Probabilistic Analysis of Hierarchical Cluster Protocols for Wireless Sensor Networks

Ingemar Kaj *

Department of Mathematics
Uppsala University
ikaj@math.uu.se

Abstract. Wireless sensor networks are designed to extract data from the deployment environment and combine sensing, data processing and wireless communication to provide useful information for the network users. Hundreds or thousands of small embedded units, which operate under low-energy supply and with limited access to central network control, rely on interconnecting protocols to coordinate data aggregation and transmission. Energy efficiency is crucial and it has been proposed that cluster based and distributed architectures such as LEACH are particularly suitable. We analyse the random cluster hierarchy in this protocol and provide a solution for low-energy and limited-loss optimization. Moreover, we extend these results to a multi-level version of LEACH, where clusters of nodes again self-organize to form clusters of clusters, and so on.

Key words: LEACH, Cluster head, Hierarchical protocol, Voronoi cluster, Low-energy sensor network

1 Introduction

Wireless sensor networks combine sensing, data processing and wireless communication into an ensemble of hundreds or thousands of small embedded units. Interconnecting protocols are designed to extract data from the network environment and provide efficient forwarding of useful information to a central processing center. Successful systems in the future are bound to meet a number of requirements of functionality, size, cost, energy handling, etc, to enable reliable monitor information to its users, possibly over the internet. An important challenge for research is to make sensor networks self-configuring, robust and maintenance-free.

Each node in a wireless sensor network is a battery driven tiny device equipped with sensoring and wireless communication capabilities. The devices enable extraction of data in the spatial environment of the nodes as well as transmission of data between the nodes in the network and between a network node and a base station. The base station could be located at a distance away from the

* Mailing adress: Department of Mathematics, Box 480, SE 751 06 Uppsala, Sweden. Telephone: +4618-471 3287

region of deployment of the sensor devices, or in the direct vicinity of these units and uses the arriving aggregated and recorded data for continuous monitoring or detection of special events. The nodes contain sophisticated layers of electronics, with a radio transceiver, antenna, computer, memory, batteries, sensors, and possibly solar panels, functionality for sensor calibration, etc. Yet, current research efforts envision a regular wireless sensor unit in the future to reach an integration size of $5 \times 5 \times 5$ mm and a manufacturing cost down to 1 euro, c.f.[7].

It is usually not an option to replace or reload battery driven sensor units. Thus, a particularly important feature for long life time sensor networks is energy efficiency. The most energy demanding part of a node is the wireless communication. Since the power consumption for transmission of data increases greatly with distance, it is important to reduce data transmission between nodes far apart and crucial to minimize the amount of upload traffic from sensor nodes to a distant base station. It has been proposed that cluster-based architectures are favorable for this purpose. Under a hierarchical clustering algorithm all nodes of the network are organized in a number of clusters, each with a designated cluster head node. The nodes within a cluster only communicate with their cluster head, which ideally is a short distance away if clusters are suitably chosen. The aggregated data from each cluster is then forwarded to the base station. In this way any substantial energy dissipation in the system is limited to the nodes currently serving as cluster heads. To avoid draining the power supply of individual sensor devices, the task of being a cluster head must rotate over time among the nodes. Also, to avoid costly central processing overheads, an effective cluster architecture protocol should be distributed in nature, so that clusters are formed only based on information already available in the nodes.

The architecture LEACH (Low-Energy Adapted Clustering Hierarchy) introduced by Heinzelman, Chandrakasan, and Balakrishnan, [4], [5], is a randomized, distributed clustering protocol, which is widely proposed and tested in wireless sensor networks. A number of variations and extensions have been discussed, see e.g. [6], [1]. Our contributions in this work include a refined probabilistic analysis of the LEACH protocol under the energy dissipation model in [5], based on a renewal-reward argument. As a further novelty, we introduce a suitable loss probability as control criteria. In addition, our approach extends the scope of modeling to a wider class of scenarios where the base station is located either inside or away from the sensor network. As in [1], stochastic geometric results of [3] are applied as a tool. From a network control perspective, our analyses yield added insight as to the role of the basic protocol parameters for minimizing energy consumption without jeopardizing functionality, hence optimizing the network life length. A similar study based on a different approach has been announced in [2]. These authors consider a scaling scenario where the sensor coverage region and the distance to the base station increase and study the energy dissipation in a scaling limit scenario.

As a main novelty in this work we propose a multi-level hierarchy version of LEACH, where the cluster head nodes of the original protocol form clusters of cluster heads, etc. Only the heads of the highest level clusters communicate directly with the base station. Our analysis shows, for example, that under loss-free conditions the one-level protocol remains competitive in comparison with the two-level version, whereas the benefits of multilevel hierarchies begin to show in lossy systems.

