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Convergence to Fractional Brownian Motion and to the Telecom Process: the Integral Representation Approach

Ingemar Kaj and Murad S. Taqqu

Abstract. It has become common practice to use heavy-tailed distributions in order to describe the variations in time and space of network traffic workloads. The asymptotic behavior of these workloads is complex; different limit processes emerge depending on the specifics of the work arrival structure and the nature of the asymptotic scaling. We focus on two variants of the infinite source Poisson model and provide a coherent and unified presentation of the scaling theory by using integral representations. This allows us to understand physically why the various limit processes arise.

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1. Introduction

Our understanding of the random variation in packet networks computer traffic has improved considerably in the last decade. Mathematical models were developed, which capture patterns observed in traffic data such as self-similarity. An essential element of these models is the use of heavy-tailed distributions at the microscopic scale. Because the mathematics can be involved, it is often difficult to understand physically why heavy-tailed distributions yield the different stochastic processes that appear at the macroscopic scale. We shall use integral representations in order to clarify this mechanism. We aim to give a *coherent and unified presentation* of a large spectrum of approximation results, so that the features and the dependence structure of the limiting processes are convincingly "explained" by the underlying model assumptions including heavy tails. This approach will also allow us to solve some open problems.

A number of different models have been suggested to capture the essential characteristics of packet traffic on high-speed links. A popular view of network traffic is an aggregate of packet streams, each generated by a source that is either in an active on-state transmitting data or an inactive off-state. In reality separate flows of packets interact because of the influence of transport protocols or other mechanisms, but in modeling work it is a standard approach to assume statistical independence between flows. This leads naturally to considering the cumulative workload as the result of adding independent on-off processes that are integrated over time. The superposition of independent renewal-reward processes have a similar interpretation, where the sources are not necessarily switching between on and off but rather change transmission rates randomly at random times. A third category of models is based on Poisson arrivals of independent sessions, where the sessions are typically long-lived and carry workload continuously or in discrete packets. Such models of Poisson shot noise type, called infinite source Poisson processes, have been specifically proposed for modeling noncongested Internet backbone links at the flow level, Barakat *et al.* 2003.

The preceding models have heavy-tailed versions, obtained by assuming that the on/off periods, the interrenewal times, or the session durations are given by heavy-tailed distributions and one can define stationary versions of these traffic models. Through detailed studies, the asympotic behavior of the workload fluctuations around its mean has been investigated and a pattern has emerged with certain generic characteristics. Taqqu (2002) and Willinger *et al.* (2003) provide summaries including details on the relevant networking concepts and observed characteristics of measured traffic. Stegeman (2002), Pipiras *et al.* (2004) and Mikosch *et al.* (2002) give a variety of results while investigating the range of possible asymptotic growth conditions. Briefly, whenever the number of multiplexing flows grows at a fast rate relative to time, fractional Brownian motion appears as a canonical limit process. If the rewards, i.e., the transmission rates, have heavy tails, then a more general stable process with dependent increments, called the Telecom process, appears instead of fractional Brownian motion, see Levy and Taqqu (2000) and Pipiras and Taqqu (2000). Whenever the degree of aggregation is slow compared to time, the natural limit process is a stable Lévy process with independent increments. In an intermediate scaling regime another type of Telecom process appears, which is neither Gaussian nor stable, Gaigalas and Kaj (2003).

Some further papers dealing with fractional Brownian limit processes under fast growth are Rosenkrantz and Horowitz (2002) and Çağlar (2004). Results on approximation by the stable Lévy motion under slow growth conditions are derived in Jedidi *et al.* (2004), and the intermediate scaling regime is further investigated in Kaj and Martin-Löf (2005). The many results in the literature use a variety of mathematical techniques, often complicated and specialized for the particular model studied, offering limited intuition as to the origin of the limit processes and their physical explanation in terms of first principles of the underlying models.

The purpose of this paper is to consider a physical model which shows clearly why these various limiting scaling processes arise. For this purpose we use integral representations and focus on two variants of the infinite source Poisson model. Because integral representations are interpretable physically, they shed light on the structure of the resulting limit processes. By using this approach, we can derive all the above asymptotics in a unified manner. We are also able to provide the solution to an open problem: finding the intermediate process when the rewards have infinite variance. Some of the approximation techniques we use have been unified in Pipiras and Taqqu (2006). In addition, our approach for the case of fixed rewards has been successfully extended in a spatial setting of Poisson germ-grain models and recast in a more abstract formulation involving random fields in Kaj *et al.* (2007), further developed in Biermé *et al.* (2006, 2007).

The paper is organized as follows. In Section 1 we develop the models and derive some basic properties. We state the main results in Section 2 and prove them in Section 3. In Section 4, the convergence in finite-dimensional distributions of the continuous flow model is extended to weak convergence in function space.

1.1. The infinite source Poisson model

Infinite source Poisson models are arrival processes with $M/G/\infty$ input obtained by integrating the standard $M/G/\infty$ queueing system size. The resulting class of Poisson shot noise processes are widely used traffic models which describe the amount of workload accumulating over time. Such models have been suggested as realistic workload processes for Internet traffic, where is is natural to assume that while web sessions are initiated according to a Poisson process, duration lengths and transmission rates could vary considerably. More exactly, the aggregated traffic consists of sessions with starting points distributed according to a Poisson process on the real time line. Each session lasts a random length of time and involves workload arriving at a random transmission rate. There are two slightly different sets of assumptions that are natural to make regarding the precise traffic pattern during a session. The first is that the workload arrives continuously at a randomly chosen transmission rate, which is fixed throughout the session and independent of the session length. The second type of model assumes that the workload arrives in discrete entities, packets, according to a Poisson process throughout the session, and such that the size of each packet is chosen independently from a given packet size distribution. The duration and the continuous or discrete rate of traffic in one session is independent of the traffic in any other session, although in general the sessions overlap. One novelty in this work is that we point out how these two types of models differ in their asymptotic behavior and that we explain the origin of the qualitative differences.

We are going to introduce the workload models using directly an integral representation with respect to Poisson measures, as in Kurtz (1996) and Çağlar (2004), rather than working with a more traditional Poisson shot noise representation, as in Kaj (2005). This approach is designed to help in understanding the scaling limit behavior of the models, and leads to useful representations of the limit processes. In formalizing the traffic pattern, the starting points of sessions will be called arrival times and the session lengths their durations. The traffic rate will be described in terms of a reward distribution, either continuous flow rewards or compound Poisson rewards. With each session in the continuous flow rate model we associate an arrival time S, a duration U and a reward R. A session in the case of compound Poisson packet arrivals is characterized by an arrival time S, a duration U, and a compound Poisson process $\Xi(t)$ constructed from copies of the reward R.

