Lecture 9

Tues, 11 Oct

We resume our development of the Hadamard factorization theorem.

$z = Re^{i\theta}$, $z_0 = re^{i\omega}$

$$|z - z_0|^2 = |Re^{i\theta} - re^{i\omega}|^2$$
$$= |R - re^{i(\omega-\theta)}|^2$$
$$= R^2 + r^2 - 2Rr\cos(\omega - \theta)$$

$f(z)$ analytic on $|z| \leq R$ $\Rightarrow$ $\boxed{\text{Poisson}}$

$f = u + iv$ as usual

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{|Re^{i\theta} - z_0|^2} f(Re^{i\theta})\, d\theta$$

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\omega - \theta)} f(Re^{i\theta})\, d\theta$$

$$\Downarrow$$

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\omega - \theta)} u(Re^{i\theta})\, d\theta .$$

Likewise if we have <u>any</u> u harmonic on $\{|z| \leq R\}$.

We will discuss this last formula quite a bit more later (for harmonic functions).

Like I said in Lec 8,

$$\mathrm{Re}\left(\frac{z+z_0}{z-z_0}\right) = \text{very interesting.}$$

$$z\bar{z} = R^2 \implies$$

$$\frac{z+z_0}{z-z_0}\,\frac{\bar{z}-\bar{z}_0}{\bar{z}-\bar{z}_0} = \frac{z\bar{z} + z_0\bar{z} - z\bar{z}_0 - z_0\bar{z}_0}{|z-z_0|^2}$$

$$= \frac{|z|^2 - |z_0|^2 + 2i\,\mathrm{Im}(z_0\bar{z})}{|z-z_0|^2}$$

$$\Downarrow$$

$$\mathrm{Re}\left(\frac{z+z_0}{z-z_0}\right) = \frac{R^2-r^2}{|z-z_0|^2} \qquad (z = Re^{i\theta})$$

$$= \frac{R^2-r^2}{R^2+r^2-2Rr\cos(\omega-\theta)}$$

Poisson's formula for u is thus:

$$\boxed{u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{Re^{i\theta}+z_0}{Re^{i\theta}-z_0}\right) u(Re^{i\theta})\,d\theta.}$$

You have to stare at this formula for a few moments, then you see a very curious fact!

Write "w" in place of "z_0". That helps!

Remember: $f(z) = u + iv$.

Let

$$F(w) \overset{\text{def.}}{=\!=} \frac{1}{2\pi}\int_0^{2\pi} \frac{Re^{i\theta}+w}{Re^{i\theta}-w}\, u(Re^{i\theta})\,d\theta .$$

$F(w)$ is analytic on $|w|<R$ by Leibnitz's rule.

But,

$$0 = \mathrm{Re}\left[f(w) - F(w)\right] \quad \text{on } |w|<R.$$

Hence, by basic analytic functions,

$$f(w) - F(w) = iA, \quad A \text{ real}.$$

So,

$$f(w) = iA + \frac{1}{2\pi}\int_0^{2\pi} \frac{Re^{i\theta}+w}{Re^{i\theta}-w}\, u(Re^{i\theta})\,d\theta .$$

Let $w=0$. Get:

$$f(0) = iA + \underbrace{u(0)}_{\text{by mean value property}},$$

hence, $iA = iv(0)$.

$$\boxed{A = v(0)}$$

Thm (Schwarz' formula)*

Let $f(z)$ be analytic on $|z| \leq R$.

Then,

$$\boxed{f(z) = iv(0) + \frac{1}{2\pi}\int_0^{2\pi} \frac{Re^{i\theta}+z}{Re^{i\theta}-z}\, u(Re^{i\theta})\, d\theta .}$$

Also, as a consequence,

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\right) u(Re^{i\theta})\, d\theta$$

$$v(z) = v(0) + \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Im}\left(\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\right) u(Re^{i\theta})\, d\theta .$$

*This is H. A. Schwarz, the same "guy" who did the famous Schwarz' lemma.

[redacted]

Let $G(z)$ be analytic on $|z| \leq R$ and <u>never zero</u>. Form some branch of $\log G(z)$.