2 The Protocol Architecture LEACH

We first recall the basic ideas of low-energy adaptive clustering hierarchies as presented in [5]. Consider a system of sensor nodes in a given spatial region. The nodes are linked with each other and with a central base station, essentially using a single-hop mode of communication. The operation of LEACH is managed over consecutive *cycles* where each cycle consists of a fixed number of *rounds*. Each round has a set-up phase and a steady-state phase. The set-up phase begins with a selection process where randomized announcements among all nodes result in a random number of cluster heads (CH). This is followed by the formation of randomly sized clusters of Voronoi type, where each non-cluster head (nonCH) node makes a decision based on minimum distance to join exactly one of the available CH nodes. During the subsequent steady-state phase, the CH coordinates the transmission of data from all sensors in its cluster to the base station. In this way, the main energy load of the system is concentrated to CH nodes. In order to distribute the energy dissipation evenly over nodes, it is essential that the nodes take turns acting as CHs. Thus, in consecutive rounds of a cycle the self-selection of CHs involves only nonCHs of previous rounds. Also, to make its decision a node does not need to have knowledge of the remaining number of nonCHs in the system. The design of LEACH guarantees that for each integer r, it is possible to operate the protocol as a sequel of independent cycles each consisting of r rounds, such that each node has the burden of acting as CH exactly once in a cycle.

2.1 Probabilistic Aspects of Cluster Selection in LEACH

Consider a network of n identical nodes deployed randomly and uniformly in some spatial region, such as a set in two-dimensional space or along a straight line. Fix an integer r, known in advance to all nodes in the network. The result of the CH selection algorithm is a sequential list $X_1, \ldots, X_r$, where

$$X_i = \text{the number of cluster head nodes in round } i, \quad i = 1, \ldots, r.$$

During the set-up of the first round in a cycle each node decides with probability $1/r$ to be a CH, so that $X_1 \sim \mathrm{Bin}(n, 1/r)$, a binomially distributed random variable with parameters n and $1/r$. The remaining $n - X_1$ nonCH nodes are potential CHs for the second round in which the selection probability is modified to $1/(r-1)$ in order to keep the average number of CHs constant at n/r. By repeating the process r times, we may recast the LEACH selection algorithm in terms of the conditional distributions

$$
\begin{aligned}
X_1 &\sim \mathrm{Bin}(n, \frac{1}{r}) \\
X_2|X_1 &\sim \mathrm{Bin}(n - X_1, \frac{1}{r-1}) \\
X_3|X_1, X_2 &\sim \mathrm{Bin}(n - X_1 - X_2, \frac{1}{r-2}) \\
&\vdots \\
X_r|X_1, \ldots, X_{r-1} &\sim \mathrm{Bin}(n - X_1 - \ldots - X_{r-1}, 1) = n - X_1 - \ldots - X_{r-1}
\end{aligned}
$$

For comparison, note that [5] uses network size n and expected cluster size $k = n/r$ as the basic parameters.

The above conditional scheme is known in statistical sampling theory to be a property of the multinomial distribution. However, it appears that the following reformulation of the cluster selection algorithm has not been observed in the context of LEACH. Since we could not find a direct reference for this particular statistical fact we also give a proof.

Proposition 1. *The joint distribution of the number of cluster heads in consecutive rounds of a cycle is given by the multinomial distribution*

$$(X_1, \ldots, X_r) \sim \mathrm{Multnom}\Big(n, \Big\{\frac{1}{r} \ldots, \frac{1}{r}\Big\}\Big)$$

Moreover, letting $Z_1, \ldots, Z_r$ be i.i.d. Poisson distributed random variables with parameter n/r,

$$(X_1, \ldots, X_r) \overset{d}{=} \Big(Z_1, \ldots, Z_r \Big| \sum_{i=1}^{r} Z_i = n\Big)$$

Proof. The conditioning scheme implies, in particular, that $X_1 + \ldots + X_r = n$, which is the property that each node serves as cluster head exactly once per cycle. The generating function of the cluster head counting variables thus have the form

$$g(s_1, \ldots, s_r) = E(s_1^{X_1} \cdots s_r^{X_r}) = s_r^n \, E((s_1/s_r)^{X_1} \cdots (s_{r-1}/s_r)^{X_{r-1}}).$$

Next,

$$E((s_{r-1}/s_r)^{X_{r-1}}|X_1, \ldots, X_{r-2}) = \Big(\frac{s_{r-1} + s_r}{2s_r}\Big)^{n - X_1 - \ldots - X_{r-2}},$$

so

$$g(s_1, \ldots, s_r) = \Big(\frac{s_{r-1} + s_r}{2}\Big)^n E\Big(\Big(\frac{2s_1}{s_{r-1} + s_r}\Big)^{X_1} \cdots \Big(\frac{2s_{r-2}}{s_{r-1} + s_r}\Big)^{X_{r-2}}\Big).$$

By repeating these steps, conditioning on $(X_1, \ldots, X_k)$ for $k = r-3$ down to $k = 1$, we obtain

$$g(s_1, \ldots, s_r) = \Big(\frac{s_2 + \ldots + s_r}{r-1}\Big)^n E\Big(\Big(\frac{(r-1)s_1}{s_2 + \ldots + s_r}\Big)^{X_1}\Big) = \Big(\frac{s_1 + \ldots + s_r}{r}\Big)^n,$$

which is the generating function of the uniform multinomial distribution with parameters n and $1/r$. It is a basic property of the multivarate distribution that it is also obtained by conditioning independent Poisson random variables on their sum.

Remark 1. To see heuristically that the dynamics of LEACH is consistent with the multivariate distribution in Proposition 1, one may consider the outcome of randomly distributing n balls uniformly and without replacement in r boxes. Let X_1 denote the number of balls in box 1. Given X_1, the remaining $n - X_1$ balls are distributed uniformly over $r-1$ boxes. In particular, $\mathrm{Bin}(n - X_1, 1/(r-1))$ balls fall in box 2, which given round 1 is the number of CH nodes X_2 in the second round according to LEACH. Now, $n - X_1$ is $\mathrm{Bin}(n, 1 - 1/r)$-distributed. Thus, the number of balls in box 2 is binomially distributed with parameters n and $(1 - 1/r)/(r-1) = 1/r$. An iteration of this argument shows that the CH distribution in LEACH agrees with the stated multivariate distribution.

