon$ stays in a narrow "ring" or "ribbon" abutting on Γ.

Sit on the locus and try to move as far as possible. If your track ever crossed itself



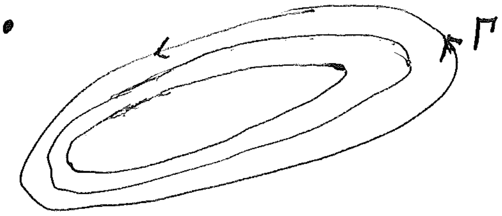
you would be able to apply property 3 with $\alpha=\varepsilon$. [You could also use property 2 to get $m \geqq 2$ and gradient $=0$.]

A simple analysis shows that what <u>must</u> happen is that your "track" travels completely around z_0, creating a nice Jordan curve.

I.E., for these "simpler generic type" ε, we immediately see $\{g=\varepsilon\}$ is a nice Jordan curve <u>near</u> Γ. [Anytime our ε is reasonably small, and not equal one of the "bad" g-values mentioned on 13 bottom!]

These Jordan curves are clearly <u>nested</u> as ε decreases to 0.

↳ cf. 1 on 11



On $\{g = \varepsilon\}$, ε small, by the "nested" property,

$$\frac{\partial g}{\partial \vec{n}} \lessapprox 0 \,. \qquad (g \text{ goes down!})$$

Since $\frac{\partial g}{\partial s} = 0$ TOO, we must have $\frac{\partial g}{\partial \vec{n}} < 0$ since $g_x - i g_y \neq 0$. $\boxed{⑧ \text{ bot}}$

We conclude:

$$\text{neg.} = \frac{\partial g}{\partial \vec{n}} = \frac{\partial h}{\partial s} \quad \text{on } \{g = \varepsilon\} = \text{"}C\text{"}$$

$$\oint_C \frac{\partial h}{\partial s}\, ds = -2\pi \quad \text{p. } 10 \,.$$

Look at $F = e^{-g-ih} = e^{-g} e^{-ih}$.

Clearly the F-image of $\{g = \varepsilon\}$ is exactly $|w| = e^{-\varepsilon}$ traversed once in the counterclockwise direction. 1–1

Apply our old "conformal mapping by edges" theorem (Lec 20, p. 6). Deduce F is univalent on $\{g > \varepsilon\}$ with image exactly $\{|w| < e^{-\varepsilon}\}$. Let $\varepsilon \to 0$. Hence:

F univalent on D, $F(D) = \mathcal{U}$.

This proves the RMT (modulo trivial issues of normalization) for D = interior of Γ. *

Normal families and taking an exhaustion for a general simply-connected domain $\mathcal{D}$ yields the RMT in full generality.

See Ahlfors, p. 251, footnote!

This approach via e^{-g-ih} is the "old approach" to RMT. It is very nice in the way it follows from the Perron process.

This method is also closer to the original thinking of Riemann (1851).

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* Notice that our proof does not show that F extends to a homeomorphism $\overline{D}$ onto $\overline{\mathcal{U}}$.

Topic D: A comment on CIT.

Very very briefly!

Take D = bounded domain $\subseteq \mathbb{C}$.
Simply-connected or not! $\underline{\underline{\partial D = \text{horrible is OK}}}$.

Though I have not discussed cycles γ (formal unions of <u>closed</u> curves) and when γ is homologous to 0 in D, I point out that the KEY TRICK in getting

$$\int_\gamma f(z)\,dz = 0 \qquad \text{as in Ahlfors p. 141}$$

{the general form of CIT}

is to simply write

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathcal{D}} \frac{f(\xi)}{\xi - z}\,d\xi \qquad \text{for } z \in \gamma$$

by CIF, wherein $\mathcal{D}$ is a <u>smoothly</u> bounded domain, $\bar{\mathcal{D}} \subseteq D$, that is part of an exhaustion of D and has $\gamma \subseteq \mathcal{D}$. (Lec 18, p. 8, vi)

Doing so allows one to use Fubini's theorem. IE,

$$\int_\gamma f(z)\,dz = \frac{1}{2\pi i} \int_{\partial \mathcal{D}} \int_\gamma \frac{f(\xi)}{\xi - z}\,dz\,d\xi$$

$$= \int_{\partial\mathcal{D}} f(\xi) \left[- \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-\xi} \right] d\xi \ .$$

This shifts all the difficulty to calculating the bracket, ie

$-$[the winding number of γ with resp to ξ].

For cycles γ homologous to 0, the foregoing winding number is 0. This is basically the definition of homologous to 0 (since $\partial\mathcal{D}$ can be taken very close to ∂D).