By page 4 [redacted], we get:

$$\ln|G(z)| = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\right)\ln|G(Re^{i\theta})|\,d\theta.$$

<u>Theorem</u> (Poisson–Jensen formula)

Let $f(z)$ be analytic on $\{|z| \leq R\}$. Let $f(Re^{i\theta}) \neq 0$. Let $\{a_j\}$ denote the zeros of f inside $\{|z| < R\}$.

For a generic w in the same open disk, we then have:

$$\ln|f(w)| + \sum_{|a_j|<R} \ln\left|\frac{R^2-\bar{a}_j w}{R(w-a_j)}\right| = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{Re^{i\theta}+w}{Re^{i\theta}-w}\right)\ln|f(Re^{i\theta})|\,d\theta.$$

(Classical <u>Jensen</u> corresponds to $w=0$.)

<u>Proof</u>

Just take $G(z) = f(z)\prod_{|a_j|<R} \frac{R^2-\bar{a}_j z}{R(z-a_j)}$ as in the proof of <u>Jensen's</u> formula. Use line 4.

$\blacksquare$

Lec 7 p. 17

We claim that by shrewd use of Poisson-Jensen formula, we can prove the HFT.

For HFT, we have an <u>entire</u> <u>function</u> $f(z)$ of order $\rho < \infty$. ($f \neq$ polynomial)

By passing to $\frac{f(z)}{z^m}$, there is no loss of generality in supposing our original f satisfies $f(0) \neq 0$.

Lec 8 p. 20 gave:

$$f(z) = \exp[\phi(z)] \prod_{a_n} E\left(\frac{z}{a_n}; p\right)$$

where $p = [\![\rho]\!]$, $\{a_n\}$ = the zeros of f (listed with multiplicity), ϕ = entire.

The product conv. <u>unif</u> + <u>abs</u> on $\mathbb{C}$-compacta.

Here

$$E(u;p) \equiv (1-u)\exp\left\{\sum_{k=1}^{p}\frac{1}{k}u^k\right\}.$$

We recall that, for $|u| < h < 1$,

$$\log E(u;p) = -\sum_{k=p+1}^{\infty}\frac{1}{k}u^k$$

$$|\log E(u;p)| \leq \frac{|u|^{p+1}}{1-|u|} \leq \frac{|u|^{p+1}}{1-h}.$$

The convergence was OK because

$$\text{Jensen} \Rightarrow \sum_n \frac{1}{|a_n|^{p+1}} < \infty.$$

Remember too that

$$n(r) = O(r^{\rho+\varepsilon}),$$

each $\varepsilon > 0$, $r \geq 1$.

OUR TASK IS TO PROVE THAT ϕ = polynomial, degree $\leq \rho$.

By considering $f \hookrightarrow Af$ (A = appropriate), WLOG $f(0) = 1$.

Hence $\phi(0) = 0$ WLOG. OK

Must prove $\phi^{(p+1)}(z) \equiv 0$ on $\mathbb{C}$. Since ϕ is entire, it suffices to check this in a tiny neighborhood of $z = 0$.

OUR PLAN IS TO <u>DIFFERENTIATE</u> BOTH SIDES OF THE POISSON-JENSEN FORMULA APPROPRIATELY.

The following facts need to be kept in mind:

$\boxed{A}$ $g(z)$ analytic, $u = \mathrm{Re}[g(z)]$

$$\Longrightarrow \quad g'(z) = u_x - i u_y \quad \text{by C-R eqs}$$

$\boxed{B}$ $k(z)$ analytic + nonzero, $u = \ln|k(z)|$

$$\Longrightarrow \quad \frac{k'(z)}{k(z)} = u_x - i u_y \quad \text{by } \boxed{A} \text{ and } g = \log k.$$

Use radii R so that $f(Re^{i\theta}) \neq 0$.

P-J $\Rightarrow$ p. 6

$$\ln|f(z)| + \sum_{|a_j|<R} \ln\left|\frac{R^2 - \bar{a}_j z}{R(z-a_j)}\right|$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\right) \ln|f(Re^{i\theta})|\, d\theta.$$

We plan to apply $\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$ to both sides.

This operator will always be applied to something "u" that is HARMONIC.