It is sometimes an advantage for energy efficiency to run the algorithm with large values of r. This will limit the average number of CHs and thus the number of costly data transmissions to the base station. As a consequence, during start-up of a round none of the potential nodes may choose to announce its intention to be a CH. The fact that LEACH does not guarantee a CH in each round may then become an issue of practical importance. In our treatment we interpret the case $X_i = 0$ for some i as the complete loss of data in the entire network in this round. To deal with such lost rounds, suppose that the network user is willing to accept a loss probability $\alpha > 0$, in the sense that the proportion of rounds over long time where no clusters form is at most α. Since $P(X_i = 0) = (1 - 1/r)^n$ it follows that the admissible range of values for r is limited to integers $1 \leq r \leq r_\alpha$ where

$$r_\alpha = [(1 - \alpha^{1/n})^{-1}]. \tag{1}$$

It might be argued that rather than incorporating lost rounds, LEACH should be altered so that each round results in the formation of at least one CH. This could be achieved by adding to the set-up phase a distributed control mechanism, which is activated if a nonCH node at the end of a short time-out period has not received any CH announcements from other nodes. If this happens to one node it happens to all. Upon activation of the control in round i, the natural consequence is to restart the cluster formation phase so that eventually $X_i \geq 1$. This modification will cause a random time delay and a conditioning of the binomial distributions to be positive. However, in this work we stay with the original LEACH protocol and analyze its performance by using lost rounds as a means to optimizing and tuning the model parameters.

2.2 Renewal-reward analysis

We begin by analyzing the energy usage in the network under the assumption that the nodes have fixed locations $\xi = (\xi_j)_{1 \le j \le n}$ in a planar region Λ. Suppose each round lasts a constant period of time of length μ. This includes the set-up of clusters in addition to the steady-state phase, which typically is the predominant mode of operation for the network. Thus, time intervals of length $r\mu$ naturally form independent cycles of a renewal process, which counts the number of complete LEACH cycles over time. With the jth such renewal cycle we associate the total energy R_j that the system must use during the entire cycle for communication and data aggregation. Now, the energy load on the network during a cycle is symmetric over rounds. In fact, as will be clarified in the next section the energy usage pattern only depends on the size and shape of clusters. It then follows from Proposition 1 that each R_j may be represented as a sum $R_j = T_{j1} + \ldots + T_{jr}$ of identically distributed (dependent) random variables $(T_{ji})_{1 \le i \le r}$ where T_{ji} is the energy dissipated in the network during round i of cycle j.

Let R and T represent the distributions of the rewards (R_j) and (T_{ji}), respectively, and let $R(t)$ denote the total energy consumed by the network up to time t. We write $\mathbf{E}_\xi$ for expectation with respect to the conditional probability $\mathbf{P}_\xi$ given spatial location ξ. Since the cycles have fixed length $r\mu$ and the rewards are nonnegative and satisfy $\mathbf{E}_\xi R \le r\mathbf{E}_\xi T$, the prerequisites for the renewal-reward theorem are satisfied as soon as $\mathbf{E}_\xi T < \infty$. In this case the average energy consumption per time unit is given asymptotically by the cycle average

$$\lim_{t\to\infty} \frac{1}{t} R(t) = \frac{\mathbf{E}_\xi(R)}{r\mu} = \frac{1}{\mu}\mathbf{E}_\xi(T), \quad \mathbf{P}_\xi - a.s.$$

As the basic measure of performance of the wireless sensor network under LEACH we take the corresponding energy dissipation per time unit averaged over the random locations of the nodes, that is $\mathbf{E}(T)/\mu$. Our next goal is to model T and compute or estimate $\mathbf{E}(T)$ by means of the cluster distribution properties.

2.3 Energy dissipation model

The energy model in [5] refers to sensors randomly distributed over the square $\Lambda_M = M \times M$ in the plane. The data is transmitted in the form of messages with fixed size ℓ bits. Data aggregation takes place in CH sensors and radio communication within the network follows the free-space model where power loss is proportional to squared distance between sender and receiver. We consider two scenarios for communication with the base station:

distBS: (model of [5]) Data transmission between CH nodes and a distant base station located outside of Λ_M follows the multipath fading model of power loss proportional to the fourth power of the distance;

nearBS: The base station is placed at a point within Λ_M, such as the center point $(M/2, M/2)$ and operates under the free-space model.

The relevant energy constants are summarized in Table 1, with values that are used for some numerical illustrations below.