The basic notation and assumptions are as follows:

Arrivals: Workload sessions start according to a Poisson process on the real line with intensity $\lambda > 0$. The arrival times are denoted $\dots, S_j, S_{j+1}, \dots$.

Durations: The session length distribution is represented by the random variable $U > 0$ with distribution function $F_U(u) = P(U \le u)$ and expected value

$$\nu = E(U) < \infty.$$

We have either

$$E(U^2) < \infty$$

or

$$P(U > u) \sim L_U(u)u^{-\gamma}/\gamma$$

as $u \to \infty$, where $1 < \gamma < 2$. We extend the parameter range to $1 < \gamma \le 2$, by letting $\gamma = 2$ represent the case $E(U^2) < \infty$.

Rewards: (1) Continuous flow rewards. The transmission rate valid during a session is given by a random variable $R > 0$ with $F_R(r) = P(R \le r)$ and

$$E(R) < \infty.$$

We suppose either

$$E(R^2) < \infty$$

or

$$P(R > r) \sim L_R(r)r^{-\delta}/\delta$$

as $r \to \infty$, where $1 < \delta < 2$. Again the parameter range extends to $1 < \delta \le 2$ by letting $\delta = 2$ be the case $E(R^2) < \infty$. Observe that the aggregated workload in a session is the product UR.

(2) Compound Poisson rewards. The packet stream in a session is a compound Poisson process

$$\Xi(t) = \sum_{i=1}^{M(t)} R_i,$$

where the packet sizes (R_i) are independent and identically distributed with distribution $F_R(dr)$ having the same properties as above for continuous flow rewards, and $\{M(t), t \ge 0\}$ is a standard Poisson process of intensity one. In this case, the aggregated workload in a session is $\sum_{i=1}^{M(U)} R_i$.

Remark 1. We use γ, δ as basic parameters for renewals and rewards for a number of reasons: (1) there will be no confusion with other works that used α, β. (2) It maintains the order used in other works: $\gamma \leftrightarrow \alpha$, $\delta \leftrightarrow \beta$. (3) Since the limit processes can be γ-stable or δ-stable, it is preferable to use indices such as γ and δ which do not have the intrinsic meaning that α and β have in relation to stable distributions. We suggest in fact that, in the future, γ and δ be used instead of α and β.

Remark 2. To simplify the presentation of our work and the statements of our results we will set $L_U = L_R = 1$. In the proofs section, however, we deal with the the modifications that one has to do when L_U and L_R are general slowly varying functions.

We are now prepared to define the infinite Poisson source workload process using integrals with respect to a Poisson measure. The aim is to define an infinite source Poisson process, W_λ^*, such that for $t \geq 0$,

$$W_\lambda^*(t) = \text{the aggregated workload in the time interval } [0, t].$$

1.1.1. The continuous flow reward model. Let $N(ds, du, dr)$ denote a Poisson point measure on $R \times R_+ \times R_+$ with intensity measure

$$n(ds, du, dr) = \lambda ds\, F_U(du)\, F_R(dr). \tag{1}$$

We use S, U, R as generic notation for the random quantities and s, u, r for a particular session outcome so that a Poisson event in (s, u, r) represents a session arriving at time s of duration u and with reward size r. With the choice of (1), we obtain a fluid model for network traffic where sessions begin successively on the (physical) time line labeled s at Poisson rate λ. A session is active during the time interval $[s, s+u]$ and transmits traffic at rate r throughout the session, where (u, r) is an outcome of independent random variables (U, R). For example,

$$\int_{-\infty}^{t} \int_0^\infty \int_0^\infty 1_{\{s<t<s+u\}}\, N(ds, du, dr) = \text{the number of active sessions at time } t.$$

To express similarly W_λ^* in terms of the point measure N, we fix $t > 0$ and partition the total traffic streams into traffic originating from sessions that began in the infinite past, $s \leq 0$, and traffic from sessions starting at a time s with $0 < s < t$. In the former case, sessions do not count if $s + u \leq 0$, the contribution to $W_\lambda^*(t)$ is $(u - |s|)r = (s + u)r$ if $0 < s + u \leq t$, and it is tr if $s + u > t$. In the latter case, the amount of traffic workload that counts for $W_\lambda^*(t)$ is ur if $u < t - s$ and $(t - s)r$ otherwise. Hence

$$\begin{aligned} W_\lambda^*(t) &= \int_{-\infty}^{0} \int_0^\infty \int_0^\infty (t \wedge (s+u)_+) r\, N(ds, du, dr) \\ &\quad + \int_0^t \int_0^\infty \int_0^\infty ((t-s) \wedge u) r\, N(ds, du, dr). \end{aligned} \tag{2}$$

Recall (Campbell Theorem, Kingman (1993), Section 3.2) that an integral of the form $I(f) = \int_S f(x)\, N(dx)$, where N is a Poisson random measure on a space S, exists with probability 1 if and only if $\int_S \min(|f(x)|, 1)\, n(dx) < \infty$ where $n(dx) = EN(dx)$. Moreover, if $\int_S |f(x)|\, n(dx) < \infty$ then the expected value of the integral equals $EI(f) = \int_S f(x)\, n(dx)$. Thus,

$$
\begin{aligned}
&EW^*_\lambda(t) \\
&\quad = E(R)\left(\int_{-\infty}^0 \int_0^\infty t \wedge (s+u)_+\, \lambda ds\, F_U(du) + \int_0^t \int_0^\infty (t-s)\wedge u\, \lambda ds\, F_U(du)\right) \\
&\quad = \lambda E(R)\left(\int_0^t \int_s^\infty P(U>u)\, duds + \int_0^t \int_0^s P(U>u)\, duds\right) \\
&\quad = \lambda \nu E(R) t, \qquad (3)
\end{aligned}
$$

by performing in each of the two terms an integration by parts in the variable u. For example,

$$
\begin{aligned}
\int_0^\infty (t-s)\wedge u\, F_U(du) &= \int_0^{t-s} u\, F_U(du) + (t-s)P(U > t-s) \\
&= \int_0^{t-s} P(U>u)\, du.
\end{aligned}
$$

The two integral terms in (2) may be combined into a single integral, by writing

$$
W^*_\lambda(t) = \int_{-\infty}^\infty \int_0^\infty \int_0^\infty ((t-s)_+ \wedge u - (-s)_+ \wedge u) r\, N(ds, du, dr). \qquad (4)
$$