Since

$$\frac{Re^{i\theta}+z}{Re^{i\theta}-z} = -1 + \frac{2Re^{i\theta}}{Re^{i\theta}-z},$$

clearly we then get by applying $\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$ to Poisson-Jensen

$$\frac{f'(z)}{f(z)} + \sum_{|a_j|<R}\left(\frac{-\bar{a}_j}{R^2-\bar{a}_j z} - \frac{1}{z-a_j}\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{2Re^{i\theta}}{(Re^{i\theta}-z)^2} \ln|f(Re^{i\theta})|\, d\theta \,.$$

We can now differentiate p more times. wrt z
($p=0$ means stop here.) GET:

$$[\log f(z)]^{(p+1)} = \sum_{|a_j|<R}\left[-\frac{p!}{(a_j-z)^{p+1}} + \frac{p!\,(\bar{a}_j)^{p+1}}{(R^2-\bar{a}_j z)^{p+1}}\right]$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} \frac{(p+1)!}{(Re^{i\theta}-z)^{p+2}}\, 2Re^{i\theta} \ln|f(Re^{i\theta})|\, d\theta \,.$$

(You can check all is OK. Remember that for differentiating integrals, we use Leibnitz's rule wrt z.)

Since $f(0)=1$, we have $|f(z)| \geq \frac{1}{2}$ ~~say~~ on

$$\{|z| \leq 2\lambda\}.$$

We propose to keep $|z| < \lambda$ and to let $R \to \infty$ in our equation for $[\log f(z)]^{(p+1)}$.

{WE USE ONLY R's WITH $f(Re^{i\theta}) \neq 0$.}

Lemma

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{2Re^{i\theta}}{(Re^{i\theta}-z)^{p+2}}\, d\theta = 0.$$

Proof

LHS is just $(\text{const}) \oint_{|\xi|=R} \frac{d\xi}{(\xi - z)^{p+2}}$.

That ξ integral is obviously zero, since $p \geq 0$.

$\blacksquare$

So, by Lemma,

$$[\log f(z)]^{(p+1)} + \sum_{|a_j|<R} \frac{p!}{(a_j - z)^{p+1}}$$

$$= \frac{(p+1)!}{2\pi} \int_0^{2\pi} \frac{2Re^{i\theta}}{(Re^{i\theta}-z)^{p+2}} \ln\left|\frac{f(Re^{i\theta})}{M(R;f)}\right| d\theta$$

$$+ \sum_{|a_j|<R} \frac{p!\,(\bar{a}_j)^{p+1}}{(R^2 - \bar{a}_j z)^{p+1}}.$$

TRICK #1

But, we've assumed $|z| < \lambda$.
This gives

$$\left| [\log f(z)]^{(p+1)} + p! \sum_{|a_j| < R} \frac{1}{(a_j - z)^{p+1}} \right|$$

$$\leq \frac{(p+1)!}{2\pi} \int_0^{2\pi} \frac{2R}{(R-\lambda)^{p+2}} \left| \ln \left| \frac{f(Re^{i\theta})}{M(R;f)} \right| \right| d\theta$$

$$+ p! \sum_{|a_j| < R} \left| \frac{\bar{a}_j^{\,p+1}}{(R^2 - \bar{a}_j z)^{p+1}} \right| .$$

remember
$|f| \leq M(R;f)$

But:

$$\left| \frac{\bar{a}_j}{R^2 - \bar{a}_j z} \right| \leq \frac{|\bar{a}_j|}{R^2 - R|z|} \leq \frac{R}{R(R-|z|)} \implies$$

$$\left| \frac{\bar{a}_j}{R^2 - \bar{a}_j z} \right| \leq \frac{1}{R - \lambda} .$$

The $|a_j| < R$ sum above with $(R^2 - \bar{a}_j z)^{p+1}$ is thus NO MORE THAN

$$p! \frac{n(R)}{(R-\lambda)^{p+1}} .$$ yes

We therefore get:

$$\left| [\log f(z)]^{(p+1)} + p! \sum_{|a_j|<R} \frac{1}{(a_j - z)^{p+1}} \right|$$

$$\leqq p! \frac{n(R)}{(R-\lambda)^{p+1}}$$

removing the big absolute value

$$+ \frac{(p+1)!}{2\pi} \int_0^{2\pi} \frac{2R}{(R-\lambda)^{p+2}} \ln \frac{M(R;f)}{|f(Re^{i\theta})|} d\theta$$

$$\leqq p! \frac{n(R)}{(R-\lambda)^{p+1}}$$

$$+ (p+1)! \frac{2R}{(R-\lambda)^{p+2}} \left[\frac{1}{2\pi} \int_0^{2\pi} \ln \frac{M(R;f)}{|f(Re^{i\theta})|} d\theta \right].$$