Table 1. Energy model in Ref [5]

energy coefficient	notation	numerical value and unit
electronic	E_{elec}	50 nJ/bit
data aggregation	E_{DA}	5 nJ/bit/signal
free-space amplifier	$\mathcal{E}_{\text{fs}}$	10 pJ/bit/m^2
multipath amplifier	$\mathcal{E}_{\text{mp}}$	0.0013 pJ/bit/m^4

We have demonstrated above, that LEACH is completely symmetric over rounds. Moreover, the performance metric does not involve the dependence between rounds within a cycle. Thus, for a given round let T_{within} be the energy loss for communication within clusters and T_{toBS} the additional energy dissipation due to uploading data from CHs to the base station. We are interested in the total energy loss $T = T_{\text{within}} + T_{\text{toBS}}$ per round and, in particular, in the expected energy dissipation $E(T)$ per round as a function of the protocol parameters n and r. In Proposition 2 below we give an approximate expression $\psi_n(r) \approx E(T)$, which is obtained by analyzing the separate parts of T. For given size n and an acceptable loss rate α in the network, LEACH should operate with the parameter r, $1 \leq r \leq r_\alpha$, tuned so that $\psi_n(r)$ is minimal.

2.4 Estimated energy loss in two versions of LEACH

Let X be the number of clusters in a given round and write $L_1, \ldots, L_X$ for the number of nonCH nodes in each of these clusters. For a nonCH node to transmit an ℓ-bit message to its CH distance d away, the radio expends the power $\ell(E_{\text{elec}} + \mathcal{E}_{\text{fs}} d^2)$. To receive this message the CH expends another ℓE_{elec}. In addition, the aggregation of data in the CH will consume ℓE_{DA} per node involved. Thus, in a cluster consisting of one CH located at ξ^{CH} and L nonCH nodes at locations $\xi_1, \ldots, \xi_L$, the power expenditure adds to

$$\ell E_{\text{elec}}(L + L) + \ell E_{\text{DA}}(L + 1) + \ell \mathcal{E}_{\text{fs}} \chi^2, \quad \chi^2 = \sum_{l=1}^{L} |\xi^{CH} - \xi_l|^2.$$

Since $\sum_{m=1}^{X} L_m = n - X$,

$$T_{\text{within}} = \ell E_{\text{elec}} 2(n - X) + \ell n E_{\text{DA}} + \ell \mathcal{E}_{\text{fs}} \sum_{m=1}^{X} \chi_m^2, \quad \chi_m^2 = \sum_{l=1}^{L_m} |\xi_m^{CH} - \xi_{lm}|^2$$

where ξ_m^{CH} and (ξ_{lm}) indicate the locations of the CH and nonCH nodes of the relevant cluster. The last term in T_{within} involves the cluster head selection distribution in Proposition 1 and the Voronoi tessalation of Λ_M, which decides the cluster size variables (L_j) and the length of the edges between nodes and cluster head in each cluster. To obtain an estimate of

$$\mathbf{E}\Big(\sum_{m=1}^{X}\chi_m^2\Big)=\mathbf{E}\sum_{m=1}^{X}\mathbf{E}(\chi_m^2|X),$$

we note that given X, for each m,

$$\chi_m^2=\sum_{j=1}^{L_m}d^2(\cdot,\cdot),\quad \sum_{\nu=1}^{X}L_\nu=n-X.$$

By alluding to the Poisson representation in Proposition 1, we may compare this situation to that in [3] for Voronoi clusters in a bivariate Poisson point process in the plane. These authors obtain the expected value over edge lengths in a typical Voronoi cell, in the precise sense of Palm distribution, in terms of the corresponding Poisson intensities. If the intensities λ_0 and λ_1 are taken to represent nonCH and CH nodes per unit area, respectively, then the expected sum of squared edge lengths in a typical Voronoi cell is given by

$$\lambda_0\int_{R^2}|x|^2e^{-\lambda_1\pi|x|^2}\,dx=\frac{\lambda_0}{\pi\lambda_1^2}.$$

To make the connection to our model, conditional on $X=k$ we may compare the cluster functional χ_m^2 to the sum of squares of the edge lengths which arise in a bivariate spatial Poisson process with intensities corresponding to $\lambda_0=(n-k)/M^2$ and $\lambda_1=k/M^2$. Hence

$$E(\chi_m^2|X=k)\approx\frac{n-k}{\pi k^2}M^2,\quad 1\le m\le k,$$

and

$$\mathbf{E}\Big(\sum_{m=1}^{X}\chi_m^2\Big)\approx\mathbf{E}\Big(\frac{n-X}{X}\Big)\frac{M^2}{\pi}=H_n(r)\frac{M^2}{\pi},\tag{2}$$

where

$$H_n(r)=\sum_{k=1}^{n}\frac{n-k}{k}\binom{n}{k}\Big(\frac{1}{r}\Big)^k\Big(1-\frac{1}{r}\Big)^{n-k}.\tag{3}$$

It follows that

$$E(T_{\text{within}})\approx\ell E_{\text{elec}}2n(1-1/r)+\ell E_{\text{DA}}\,n+\ell\mathcal{E}_{\text{fs}}\pi^{-1}\,H_n(r)\,M^2.$$

Next we turn to the energy loss associated with transmitting the data in the system from CH nodes to the base station, which we assume is located in the point $\xi_{BS}=(u,v)M$. For the case $(u,v)\notin[0,1]^2$ that is $\xi_{BS}\notin\Lambda_M$, we adopt the model *distBS* so that multipath transmission of an ℓ-bit message requires the additional energy dissipation

$$T_{\mathrm{distBS}} = \ell E_{\mathrm{elec}} X + \ell \mathcal{E}_{\mathrm{mp}} \sum_{k=1}^{X} |\xi_k^{CH} - \xi_{\mathrm{BS}}|^4.$$