The kernel

$$
K_t(s,u) = (t-s)_+ \wedge u - (-s)_+ \wedge u \qquad (5)
$$

is such that

$$
0 \le K_t(s,u) = \begin{cases} 0 & \text{if } s+u \le 0 \text{ or } s \ge t \\ s+u & \text{if } s \le 0 \le s+u \le t \\ t & \text{if } s \le 0,\ t \le s+u \\ u & \text{if } 0 \le s,\ s+u \le t \\ t-s & \text{if } 0 \le s \le t \le s+u. \end{cases}
$$

Hence $K_t(s,u)$ is a function of the starting time s and the duration u of a session that measures the length of the time interval contained in $[0,t]$ during which the session is active. Figure 1 indicates the shape of $K_t(s,u)$ defined on the (s,u)-plane when we have fixed a value of t. Write

$$
\widetilde{N}(ds, du, dr) = N(ds, du, dr) - n(ds, du, dr) \qquad (6)
$$

for the compensated Poisson measure with intensity measure $n(ds, du, dr)$. By (4) and (3),

$$
W^*_\lambda(t) = \int_{-\infty}^\infty \int_0^\infty \int_0^\infty K_t(s,u) r\, N(ds, du, dr) \qquad (7)
$$

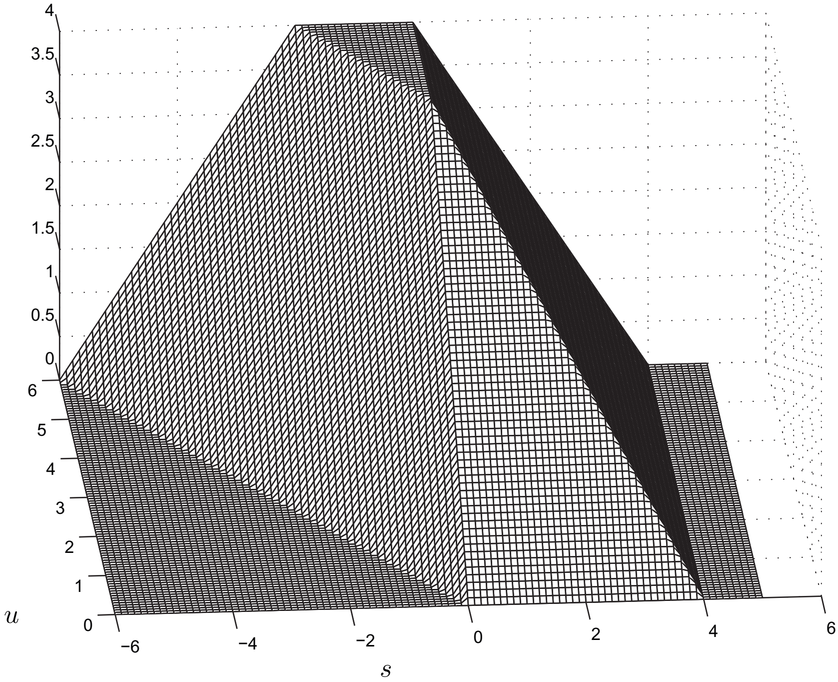

FIGURE 1. The kernel function $K_t(s,u)$, $t=4$, $-6 \le s \le 6$, $0 \le u \le 6$

with

$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} K_t(s,u)r\,n(ds,du,dr) < \infty,$$

and

$$W_\lambda^*(t) = \lambda\nu E(R)t + \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} K_t(s,u)r\,\widetilde{N}(ds,du,dr), \tag{8}$$

which represents the workload in the form of a linear drift and random Poisson fluctuations.

Note that the case of fixed unit rewards, $R \equiv 1$, is contained as a special case of the above by setting F_R equal to the Dirac measure

$$F_R(dr) = \delta_1(dr),$$

which then gives

$$W_\lambda^*(t) = \int_{-\infty}^{\infty}\int_0^{\infty} K_t(s,u)\,N(ds,du)$$

with

$$\int_{-\infty}^{\infty}\int_0^{\infty} K_t(s,u)\,\lambda ds\,F_U(du) = \lambda\nu t < \infty.$$

Here $N(ds, du)$ is the marginal of the Poisson measure $N(ds, du, dr)$ restricted to its first two coordinates.

1.1.2. The compound Poisson arrival workload model. This model results from nesting two Poisson measures as follows. During each session, we allow packets to be generated at discrete Poisson time points. More precisely, consider the compound Poisson process

$$\Xi(t) = \sum_{i=1}^{M(t)} R_i, \quad t \geq 0, \tag{9}$$

where $(R_i)_{i\geq 1}$ is an i.i.d. sequence from the distribution F_R and $M(t)$ is a unit rate Poisson process on R_+. The paths of Ξ are elements in the space D of right-continuous functions with left limits, $t \mapsto \xi(t)$, $t \geq 0$, and we let μ denote the distribution of Ξ defined on D. Let $N_\sharp(ds, du, d\xi)$ be a Poisson measure on $R \times R_+ \times D$ with intensity measure

$$n_\sharp(ds, du, d\xi) = \lambda ds\, F_U(du)\, \mu(d\xi). \tag{10}$$

A Poisson event of $N_\sharp$ at (s, u, ξ) represents a session that starts at s, has duration u, and generates packets according to ξ. The length of time in $[0, t]$ during which the session is active is given by $K_t(s, u)$ defined in (5), and the resulting workload is therefore given by $\xi(K_t(s, u))$. Thus, the accumulated workload $W^*_\lambda(t)$ under compound Poisson packet generation is

$$W^*_{\sharp,\lambda}(t) = \int_{-\infty}^{\infty} \int_0^{\infty} \int_D \xi(K_t(s, u))\, N_\sharp(ds, du, d\xi). \tag{11}$$

Since

$$E\Xi(t) = EM(t)E(R) = tE(R), \tag{12}$$

the expected value of $W^*_{\sharp,\lambda}(t)$ equals

$$\begin{aligned} EW^*_{\sharp,\lambda}(t) &= \int_{-\infty}^{\infty} \int_0^{\infty} \int_D \xi(K_t(s, u))\, \lambda ds\, F_U(du)\, \mu(d\xi) \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} K_t(s, u)\, \lambda ds\, F_U(du)\, E(R) \\ &= \lambda\nu E(R)t, \end{aligned}$$

just as in the continuous flow model. By analogy with (8) we have the representation

$$W^*_{\sharp,\lambda}(t) = \lambda\nu E(R)t + \int_{-\infty}^{\infty} \int_0^{\infty} \int_D \xi(K_t(s, u))\, \widetilde{N}_\sharp(ds, du, d\xi) \tag{13}$$

in terms of the compensated Poisson measure

$$\widetilde{N}_\sharp(ds, du, d\xi) = N_\sharp(ds, du, d\xi) - n_\sharp(ds, du, d\xi). \tag{14}$$

Kurtz (1996) introduced general workload input models of this form, Çağlar (2004) considers the above model with a specific choice of duration distribution F_U. Poissonized integral representations are discussed in Cohen and Taqqu (2003) and Wolpert and Taqqu (2004).