Remember Jensen's formula:

$$\ln |f(0)| + \sum_{|a_j|<R} \ln \frac{R}{|a_j|} = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(Re^{i\theta})| d\theta.$$

$f(0)=1$

The bracketed $d\theta$ integral is thus:

$$0 \leqq \frac{1}{2\pi} \int_0^{2\pi} \ln \frac{M}{|f|} d\theta = \ln M - \sum_{|a_j|<R} \ln \frac{R}{|a_j|}$$

$$\leqq \ln M(R;f).$$

TRICK #2

We conclude:

$$\left| [\log f]^{(p+1)} + p! \sum_{|a_j|<R} \frac{1}{(a_j - z)^{p+1}} \right|$$

$$\lesssim \frac{p!\, n(R)}{(R-\lambda)^{p+1}} + (p+1)! \frac{2R}{(R-\lambda)^{p+2}} \ln M(R;f).$$

But,

$p = [\![\rho]\!]$

$p+1 > \rho$ automatically

$\sum \frac{1}{|a_j|^{p+1}} < \infty$

$n(R) = O(R^{\rho+\varepsilon})$, each ε

$\ln M(R;f) \lesssim R^{\rho+2\varepsilon}$ for giant R.

Keep $\rho + 3\varepsilon < p+1$. <u>Also keep R giant.</u>

The above right-hand-side is thus

$$< \frac{R^{\rho+3\varepsilon}}{R^{p+1}} + \frac{R^{1+\rho+3\varepsilon}}{R^{p+2}}.$$

$\boxed{\varepsilon \text{ tiny}}$

Letting $R \to \infty$, we get

$$[\log f(z)]^{(p+1)} + p! \sum_{\substack{\text{all} \\ a_j}} \frac{1}{(a_j - z)^{p+1}} = 0$$

on $|z| < \lambda$. NICE!!!

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But, by p.,

$$\log f = \phi(z) + \sum_{\substack{\text{all} \\ a_n}} \left[\log\left(1 - \frac{z}{a_n}\right) + \sum_{k=1}^{p} \frac{1}{k} \left(\frac{z}{a_n}\right)^k \right]$$

Σ empty if $p = 0$

{take $p+1$ derivatives by Weierstrass' theorem}

$p+1$ kills the polynomial

$$\Rightarrow [\log f]^{(p+1)} = \phi^{(p+1)}(z) + \sum_{\substack{\text{all} \\ a_n}} \left[-\frac{p!}{(a_n - z)^{p+1}} + 0 \right]$$

easy to check

$$\Rightarrow [\log f(z)]^{(p+1)} = \phi^{(p+1)}(z) - p! \sum_{\substack{\text{all} \\ a_j}} \frac{1}{(a_j - z)^{p+1}}.$$

$\{ |z| < \lambda \}$

We conclude that:

$$\phi^{(p+1)}(z) \equiv 0 \quad \text{on} \quad |z| < \lambda.$$

Since ϕ is entire, we conclude that

$$\phi(z) = \text{polynomial, degree} \leqq p.$$

QED for HFT!! ■

This proof is like the one in Ahlfors' book. It is clearly <u>very</u> <u>slick</u>; but it relies on a bit more structure than, e.g., the "elementary" proof of Landau.

For Landau's "elementary" proof, see, e.g.,

E. Landau, *Math. Zeitschrift* 26 (1927) 170–175

OR

E. Landau, <u>Vorlesungen über Zahlentheorie</u>,
Sätze 423 + 225

Before pushing onward, we need to take care of a few "odds and ends". These items are much easier than HFT.

It is customary to say that any non identically vanishing polynomial $P(z)$ has order 0.

Some trivial facts then hold.

$$\boxed{\text{Here "Order" means finite order}}$$

Fact 1

Let f have order ρ. Let g have order $\rho' < \rho$. Then $f+g$ has order ρ.

Fact 2

Let f have order ρ_1. Let g have order ρ_2. Then fg has order $\leqq \max(\rho_1, \rho_2)$.

Fact 3

Let $p(z) \not\equiv 0$ be a polynomial. Let f have order ρ. Then $p(z)f(z)$ has order ρ as well.