Since the CH locations (ξ_k^{CH}) are uniformly scattered over Λ_M and independent of the number of CH nodes, the expected energy loss is

$$E(T_{\mathrm{distBS}}) = \ell E_{\mathrm{elec}} \frac{n}{r} + \ell \mathcal{E}_{\mathrm{mp}} \frac{n}{r} M^4 \, d^4_{\mathrm{distBS}}(u,v),$$

with

$$d^4_{\mathrm{distBS}}(u,v) = \int_{[0,1]^2} ((x-u)^2 + (y-v)^2)^2 \, dxdy.$$

For the case $\xi_{BS} \in \Lambda_M$, the modeling assumption *nearBS* stipulates free-space transmission for which the energy dissipation per ℓ-bit message amounts to

$$T_{\mathrm{nearBS}} = \ell E_{\mathrm{elec}} X + \ell \mathcal{E}_{\mathrm{fs}} \sum_{k=1}^{X} |\xi_k^{CH} - \xi_{\mathrm{BS}}|^2.$$

with expected value

$$E(T_{\mathrm{nearBS}}) = \ell E_{\mathrm{elec}} \frac{n}{r} + \ell \mathcal{E}_{\mathrm{fs}} \frac{n}{r} M^2 \, d^2_{\mathrm{nearBS}}(u,v),$$

and now

$$d^2_{\mathrm{nearBS}}(u,v) = \int_{[0,1]^2} ((x-u)^2 + (y-v)^2) \, dxdy.$$

In particular, if the base station is placed at the middle point of the deployment region, $u = v = 1/2$, then $d^2_{\mathrm{nearBS}}(u,v) = 1/6$.

By summing up the terms for energy consumption within clusters and between cluster heads and base, we obtain an approximate expression for the performance measure $E(T)$ in terms of network size n and the protocol parameter r. Based on the above arguments we summarize these findings as follows.

Proposition 2. *Consider the sensor network model as above with n nodes deployed uniformly within the square $[0,M]^2$, the base station placed in the point (uM, vM), and with energy use specified by the parameters in Table 1. Asymptotically over many cycles of LEACH, the average energy dissipation per round and per ℓ-bit message is given approximately by*

$$\psi_n(r) = \ell\Big(E_{\mathrm{elec}} n \Big(2 - \frac{1}{r}\Big) + E_{\mathrm{DA}} n + \pi^{-1} \mathcal{E}_{\mathrm{fs}} M^2 \, H_n(r) + C_M \frac{n}{r}\Big), \quad 1 \le r \le n,$$

where $H_n(r)$ is introduced in (3) and

$$C_M = \begin{cases} \mathcal{E}_{\mathrm{mp}} \, M^4 \, \int_{[0,1]^2} ((x-u)^2 + (y-v)^2)^2 \, dxdy & \text{for distBS}, \\ \mathcal{E}_{\mathrm{fs}} \, M^2 \, \int_{[0,1]^2} ((x-u)^2 + (y-v)^2) \, dxdy. & \text{for nearBS}. \end{cases} \tag{4}$$

To optimize LEACH for energy efficiency while accepting lost rounds at a rate of α, use

$$\tilde{r} = \arg\min\{\psi_n(r) : 1 \leq r \leq r_\alpha\},$$

with r_α defined in (1).

Remark 2. The following series expansion for $H_n(r)$, which is extracted from [8], Corollary 1, is accurate for not too large values of r:

$$H_n(r) = (r-1)\Big(1 + \frac{r}{n} + \frac{r(2r-1)}{n^2} + \mathcal{O}\Big(\frac{r^3}{n^3}\Big)\Big).$$

Remark 3. The analysis in [5] involves the (lower bound) approximation

$$\mathbf{E}(X^{-1}) \approx \frac{r}{n} = \frac{1}{\mathbf{E}(X)}$$

and corresponds to taking $H_n(r) \approx r$ in our setting. By ignoring one additional term we obtain for the case *distBS*

$$\psi_n(r) \approx \ell\Big(2E_{\mathrm{elec}}n + E_{\mathrm{DA}}n + \pi^{-1}\mathcal{E}_{\mathrm{fs}}M^2\,r + \mathcal{E}_{\mathrm{mp}}\,M^4\,d^4_{\mathrm{distBS}}(u,v)\,\frac{n}{r}\Big).$$

In their study [5] takes $(u,v) = (1/2, 7/4)$ and varies d_{BS} over min and max distance to the CHs. We obtain $d_{\mathrm{distBS}}(1/2, 7/4) \approx 1.3699$. Apart from this and from their choice of prefactor of the free-space transmission term, $1/(2\pi)$ instead of our $1/\pi$, we recover the corresponding expression in [5] by setting $k = n/r$. It leads to the conclusion that k, the expected number of clusters in the system, should be proportional to the square root of n.