1.2. Preliminary observations

We now represent the continuous flow model as an integral of an instantaneous arrival rate process, show that the workload models have stationary increments, and provide alternative representations which do not involve the presence of an infinite stretch of past arrivals.

1.2.1. Instantaneous arrival rate for continuous flow workload. The integration kernel K_t in (5) has several useful alternative representations. The relation

$$(t-s)_+ \wedge u - (-s)_+ \wedge u = \int_{-s}^{t-s} 1_{\{0<y<u\}}\, dy$$

yields

$$K_t(s,u) = \int_0^t 1_{\{s<y<s+u\}}\, dy \tag{15}$$

and the geometric interpretation

$$K_t(s,u) = |(0,t) \cap (s, s+u)|.$$

The resulting bounds

$$0 \le K_t(s,u) \le t \wedge u \tag{16}$$

are used repeatedly in the proofs below. A further equivalent representation of the kernel function $K_t(s,u)$ is given by

$$K_t(s,u) = \int_0^u 1_{\{0<y+s<t\}}\, dy. \tag{17}$$

As a consequence of relation (15) applied to (7), one can represent the accumulated workload of the continuous flow model as

$$W_\lambda^*(t) = \int_0^t W_\lambda(y)\, dy, \quad W_\lambda(y) = \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} 1_{\{s<y<s+u\}}\, r\, N(ds,du,dr). \tag{18}$$

Here the integrand $W_\lambda(y)$, $-\infty < y < \infty$, is itself a well-defined random instantaneous workload arrival rate process and $W_\lambda^*(t)$ is the corresponding cumulative workload. The expressions (18) provide a physical interpretation of W_λ and W_λ^*. The instantaneous rate $W_\lambda(y)$ is the Poisson aggregation of rewards of all sessions that are active at time y, and the cumulative workload W_λ^* builds up accordingly during the time integration over $[0,t]$.

1.2.2. Stationarity of the increments of the workloads.

Lemma 1. *In the continuous flow workload model, the instantaneous arrival rate process* $\{W_\lambda(y), -\infty < y < \infty\}$ *is stationary and the cumulative workload process* $\{W_\lambda^*(t), t \ge 0\}$ *has stationary increments.*

Proof. Because of the time-homogeneity of $n(ds, du, dr)$ in the variable s the shifted process

$$W_\lambda(y+\tau) = \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} 1_{\{s<y+\tau<s+u\}}\, r\, N(ds, du, dr)$$

has the same finite-dimensional distributions as

$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} 1_{\{s<y<s+u\}}\, r\, N(ds, du, dr) = W_\lambda(y). \qquad \square$$

Remark 3. Consider a link of maximal traffic capacity $C > 0$. The process

$$C_\lambda(t) = \int_0^t (W_\lambda(y) - C)_+\, dy, \quad t \geq 0,$$

represents the cumulative workload loss up to time t on the congested link where any traffic of instantaneous rate in excess of C is lost.

Remark 4. In the case $R \equiv 1$, the stationary process W_λ measures the system size of the standard M/G/∞ service model running on the real line with service distribution $G = F_U$. For each fixed y, $W_\lambda(y)$ is Poisson distributed with expected value $\lambda\nu$ because for $R \equiv 1$,

$$W_\lambda(y) = \int\int 1_{\{s<t<s+u\}}\, N(ds, du) = N(A), \quad A = \{(s,u) : s < t < s+u\},$$

with

$$EN(A) = \int\int_A \lambda ds\, F_U(du) = \lambda\int_{-\infty}^t P(U > t-s)\, ds = \lambda\int_0^\infty P(U > s)\, ds = \lambda\nu.$$

For the discrete packet generation workload model we apply a similar but slightly different argument.

Lemma 2. *The compound Poisson arrival workload model* $\{W^*_{\sharp,\lambda}, t \geq 0\}$ *has stationary increments.*

Proof. By (17),

$$K_{t+\tau}(s,u) - K_t(s,u) = \int_0^u 1_{\{t<y+s<t+\tau\}}\, dy = K_\tau(s-t, u),$$

and hence by (9),

$$\Xi(K_{t+\tau}(s,u)) - \Xi(K_t(s,u)) \overset{d}{=} \Xi(K_\tau(s-t,u)).$$

Since $n_\sharp(ds, du, d\xi)$ is time-homogeneous in the variable s, it now follows from (11) that

$$\begin{aligned} W^*_{\sharp,\lambda}(t+\tau) - W^*_{\sharp,\lambda}(t) &\overset{d}{=} \int_{-\infty}^{\infty}\int_0^{\infty}\int_D \xi(K_\tau(s-t,u))\, N_\sharp(ds, du, d\xi) \\ &\overset{d}{=} \int_{-\infty}^{\infty}\int_0^{\infty}\int_D \xi(K_\tau(s,u))\, N_\sharp(ds, du, d\xi) \\ &\overset{d}{=} W^*_{\sharp,\lambda}(\tau) \end{aligned} \qquad \square$$

Here and elsewhere, the notation $\stackrel{d}{=}$ denotes equality in the sense of the finite-dimensional distributions.

At this point, having established the property of stationary increments for $W^*_\lambda(t)$ and $W^*_{\sharp,\lambda}(t)$, we comment on the special case $E(R^2) < \infty$ when the reward distribution F_R has a finite second moment. Then, for $s < t$,

$$\mathrm{Cov}(W^*_\lambda(s), W^*_\lambda(t)) = \frac{1}{2}\left(\mathrm{Var}(W^*_\lambda(s)) + \mathrm{Var}(W^*_\lambda(t)) - \mathrm{Var}(W^*_\lambda(t-s))\right)$$

where

$$\mathrm{Var}(W^*_\lambda(t)) = E(R^2)\int_{-\infty}^{\infty}\int_0^{\infty} K_t(s,u)^2\, ds\, F_U(du).$$

Also,

$$\mathrm{Var}(W^*_{\sharp,\lambda}(t)) = \lambda\nu E(R^2)\, t + (ER)^2\int_{-\infty}^{\infty} K_t(s,u)^2\, \lambda ds\, F_U(du).$$