We get:

$$\int_{\gamma} f(z)\,dz = \int_{\partial\mathcal{D}} f(\xi)\,[0]\,d\xi = 0 .$$

Compare Ahlfors, 3rd edition, 143 (bot) + 144 (top).

This "TRANSFER" TRICK (or "TRANSFER TO THE FRACTION TRICK") is well worth remembering in other contexts...

"WHAT AHLFORS SHOULD HAVE INCLUDED" (19)

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, Number 3, August 1971

A BRIEF PROOF OF CAUCHY'S INTEGRAL THEOREM

JOHN D. DIXON[1]

ABSTRACT. A short proof of Cauchy's theorem for circuits homologous to 0 is presented. The proof uses elementary local properties of analytic functions but no additional geometric or topological arguments.

The object of this note is to present a very short and transparent proof of Cauchy's theorem for circuits homologous to 0. The proof is based on simple 'local' properties of analytic functions that can be derived from Cauchy's theorem for analytic functions on a disc, and it may be compared with the treatment in Ahlfors [1, pp. 137–145]. It is apparent from this proof that this version of Cauchy's theorem is not only much more natural than the homotopic version which appears in several recent textbooks; it is also much easier to prove (contra Dieudonné [2, p. 192]). It is reasonable to argue that the concept of homotopy in connection with Cauchy's theorem is as extraneous as the notion of Jordan curve.

We recall that if γ is a circuit (="continuous, piecewise smooth, closed curve"), and $w\in\mathbf{C}$ does not lie on γ, then the *index* of w with respect to γ is $\operatorname{Ind}(\gamma, w)=(2\pi i)^{-1}\int_\gamma (z-w)^{-1}dz$. It is easily proved that $E=\{w\in\mathbf{C}\mid \operatorname{Ind}(\gamma, w)=0\}$ contains a neighbourhood of ∞ and is open (see [1, p. 116]). In the following proof we give full references to the 'local' properties used in order to emphasize the elementary nature of the proof.

CAUCHY'S THEOREM. *Let D be an open subset of $\mathbf{C}$ and let γ be a circuit in D. Suppose that γ is homologous to 0 in D, i.e. each $w\notin D$ lies in the set E defined above. Then, for each f analytic on D:*

(i) $\int_\gamma f(z)dz=0$;

(ii) $\operatorname{Ind}(\gamma, w)f(w)=(2\pi i)^{-1}\int_\gamma (z-w)^{-1}f(z)dz$ *for all $w\in D$ not lying on γ.*

PROOF. Consider $g:D\times D\to\mathbf{C}$ defined by $g(w, z)=(f(z)-f(w))/(z-w)$ for $z\neq w$ and $g(w, w)=f'(w)$. Then g is continuous, and for each fixed z, $w\mapsto g(w, z)$ is analytic [1, p. 124]. Define $h:\mathbf{C}\to\mathbf{C}$ by $h(w)$

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$=\int_\gamma g(w, z)dz$ on D and $h(w)=\int_\gamma (z-w)^{-1}f(z)dz$ on E. Note that $C=D\cup E$ by hypothesis, and that these two expressions for $h(w)$ are equal on $D\cap E$ because $\operatorname{Ind}(\gamma, w)=0$ there.

Now h is differentiable on both D and E ([1, p. 123] or [3, p. 137]), and so h is an entire function. Since the image of γ is bounded, and E contains a neighbourhood of ∞, $h(w)\to 0$ as $w\to\infty$. This implies firstly that h is constant (Liouville's theorem), and secondly that h is 0. Thus $\int_\gamma g(w, z)dz=0$ for all $w\in D$ not lying on γ; and (ii) follows. Finally, let u be some fixed point of D not lying on γ. Then applying (ii) to the function $z\mapsto f(z)(z-u)$ in place of f, and evaluating at $w=u$, we obtain (i).

REMARK. The proof goes through word for word when γ is a cycle (see [1, p. 138]) rather than a circuit. Then, as in Ahlfors' treatment, the general form of the residue theorem follows immediately.

REFERENCES

1. L. V. Ahlfors, *Complex analysis: An introduction to the theory of analytic functions of one complex variable*, 2nd ed., McGraw-Hill, New York, 1966. MR **32** #5844.

2. J. Dieudonné, *Foundations of modern analysis*, Pure and Appl. Math., vol. 10, Academic Press, New York, 1960. MR **22** #11074.