2.5 Numerical illustration

To illustrate the optimization problem in Proposition 2, we consider again the experimental and numerical setup in [5]. Thus, we take $M = 100$, $\xi_{BS} = (1/2, 7/4)M = (50, 175)$, $\ell = 4\,200$, and adopt the energy parameter values listed in Table 1. By Proposition 1 for *distBS*,

$$\begin{aligned}\psi_n(r) &= \ell\Big(E_{\mathrm{elec}}n\Big(2 - \frac{1}{r}\Big) + E_{\mathrm{DA}}n + \pi^{-1}\mathcal{E}_{\mathrm{fs}}M^2\,H_n(r) + \mathcal{E}_{\mathrm{mp}}\,M^4\,3.522\,\frac{n}{r}\Big)\\ &= 4.2\cdot 10^{-4}\Big(\frac{n}{2}\Big(2 - \frac{1}{r}\Big) + \frac{n}{20} + \pi^{-1}H_n(r) + 4.5785\frac{n}{r}\Big) \quad \text{(Joule)}\end{aligned}$$

The example in [5] is $n = 100$. Figure 1, lower curve, shows the graph of $\psi_{100}(r)$. For comparison, the upper curve represents the energy loss with twice as many nodes in the same region, $\psi_{200}(r)$. These curves have no distinct minimum point. If the number of rounds per cycle is less than 15 or so, then the multipath fading term is dominant with increased risk of draining the energy supply in the nodes. Cycles of length $r \approx 20$ and above appear to be equivalent energy wise. However, with increasing r there is a trade-off in terms of losses. By (1), a 5% loss rate in the network of size $n = 100$ will limit the length of cycles to $r_{0.05} = 33$ and a loss rate of 1% implies a maximum of $r_{0.01} = 22$. Hence for this example, LEACH achieves its optimal performance at $n/r \approx 5$.

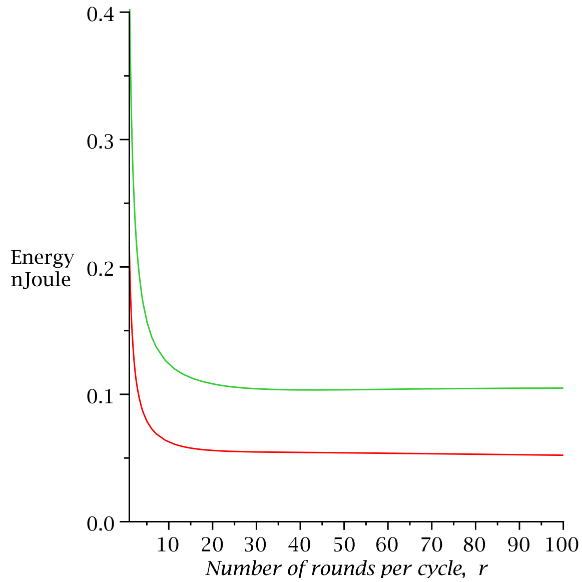

Figure 1. Energy loss per round as function of cycle length r, distant base station and energy model of [5], lower curve: $n = 100$, upper curve: $n = 200$

In case the base station can be placed in the same region as the sensors themselves, the energy balance shifts. It is no longer obvious that short cycles - many cluster heads on average - is a disadvantage. Indeed, if the base station is a conveniently located centre point of the network region, it may be counterproductive to use only a few cluster heads. Figure 2 applies to fixed size $n = 100$ and the same situation as above except that we use *nearBS* in Proposition 1 with the base station either in a corner point $(0, 0)$ (upper curve) or the middle point, that is $u = v = 1/2$ (lower curve). It is noteworthy that if the base station is right in the middle of the network then optimal performance of LEACH is obtained for $r = 1$, which is the case when each node forms its own cluster of size one. If instead the CHs must establish data transmission with a corner point, then the upper curve shows that the optimal (integer) value is $r = 7$.

3 A multi-level hierarchical algorithm

A limiting assumption in LEACH is that all nodes are in within communication range of each other and the base station. As discussed already in [5], to relax the impact of this assumption one alternative is to develop further the hierarchy of clusters by forming "super clusters" out of the CH nodes. It is natural to follow up on this line of thought and investigate the possible gain in terms of energy consumption if we let the cluster heads which have been selected in a round, again act according to the same protocol principles and form new clusters of cluster heads. This will require a node to store further information about its prehistory as CH and prolonge the set-up phase, but should not impose any significant additional work in the system since the same mechanisms as in the one-level version are iterated.

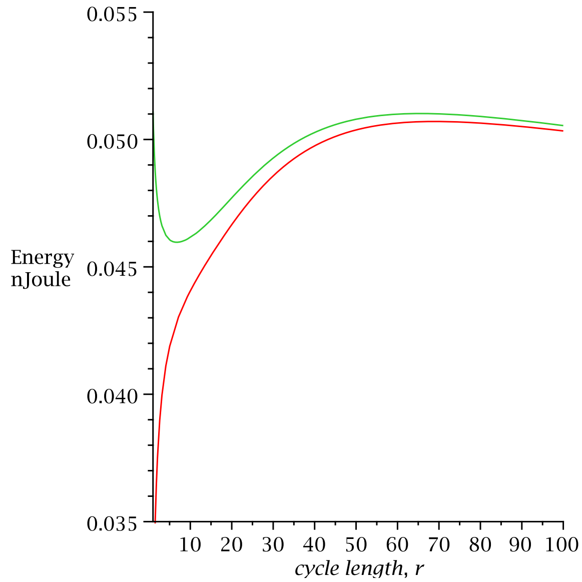

Figure 2. Energy loss per round for nearby base station at a corner point (upper curver) or middle point (lower curve)