The crucial property of regular variation which determines the large time behavior of these processes in the finite variance case ($\delta = 2$) is the asymptotic power law

$$\mathrm{Var}(W^*_\lambda(t)) \sim \mathrm{Var}(W^*_{\sharp,\lambda}(t)) \sim \mathrm{const}\, t^{2H}, \quad t \to \infty,$$

where we apply the convention of using a Hurst index H, which in our case is related to the tail parameter γ as

$$H = \frac{3-\gamma}{2} \in (1/2, 1).$$

Our limit results will show that the parameter H appears as a self-similarity index in those cases where the limit process is Gaussian. However, our results cover several other cases as well and hence we will keep γ and δ as basic parameters. A line of research of current interest is that of estimation of such key parameters based on observations of the process. For example, Fay, Roueff and Soulier (2007), study a wavelet-based estimator of the Hurst index for the continuous rate flow model based on the infinite source Poisson process and of the corresponding instantaneous arrival rate process described above. They show consistency of the estimator and study the rate of convergence. Some of the results allow for specific dependencies between durations and rewards. Simulation technique for these processes is a related and relevant direction of research. Here, we restrict to mentioning the references Bardet *et al.* (2003a, 2003b) which survey estimation and simulation techniques for long-range dependent random processes.

1.2.3. Representations based on an equilibrium distribution. The workload processes $W^*_\lambda(t)$ and $W^*_{\sharp,\lambda}$, as defined in (2) and (11), involve sessions arriving at any time s in the infinite past. We now provide an alternative representation of the workload, such that for each t the underlying random mechanism generating $W^*_\lambda(t)$ or $W^*_{\sharp,\lambda}(t)$ consists of sessions with arrival times restricted to the time interval $[0, t]$. To do this, recall the two terms leading to (2). One term

$$\int_0^t\int_0^{\infty}\int_0^{\infty} ((t-s)\wedge u) r\, N(ds, du, dr)$$

represents a nonstationary workload process only governed by session arrivals in $(0,t]$. We focus here on the other term, which represents arrivals in the past, is a Poisson integral with expected value

$$\int_{-\infty}^{0}\int_{0}^{\infty}\int_{0}^{\infty}(t\wedge(s+u)_{+})r\,n(ds,du,dr)=\lambda E(R)\int_{0}^{t}\int_{u}^{\infty}P(U>v)\,dvdu.$$

To express this as an integral of sessions starting at $s=0$ and with respect to a different Poisson measure, we introduce the notation $\widetilde{U}$ for the equilibrium distribution associated to U having distribution function $F_{\widetilde{U}}(u)=P(\widetilde{U}\le u)$ such that

$$1-F_{\widetilde{U}}(u)=\frac{1}{\nu}\int_{u}^{\infty}P(U>v)\,dv\sim\frac{1}{\nu\gamma(\gamma-1)u^{\gamma-1}},\quad u\to\infty. \tag{19}$$

Let $M(dv,du,dr)$ be a Poisson measure on $[0,1]\times R_{+}\times R_{+}$ with intensity measure

$$m(dv,du,dr)=\lambda\nu dv\,F_{\widetilde{U}}(du)\,F_{R}(dr)$$

and independent of $N(ds,du,dr)$. The measure $M(dv,du,dr)$ produces a Poisson distributed number of independent sessions each with duration taken from $\widetilde{U}$ and reward R. One has

$$\int_{-\infty}^{0}\int_{0}^{\infty}\int_{0}^{\infty}(t\wedge(s+u)_{+})r\,N(ds,du,dr)\stackrel{d}{=}\int_{0}^{1}\int_{0}^{\infty}\int_{0}^{\infty}(t\wedge u)r\,M(dv,du,dr) \tag{20}$$

To see this, use the fact that the characteristic function of a Poisson integral satisfies

$$\ln E\exp\Big\{i\theta\int f(x)N(dx)\Big\}=\int(e^{i\,f(x)}-1)\,n(dx)$$

and observe that

$$\begin{aligned}
&\log E\exp\Big\{\sum_{j=1}^{n}\theta_j\int_{-\infty}^{0}\int_{0}^{\infty}\int_{0}^{\infty}(t_j\wedge(s+u)_{+})r\,N(ds,du,dr)\Big\}\\
&=\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}(e^{i\sum_{j=1}^{n}\theta_j(t_j\wedge(u-s)_{+})r}-1)\lambda ds\,F_U(du)\,F_R(dr)\\
&=\lambda\int_{0}^{\infty}F_U(du)\int_{0}^{u}ds\int_{0}^{\infty}F_R(dr)(e^{i\sum_{j=1}^{n}\theta_j(t_j\wedge s)r}-1)\\
&=\lambda\int_{0}^{\infty}ds\int_{s}^{\infty}F_U(du)\int_{0}^{\infty}F_R(dr)(e^{i\sum_{j=1}^{n}\theta_j(t_j\wedge s)r}-1)\\
&=\lambda\nu\int_{0}^{\infty}F_{\widetilde{U}}(ds)\int_{0}^{\infty}F_R(dr)(e^{i\sum_{j=1}^{n}\theta_j(t_j\wedge s)r}-1)\\
&=\log E\exp\Big\{\sum_{j=1}^{n}\theta_j\int_{0}^{1}\int_{0}^{\infty}\int_{0}^{\infty}(t_j\wedge s)r\,M(dv,ds,dr)\Big\},
\end{aligned}$$

where M has intensity measure $m(dv, ds, dr)$. Therefore we can express $W^*_\lambda(t)$ as

$$\begin{aligned} W^*_\lambda(t) &\stackrel{d}{=} \int_0^1 \int_0^\infty \int_0^\infty (t \wedge u) r\, M(dv, du, dr) \\ &\quad + \int_0^t \int_0^\infty \int_0^\infty ((t-s) \wedge u) r\, N(ds, du, dr). \end{aligned}$$

The expected number of sessions contributing to the first term in this alternative representation is $\lambda\nu$ and we have the following interpretation. A random number of sessions, Poisson distributed with mean $\lambda\nu$, arrive at time $s = 0$. They last independently over time durations $\widetilde{U}$ and transmit independently at rate R, hence a Poisson event at (v, u, r) contributes the workload $(t \wedge u)r$ to $W^*_\lambda(t)$. The number $v \in [0, 1]$ assigned to each session is an auxiliary part of the construction for generating the correct number of initial sessions at time $s = 0$, and has no physical meaning in itself.