Fact 4

Let f have order ρ. Let the zeros of f include $\{a_1, \dots, a_m\}$ (with multiplicity). Then

$$\frac{f(z)}{\prod_{j=1}^{m}(z-a_j)}$$

has order ρ.

FAMOUS COROLLARY (to Hadamard's theorem)

Let $f(z)$ be entire, of finite order ρ. Suppose $\rho \notin \mathbb{Z}$ and $a \in \mathbb{C}$. Then $f(z) = a$ has infinitely many roots.

Proof

Clearly $\rho > 0$. Let $g(z) = f(z) - a$. Suppose $g(z)$ has only finitely many zeros. Let $p = [\![\rho]\!]$. Know $\rho(g) = \rho$. Write

$$g(z) = z^{\gamma} \exp[\phi(z)] \prod_{n=1}^{N} E\left(\frac{z}{a_n}; p\right) \qquad (N < \infty)$$

à la Hadamard. Note that

$$p < \rho < p+1 .$$

In the expansion

$$z^{q} \exp[\phi(z)] \cdot \prod_{n=1}^{N} \left(1 - \frac{z}{a_n}\right) \exp\left\{ \sum_{k=1}^{p} \frac{1}{k} \left(\frac{z}{a_n}\right)^k \right\},$$

$\deg \phi \leqq p$, and each "chunk" has order $\leqq p$.
(Remember that $N =$ finite!) See Fact 2.

<u>Hence</u> $g(z)$ has order $\leqq p$.
This contradicts

$$\rho(g) = \rho(f) = \rho .$$

Hence $N = \infty$. $\blacksquare$

Another Corollary of Hadamard

Let $f(z)$ be entire, finite order $\rho > 0$, $\rho =$ integer. Then $f(z)$ assumes every $a \in \mathbb{C}$ with at most one exception.

$\langle$ eg $f = e^z$ and 0 $\rangle$

Proof

IE 2 exceptions

Suppose that $f(z) \neq \alpha$, $f(z) \neq \beta$.

Look at $g(z) = f(z) - \alpha$.

$\rho(g) = \rho$.

Apply Hadamard to g. We know $g(z) \neq 0$.

Therefore, we get

$$g(z) = \exp[\phi(z)]$$

where ϕ is a polynomial of degree $\leq \rho$.

Since the order of g is ρ, we must have $\deg \phi = \rho$.

$\rho =$ positive

But now solve $\phi(z) = \log(\beta - \alpha)$ (any branch). At such z, clearly $g(z) = \beta - \alpha$ so $f(z) = \beta$. Contrad. ∎

Example of Hadamard Factorization

Let $f(z) = \sin(\pi z)$.
There are simple zeros at $z = k \in \mathbb{Z}$.
Order is 1. Get [by Hadamard]:

$$f(z) = \pi z \exp(a + bz) \cdot \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}.$$

Regroup the terms (via $n, -n$). Get

$$f(z) = \pi z \exp(a + bz) \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

So,

$$\frac{\sin(\pi z)}{\pi z} = \exp(a + bz) \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

unif + abs conv on $\mathbb{C}$-compacta

Let $z \to 0$. Get $\exp(a) = 1$ or $a = 0$, WLOG. Get:

$$\frac{\sin \pi z}{\pi z} = \exp(bz) \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

Therefore $\exp(bz)$ is EVEN.
This forces $b = 0$. So,

$$\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

Nice!

Quick Rehash:

$|u| < 1 \Rightarrow$

$$\log E(u;p) = -\sum_{k=p+1}^{\infty} \frac{u^k}{k} \quad \text{trivially}$$

$$|\log E(u;p)| \leqq \frac{1}{p+1} \frac{|u|^{p+1}}{1-|u|} \leqq \frac{|u|^{p+1}}{1-h}$$

for $|u| \leqq h < 1$

$$\ln|E(u;p)| \leqq \frac{|u|^{p+1}}{1-h} \quad \text{for } |u| \leqq h < 1$$

AHLFORS suggests that it is wise to get an estimate on $\ln|E(u;p)|$ for <u>ALL</u> $u \in \mathbb{C}$. I agree!! ☆

(It just involves some fussing around.)

Next Lecture!

☆ E.G., one might want to study $\prod_n E(\frac{z}{a_n};p)$ for its own sake on $\mathbb{C}$.