We introduce m-level LEACH by specifying m parameters $r_1, \dots, r_m$ and an m-level CH selection algorithm as follows. In setting up the first round, the ensemble of n nodes make m decisions. First of all, as in regular 1-level LEACH, the nodes pick $X_{1,1}$ CHs according to $\mathrm{Bin}(N, 1/r_1)$. These selected CHs immediately go on to announce among themselves with probability $1/r_2$ their willingness to be CH-heads (2-CH). Given $X_{1,1}$, the resulting number, $X_{2,1}$, of 2-CH nodes is $\mathrm{Bin}(X_{1,1}, 1/r_2)$. This is repeated m times and yields in the final step $X_{m.1}$ m-CH nodes. The network is now ready to enter the steady state phase in which data will be transmitted from nonCH to CH to 2-CH and so on, until the m-CH nodes complete the round by handling data transmission to the base station. In setting up round 2, the cluster heads of levels 1 to $m-2$ remain the same. Only the (m-1)-CH nodes must select a new set of m-CHs among the $X_{m-1,1} - X_{m,1}$ available candidates. The selection probability changes to $1/(r_m - 1)$ and we obtain $X_{m,2}$ mCH nodes in charge of exchange with the base station in round 2. After completion in this manner of the first sub-cycle of r_m rounds, all $\sum_{j=1}^{r_m} X_{m,j} = X_{m-1,1}$ CHs on level $m-1$ are considered used up and will be replaced in the beginning of round $r_m + 1$. The suitable selection probability is $1/(r_{m-1} - 1)$. After $r_m r_{m-1}$ rounds it is time to go one level further down in the hierarchy and make the appropriate update. The subsequent decisions and updates of CHs at the various levels follow the path of a contour around the branches of a regular rooted tree of depth m where all tree-nodes at distance j from the root have r_j branches. Finally, after a total of $r_1 \cdots r_m$ rounds the system has completed a full cycle in which each node has been a CH on each level exactly once.

The analogous result to Proposition 1 for m-level LEACH yields that the collection of cluster sizes on level j, that is $(X_{j,1},\dots X_{j,r_1\cdots r_j})$, $j=1,\dots,m$ has the uniform multinomial distribution with parameters n and $\frac{1}{r_1\cdots r_j}$. Since the full cycles of length $r_1\cdots r_m$ are independent, we may apply the renewal-reward theorem as before and measure performance of the system as the expected energy loss per round.

Theorem 1. *Suppose a network with n nodes is deployed uniformly in $[0,M]^2$. The base station is located in (uM,vM), and the energy model in Proposition 2 applies. For the m-level LEACH model introduced above with parameters $\mathbf{r}=(r_1,\dots,r_m)$, the average energy dissipation per round and ℓ-bit message is given approximately by*

$$\psi_n^{(m)}(\mathbf{r}) = \ell\Big(2E_{\mathrm{elec}}\Big(1-\frac{1}{r_1\cdots r_m}\Big)n + E_{\mathrm{elec}}\sum_{j=1}^{m}\frac{n}{r_1\cdots r_j}$$
$$+E_{\mathrm{DA}}\Big(1+\sum_{j=1}^{m-1}\frac{1}{r_1\cdots r_j}\Big)n + \pi^{-1}\mathcal{E}_{\mathrm{fs}}M^2\,H_n^{(m)}(\mathbf{r}) + C_M\frac{n}{r_1\cdots r_m}\Big),$$

where

$$H_n^{(m)}(\mathbf{r}) = \sum_{j=1}^{m}\frac{r_j-1}{r_1\cdots r_j-1}H_n(r_1\cdots r_j),\quad H_n^{(1)}=H_n \tag{5}$$

$H_n(r)$ is introduced in (3), and C_M is defined in Proposition 2.

For a given acceptable loss rate α, optimal performance of m-LEACH is achieved by minimizing $\psi_n^{(m)}(\mathbf{r})$ over all m-tuples $\mathbf{r}=(r_1,\dots,r_m)$ of integers such that

$$r_1\cdots r_m \le r_\alpha.$$

Proof. The total expected energy dissipation is obtained by adding over all m levels and each time use Proposition 2 with the appropriate parameter values r_j inserted. In this way all terms in $\psi_n^{(m)}$ arise straightforwardly, except the free-space energy within clusters which has the form

$$\frac{1}{\pi}\mathcal{E}_{\mathrm{fs}}M^2\sum_{j=1}^{m}\mathbf{E}\Big(\frac{X_{j-1,1}-X_{j,1}}{X_{j,1}}\Big),$$

where we use the cluster size variables $(X_{j,1})$ which arise in the first r_1 rounds of m-LEACH and we have put $X_{0,1}=n$. To verify (5) we must show

$$\mathbf{E}\Big(\frac{X_{j-1,1}-X_{j,1}}{X_{j,1}}\Big) = \frac{r_j-1}{r_1\cdots r_j-1}H_n(r_1\cdots r_j),\quad 2\le j\le m.$$