With $\widetilde{M}(dv, du, dr) = M(dv, du, dr) - m(dv, du, dr)$, this can also be expressed as

$$\begin{aligned} W^*_\lambda(t) - \lambda\nu E(R)t &\stackrel{d}{=} \int_0^1 \int_0^\infty \int_0^\infty (t \wedge u) r\, \widetilde{M}(dv, du, dr) \\ &\quad + \int_0^\infty \int_0^\infty \int_0^\infty ((t-s)_+ \wedge u) r\, \widetilde{N}(ds, du, dr). \end{aligned} \tag{21}$$

Similarly, the compound Poisson arrival workload process (11) has the representation

$$\begin{aligned} W^*_{\sharp,\lambda}(t) &\stackrel{d}{=} \lambda\nu E(R)t + \int_0^1 \int_0^\infty \int_D \xi(t \wedge u)\, \widetilde{M}_\sharp(dv, du, d\xi) \\ &\quad + \int_0^t \int_0^\infty \int_D \xi((t-s) \wedge u)\, \widetilde{N}_\sharp(ds, du, d\xi), \end{aligned} \tag{22}$$

where $\widetilde{M}_\sharp(dv, du, d\xi) = M_\sharp(dv, du, d\xi) - \lambda\nu dv\, F_{\widetilde{U}}(du)\, \mu(d\xi)$.

2. Scaling behavior of the workload process

We are interested in the various limit processes that arise when the speed of time increases in proportion to the intensity of traffic sessions. Heuristically, these approximation results describe the random variation in traffic patterns that correspond to larger and larger volumes of Internet traffic being transmitted over networks of higher and higher capacity.

The traffic fluctuations in an infinite source Poisson system are expressed by the workload process centered around its average value, $W^*_\lambda(t) - \lambda\nu E(R)t$. To balance the increasing session intensity λ, we will speed up time by a factor a and simultaneously normalize the size using a factor b. It follows from (8) and (18) that the scaled continuous flow workload process has the form

$$\begin{aligned}\frac{1}{b}(W_\lambda^*(at) - \lambda\nu E(R)at) &= \frac{1}{b}\int_0^{at} (W_\lambda(y) - \lambda\nu E(R))\,dy \\ &= \frac{1}{b}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} K_{at}(s,u)r\,\widetilde{N}(ds,du,dr), \quad t\ge 0.\end{aligned} \tag{23}$$

Similarly, the scaled compound Poisson workload process is given by

$$\frac{1}{b}(W_{\sharp,\lambda}^*(at) - \lambda at\nu E(R)) = \frac{1}{b}\int_{-\infty}^{\infty}\int_0^{\infty}\int_D \xi(K_{at}(s,u))\widetilde{N}_\sharp(ds,du,d\xi). \tag{24}$$

We are going to study both as λ tends to infinity with a and b appropriately chosen functions of λ, which also tend to infinity. Observe that there are several ways to change variables in the integrals. We will use

$$\frac{1}{b}(W_\lambda^*(at) - \lambda at\nu E(R))$$

$$\stackrel{d}{=} \frac{1}{b}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} K_{at}(as,u)r\,(N(ads,du,dr) - \lambda ads\,F_U(du)\,F_R(dr)) \tag{25}$$

$$\stackrel{d}{=} \frac{a}{b}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} K_t(s,u)r\,(N(ads,adu,dr) - \lambda ads\,F_U(adu)\,F_R(dr)) \tag{26}$$

$$\stackrel{d}{=} \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} K_t(s,u)r\Big(N(ads,adu,\frac{b}{a}dr) - \lambda ads\,F_U(adu)\,F_R(\frac{b}{a}dr)\Big), \tag{27}$$

and other variations. We used here the scaling property

$$K_{at}(as,au) = aK_t(s,u). \tag{28}$$

Thus, turning to the compound Poisson arrival model (24) we obtain, e.g.,

$$\frac{1}{b}(W_{\sharp,\lambda}^*(at) - \lambda at\nu E(R)) \stackrel{d}{=} \frac{1}{b}\int_{-\infty}^{\infty}\int_0^{\infty}\int_D \xi(aK_t(s,u))\widetilde{N}_\sharp(ads,adu,d\xi)$$

instead of (26). An interesting feature of our approximation results is that the choice of either continuous flow rate or compound Poisson packet generation during sessions *does affect* the limit process. In fact, we will see that for the compound Poisson model there is an additional averaging effect that takes place during sessions, which changes the asymptotic behavior relative to that of the continuous flow model. This means that the influence of heavy-tailed distributions acting over long time scales alone does not dictate limit results. Rather, the local workload structure over short time scales has an impact on the asymptotics.

Remark 5. To simplify the notation the following useful convention will be used in the sequel: the presence of the term $N(dx) - n(dx)$ will imply, in particular, that $N(dx)$ is a Poisson random measure with intensity measure $n(dx)$.

2.1. Gaussian and stable random measures and processes

We will be using Gaussian and α-stable random measures $M(dx)$ with control measure $m(dx)$ defined for $x \in R^d$. The measure M has the following properties. If $A_1, \ldots, A_n$ are disjoint Borel sets in R^d, then $M(A_1), \ldots, M(A_n)$ are independent random variables. If $\alpha = 2$ (Gaussian case), then for any Borel set A in R^d, the random variable $M(A)$ is normal with mean 0 and variance $m(A)$. If $\alpha < 2$, then

$$\sigma_\alpha M(A) \stackrel{d}{=} \int_A \int_0^\infty r \left(N(dv, dr) - m(dv)\, r^{-(1+\alpha)} dr \right) \tag{29}$$

where

$$\sigma_\alpha = \left(\frac{2\Gamma(2-\alpha)}{\alpha(\alpha-1)} (-\cos \pi\alpha/2) \right)^{1/\alpha}, \tag{30}$$

and thus $M(A)$ has an α-stable distribution which is totally skewed to the right (this is because $r > 0$).

The characteristic function of $M(A)$ is given by

$$\ln E(e^{i\theta M(A)}) = -m(A)|\theta|^\alpha k_\alpha(\theta), \tag{31}$$

where

$$k_\alpha(\theta) = 1 - i(\operatorname{sign}\theta) \tan \pi\alpha/2 \tag{32}$$

(For more details, see Samorodnitsky and Taqqu (1994), pages 156, 119 and 5.) We will write M_2 to denote a Gaussian random measure and M_α to denote an α-stable random measure with $\alpha < 2$. The index α will be either γ or δ.