3. R. Nevanlinna and V. Paatero, *Einführung in die Funktionentheorie*, Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, Mathematische Reihe, Band 30, Birkhäuser Verlag, Basel, 1965; English transl., Addison-Wesley, Reading, Mass., 1969. MR **34** #1491; MR **39** #415.

CARLETON UNIVERSITY, OTTAWA 1, ONTARIO, CANADA

LAST HW PROBLEM (due Wed, 14 Dec)

Ahlfors, p. 244 prob 1 $\left\{ \begin{array}{l} \text{about} \\ u(z_2) \lesssim M u(z_1) \end{array} \right\}$

Warning: be very careful about connectedness.
Hint: one might try to use Harnack's principle.

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Some Comments and Pointers About The
Written 5-hour Exam

"Analytic Functions" is a graduate course. The exam will reflect that. The basic theorems from undergraduate complex analysis are assumed as known.

Plan on 5 problems:
a) 2 modest type problems (eg, state Theorem X,Y,Z or do something similar to an easy HW problem)
b) 2 harder type problems (similar to HW problems or else utilizing some key theorems that we proved in lecture)
c) STATE AND PROVE the Riemann Mapping Theorem.

I *will* ask (c).

It is smart to review Lectures 1--25 and note which HW problems I assigned.

To help you study, here is a list of topics for this written exam.
The code letters mean
A = know the statement
B = know the proof
C = be able to apply in problems

You will see that many topics have "B" OMITTED. Sometimes even "C" OMITTED. Take that for what it's worth. (Yes, I know some topics are completely omitted on this test.)

On the exam, you may have a choice in one of the problems; i.e., choose 1 of problems X1,X2,...,Xn.

NOTE: a **modified** listing will be posted for the oral exam. (BTW: schedule for the oral exam will be decided **via email**.)

written exam topics

Sum of things like n^(-2)	A,C
Cauchy-Goursat theorem for Rectangle	A,B
All standard Maclaurin series	A,C
Laurent Series Expansion for f(z)	A,C
Mittag-Leffler principal parts theorem	A
Euler's limit formula for Gamma(z)	A,C
Gamma(z) meromorphic on C, with known simple poles	A,C
The product formula for 1/Gamma(z)	A
Definition of unif and abs conv for imfinite products	A,C
Weierstrass M-test for infinite products	A,C
Jensen's formula	A
Poisson-Jensen formula	A
Hurwitz's theorem for nonzero analytic functions	A,C
Order and type of entire f(z)	A,C (B in examples)
Hadamard Factorization Theorem	A,C
Maximum Modulus Principle (correct form!) on bounded D	A,C
Schwarz's lemma (correct form!)	A,B,C

Maximum Principle for harmonic and continuous subharmonic functions (correct form!) on bounded D	A,C
Phragmen-Lindelof Maximum Modulus Principle for f on bounded D	A,B,C
P-L max principle for harmonic and continuous subharmonic fcns on bounded D	A,B,C
Poisson integral repr for harmonic functions and analytic fcns. on a disk	A,C
Schwarz's "converse" theorem for Poisson integrals with given edge value	A,B,C
Schwarz's formula for f(z) in terms of u	A,C
Stirling's Formula for log Gamma	A,C
Definition of Normal Family	A,C
Definition of Uniform Equicontinuity	A,C
Arzela-Ascoli Theorem	A,C
Normal family criterion for analytic functions on D	A,C
Hurwitz's Corollary for univalent analytic functions on D	A,C
Riemann Mapping Theorem	A,B,C
Three forms of the Schwarz Reflec Principle (harmonic, analytic, univalent+analytic)	A,C
Identical Vanishing Condition for analytic f(z)	A,C
Harnack's Inequality and Principle	A,C
One-sided free boundary arcs	A,C
Analytic Jordan arcs	A,C
Conformal Invariance of Dirichlet's Problem, e.g. under LF maps	A,C
Solvability of Dirichlet Problem (for bdd harmonic u) on domains with barriers	A,C
Explicit Solution of Dirichlet Problem on a disk	A,C
Mean-value characterization of harmonic functions	A,C
Green's 1st and 2nd identities	A,C
Mean-value property for harmonic u in an annulus	A,C
Conjugate differential *du	A,C
Little Picard Theorem (as in Lec 25)	A,C

SOME SAMPLE GOOD PROBLEMS
FROM AHLFORS, THIRD EDITION:

p.166 (prob 1) p.171 (prob 1,2)
p.174 (prob 4,5) p.190 (prob 4)
p.193 (prob 3; also the similar HW prob)
p.197 (prob 2, corrected) p.244 (prob 1)