Fix a j, $2\le j\le m$, and write $p=1/r_1\cdots r_{j-1}$ and $q=1/r_j$. We have

$$X_{j-1,1}\sim \mathrm{Bin}(n,p),\qquad X_j|X_{j-1,1}\sim\mathrm{Bin}(X_{j-1,1},q)$$

so that

$$\mathbf{E}\Big(\frac{X_{j-1,1}-X_{j,1}}{X_{j,1}}\Big) = \sum_{k=1}^{n}\sum_{\ell=1}^{k}\frac{k-\ell}{\ell}\binom{n}{k}p^k(1-p)^{n-k}\binom{k}{\ell}q^\ell(1-q)^{k-\ell}$$
$$= \frac{p(1-q)}{1-pq}\sum_{\ell=1}^{n}\frac{n-\ell}{\ell}\binom{n}{\ell}(pq)^\ell(1-pq)^{n-\ell}$$
$$= \frac{p(1-q)}{1-pq}H_n(\frac{1}{pq}) = \frac{r_j-1}{r_1\cdots r_j-1}H_n(r_1\cdots r_j),$$

which is (5). The loss probability is given by

$$P(X_{1,1}=0)+P(X_{1,1}>0,X_{2,1}=0)+\dots$$
$$\dots+P(X_{1,1}>0,\dots,X_{m-1,1}>0,X_{m,1}=0)$$
$$=\sum_{j=1}^{m}(P(X_{j,1}=0)-P(X_{j-1,1}=0)) = \Big(1-\frac{1}{r_1\cdots r_m}\Big)^n,$$

which completes the proof.

3.1 Numerical illustration of 2-level LEACH

To evaluate the performance of m-level LEACH with that of the original one-level protocol under comparable loss conditions, it is natural to consider the difference in energy dissipation, which according to Theorem 1 is given by

$$\psi_n^{(m)}(r_1,\dots,r_m)-\psi_n^{(1)}(r_1\cdots r_m) = \ell\Big((E_{\text{elec}}+E_{\text{DA}})\sum_{j=1}^{m-1}\frac{n}{r_1\cdots r_j}$$
$$+\pi^{-1}\mathcal{E}_{\text{fs}}M^2\,H_n^{(m-1)}(r_1,\dots,r_{m-1})-\frac{r_1\cdots r_m-r_m}{r_1\cdots r_m-1}H_n(r_1\cdots r_m)\Big).$$

In particular, if we compare 1-level and 2-level LEACH applied to the same numerical example as above, this yields

$$\psi_n^{(2)}(r_1,r_2)-\psi_n(r_1\cdot r_2)$$
$$= 4.2\cdot 10^{-4}\Big(0.55\frac{n}{r_1}+\pi^{-1}\,H_n(r_1)-\pi^{-1}\frac{(r_1-1)r_2}{r_1\cdot r_2-1}H_n(r_1\cdot r_2)\Big). \qquad (6)$$

Figure 3 shows the differencing energy dissipation for the case $n=200$, for which loss rates of 1% and 5% yield $r_{0.01}=43$ and $r_{0.05}=67$. The merits of 2-level LEACH begin to show in the approximate range of parameters $r_1r_2\geq 50$, which as we have seen is also the parameter regime where lost rounds come into play regularly.

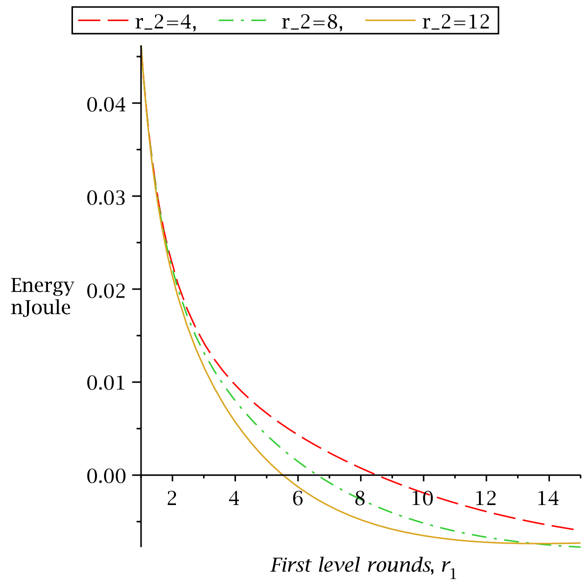

Figure 3. Energy difference according to (6)

References

1. Bandyopadhyay, S., Coyle, E.J.: An energy efficient hierarchical clustering algorithm for wireless sensor networks. Proceedings of IEEE INFOCOM, 2003.
2. Banerjee, P., Lahiri, S.N.: Optimal configuration of the LEACH: A statistical analysis. Presentation available at http://www.samsi.info/200708/sensor/presentations/Tue/Soumen Lahiri.pdf
3. Foss, S.G., Zuyev, S.A.: On a Voronoi aggregative process related to a bivariate Poisson process. Adv. Appl. Probab. 28:4, 965–981 (1996).
4. Heinzelman, W.B., Chandrakasan, A.P., Balakrishnan, H.: Energy-efficient communication protocol for wireless microsensor networks. In Proceedings of the Hawaii International Conference on System Sciences, Maui, Hawaii, January 2000.
5. Heinzelman, W.B., Chandrakasan, A.P., Balakrishnan, H.: An application-specific protocol architecture for wireless microsensor networks. IEEE Trans. Wireless Commun. 1:4, 660–670 (2002).
6. Haiming Yang, Sikdar, B: Optimal Cluster Head Selection in the LEACH Architecture. Proceedings of IPCCC 2007, IEEE Performance, Computing, and Communications Conference, 93–100 (2007).
7. WISENET, Uppsala VINN Excellence Center for Wireless Sensor Networks, Uppsala University, Vision statement, http://www.wisenet.uu.se.
8. Wuyungaowa, Wang, T.: Asymptotic expansions for inverse moments of binomial and negative binomial. Statistics and Probability Letters 78, 3018–3022 (2008).