We will also consider a Lévy-stable process $\Lambda_\alpha(t)$ with index $1 < \alpha < 2$ totally skewed to the right (here again α will be either γ or δ). This is a process with independent increments which can be represented as

$$\Lambda_\alpha(t) = \sigma_\alpha \int_0^t M_\alpha(ds) \stackrel{d}{=} \int_0^t \int_0^\infty r(N(ds, dr) - ds\, r^{-(1+\alpha)} dr), \tag{33}$$

where σ_α is given by (30) and $M(ds)$ is an α-stable random measure with control measure ds and $N(ds, dr)$ is a Poisson random measure with intensity measure $ds\, r^{-1-\alpha} dr$ (see Samorodnitsky and Taqqu, Theorem 3.12.2).

We will also use (standard) fractional Brownian motion $B_H(t)$, which is a Gaussian, mean 0 process, with stationary increments and covariance

$$EB_H(t_1)B_H(t_2) = \frac{1}{2} \left\{ |t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H} \right\},$$

where $0 < H < 1$. Fractional Brownian motion is H-self-similar, that is, for any $a > 0$, the processes $B_H(at)$, $t \geq 0$ and $a^H B_H(t)$, $t \geq 0$ have identical finite-dimensional distributions. Fractional Brownian motion reduces to Brownian motion when $H = \frac{1}{2}$.

2.2. Results on fast, intermediate, and slow connection rates

When we let the session intensity λ increase to infinity and simultaneously scale time, letting a tend to infinity, and scale size, letting b tend to infinity, it is possible to obtain several different limit processes in (23) and (24). A crucial feature of these limiting schemes is the relative speed at which λ and a increase. Namely, in most cases it is the asymptotic behavior of the ratio

$$\lambda/a^{\gamma-1}$$

which determines the proper normalizing sequence b and the limit process. More precisely, we will see that this is the case for the continuous flow model in the situation $1 < \gamma < \delta \leq 2$, when the durations length has a heavier tail than that of the rewards, and for the compound Poisson model for any set of parameters $1 < \gamma, \delta \leq 2$ except $\gamma = \delta = 2$. To understand why the ratio $\lambda/a^{\gamma-1}$ enters in the picture, consider the representations (21) and (22) of the workload using the equilibrium session lengths $\widetilde{U}$. At time zero, or at any fixed time point, there is a Poisson number of independent sessions of mean $\lambda\nu$. The remaining length of each session has the distribution $\widetilde{U}$. Hence, letting M be a Poisson random measure with mean $\lambda\nu$, we have

$$\#(\lambda, a) = \text{number of initial sessions still active at time } a \stackrel{d}{=} \sum_{i=1}^{M} 1_{\{\widetilde{U}_i > a\}}.$$

For a given choice of sequences λ and a, $\#(\lambda, a)$ measures the degree to which very long sessions are present and contribute to the total workload. The expected value of the random variable $\#(\lambda, a)$ is

$$E(\#(\lambda, a)) = \lambda\nu P(\widetilde{U} > a) \sim \frac{1}{\gamma(\gamma-1)} \frac{\lambda}{a^{\gamma-1}}, \tag{34}$$

in view of (19). This makes it natural to distinguish three limit regimes based on whether $E(\#(\lambda, a))$ tends to a finite and positive constant, tends to infinity, or vanishes to zero as λ and a goes to infinity. We will introduce a parameter c to quantify the relative speed in the scaling of time and size, and refer to the three cases as:

intermediate connection rate:	$\lambda/a^{\gamma-1} \to c^{\gamma-1}, \quad 0 < c < \infty,$
fast connection rate:	$\lambda/a^{\gamma-1} \to \infty,$
slow connection rate:	$\lambda/a^{\gamma-1} \to 0.$

2.2.1. Intermediate connection rate (ICR). We consider the asymptotics

$$E(\#(\lambda, a)) \sim \text{const}, \quad \lambda, a \to \infty,$$

in which case the number of very long sessions stays bounded. In this situation two kinds of summation schemes influence the workload. First, the aggregation of traffic corresponding to a large value of λ consists of many overlapping sessions, all active at the same fixed time. Secondly, for large a the accumulated traffic in the interval $[0, at]$ involves many sessions that were active during some period in the past. To clarify this structure using heuristic arguments before stating the precise results, let us consider the case $E(R^2) < \infty$, take $c = 1$, and recall the representation (18) of $W^*_\lambda(t)$. We have

$$\frac{1}{a}(W^*_\lambda(at) - \lambda\nu E(R)at) = \frac{1}{a}\int_0^{at} (W_\lambda(y) - \lambda\nu E(R))\,dy$$
$$\sim \frac{1}{a^{(3-\gamma)/2}}\int_0^{at} \frac{W_\lambda(y) - \lambda\nu E(R)}{\sqrt{\lambda}}\,dy,$$

since $\lambda a^{3-\gamma} \sim a^2$. For each y, $W_\lambda(y)$ has a compound Poisson distribution with finite variance and hence for large λ the distribution of the integrand $(W_\lambda(y) - \lambda\nu E(R))/\sqrt{\lambda}$ is approximately Gaussian. The subsequent integration over y affects the covariance structure but preserves the Gaussian distribution. On the other hand, the following argument indicates that we should expect a stable distribution in the limit. Suppose for convenience that λ is an integer and decompose $W_\lambda = \sum_{i=1}^{\lambda} W_1^i$ as a sum of i.i.d. components W_1^i, $1 \le i \le \lambda$. Then

$$\frac{1}{a}(W^*_\lambda(at) - \lambda\nu E(R)at) = \frac{1}{a}\int_0^{at} \sum_{i=1}^{\lambda}(W_1^i(y) - \nu E(R))\,dy$$
$$\sim \frac{1}{\lambda^{1/\gamma}}\sum_{i=1}^{\lambda}\frac{1}{a^{1/\gamma}}\Big(\int_0^{at} W_1^i(y)\,dy - \nu E(R)at\Big),$$

where we use $\lambda a \sim a^\gamma$. The integral process $\int_0^t W_1^i(y)\,dy$, $t \ge 0$, that appears in the last expression is increasing with expected value νt, but since the integrand $W_1^i(y)$ typically stays constant for intervals of length U and the distribution of U has infinite variance, there is no Gaussian central limit law for the corresponding centered process. Instead, we note that $\int_0^t W_1^i(y)\,dy$, after centering and scaling by a as above, should behave as a renewal process having interrenewal times with the heavy-tailed distribution F_U of index γ. For such processes it is known that the limit distribution as $a \to \infty$ is stable with stable index γ. The additional summation over i preserves the stable distribution. For a more detailed discussion in a similar case (of inverse Lévy processes), see Kaj and Martin-Löf (2004).

Turning now to the statement of our first result, it turns out that the limit processes under ICR scaling are neither Gaussian nor stable. In fact new limit processes arise. A further interesting consequence is that the limits are different for the continuous flow rate model and for the compound Poisson model.

Theorem 1. *Consider a pair of parameters* $1 < \gamma < 2$ *and* $1 < \delta \le 2$*, fix an arbitrary constant* c*,* $0 < c < \infty$*, and assume*

$$\lambda \to \infty, \quad a \to \infty, \quad \frac{\lambda}{a^{\gamma-1}} \to c^{\gamma-1}.$$

Take $b = a$ *as size factor.*

(i) *If $1 < \gamma < \delta \le 2$, the continuous flow rate model, scaled and normalized as in* (23)*, has the limit*

$$\frac{1}{a}(W^*_\lambda(at) - \lambda\nu E(R)at) \Rightarrow c\, Y_{\gamma,R}(t/c),$$

where

$$\begin{aligned} Y_{\gamma,R}(t) &= \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} K_t(s,u) r \left(N(ds,du,dr) - ds\, u^{-(1+\gamma)} du\, F_R(dr)\right) \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} K_t(s,u)\Big(\int_0^{\infty} r\, N(ds,du,dr) - E(R)\, ds\, u^{-(1+\gamma)} du\Big). \end{aligned} \tag{35}$$

In the special case of fixed rewards, $R \equiv 1$, the limit process is

$$Y_\gamma(t) = \int_{-\infty}^{\infty}\int_0^{\infty} K_t(s,u)\left(N(ds,du) - ds\, u^{-(1+\gamma)} du\right). \tag{36}$$

(ii) *The compound Poisson workload model in* (24) *has the limit process*

$$\frac{1}{a}(W^*_{\sharp,\lambda}(at) - \lambda\nu E(R)at) \Rightarrow E(R)\, c\, Y_\gamma(t/c),$$

where Y_γ is defined in (36).

Convention. The convergence is in the sense of the finite-dimensional distributions in this theorem and in the following one. Weak convergence in function space will be established in Section 4.

Remark 6. The limit process $Y_{\gamma,R}$ is not self-similar, because N does not have the scaling properties that a Gaussian or a stable process has. However, if we assume that the reward distribution $F_R(dr)$ has finite variance then

$$\begin{aligned} \mathrm{Var}(Y_{\gamma,R}(t)) &= E(R^2)\int_{-\infty}^{\infty}\int_0^{\infty} K_t(s,u)^2\, ds\, u^{-(1+\gamma)} du \\ &= E(R^2)\,\sigma^2\, t^{2H}, \qquad H = \frac{3-\gamma}{2}, \end{aligned}$$

where σ^2 is given in (38). Thus, in this case $Y_{\gamma,R}$ is second order self-similar with Hurst index H.

Benassi *et al.* (1997) introduced local asymptotic self-similarity as another means of generalizing the class of self-similar processes. It is shown in Gaigalas and Kaj (2003) and with a proof more adapted to the present setting in Gaigalas (2006), that the process Y_γ is locally asymptotically self-similar with index H and with fractional Brownian motion as tangent process, in the sense that

$$\left\{\frac{Y_\gamma(t+\lambda u) - Y_\gamma(t)}{\lambda^H},\ u \in \mathbf{R}\right\} \Rightarrow \{B_H(u),\ u \in \mathbf{R}\}, \quad \text{as} \quad \lambda \to 0.$$

Benassi *et al.* (2002) defined a stochastic process $X(t)$ to be asymptotically self-similar at infinity with index H if there exists a process $R(t)$ such that

$$\lambda^{-H} X(\lambda t) \to R(t), \quad \text{as } \lambda \to \infty.$$

The intermediate limit process $Y_\gamma(t)$ is asymptotically self-similar at infinity with index $H = 1/\gamma$ and asymptotic process $R(t)$ given by an γ-stable Lévy process, totally skewed to the the right, see Gaigalas (2006).

Remark 7. The difference between the representation of $W^*_\lambda(t) - EW^*_\lambda(t)$ in (8) and that of $Y_{\gamma,R}(t)$ in (35) is that the control measure $F_U(du)$ is now replaced by $u^{-(1+\gamma)}$ which is not a probability measure anymore.

Remark 8. The process $Y_{\gamma,R}$ will be called the *Intermediate Telecom process*. We are thus able to identify the limit process in the case of general reward distributions, which has been an open problem in Pipiras *et al.* (2004). The special case of fixed rewards $R \equiv 1$, has been solved earlier. It can be obtained by combining results in Gaigalas (2006), Kaj (2005) and Gaigalas and Kaj (2003).

Remark 9. The limit for the compound Poisson workload model is a scaled version of Y_γ defined in (36) and, as noted in the theorem, Y_γ is $Y_{\gamma,R}$ in (35) in the special case of fixed rewards $R \equiv 1$.

2.2.2. Fast connection rate (FCR). In this case, a large number of very long sessions contribute in the asymptotic limit of aggregating the traffic workload. Essentially, we will have a summation scheme for processes as in the ordinary central limit theorem, but with strong dependencies building up over time. For the continuous flow model the limit is Gaussian in the case of finite variance rewards and the limit is stable if the reward distribution does not possess finite variance. For the compound Poisson packet generation model, the limit is Gaussian whether the rewards have finite variance or not.

Theorem 2. *Let* $1 < \gamma < 2$, $1 < \delta \le 2$, *and assume*

$$\lambda \to \infty, \quad a \to \infty, \quad \frac{\lambda}{a^{\gamma-1}} \to \infty. \tag{37}$$

Set

$$b = \lambda^{1/\delta}\, a^{(\delta+1-\gamma)/\delta} \quad \textit{so that} \quad b/a = (\lambda/a^{\gamma-1})^{1/\delta} \to \infty.$$

(i) *In the case of finite variance rewards,*

$$1 < \gamma < \delta = 2,$$

so $b = \lambda^{1/2} a^{(3-\gamma)/2}$, *then the limit process for* (23) *is the fractional Brownian motion*

$$\sqrt{E(R^2)}\,\sigma\, B_H(t)$$

with index

$$H = \frac{3-\gamma}{2} \in (1/2, 1),$$

where

$$\sigma^2 = \int_{-\infty}^{\infty}\int_0^{\infty} K_1(s,u)^2\, ds\, u^{-(\gamma+1)} du = \frac{2}{\gamma(\gamma-1)(2-\gamma)(3-\gamma)}. \tag{38}$$

Alternatively, the limit process can be represented as

$$E(R^2)^{1/2} \int_{-\infty}^{\infty} \int_0^{\infty} K_t(s,u)\, M_2(ds,du), \tag{39}$$

where $K_t(s,u)$ *is the kernel defined in* (5) *and* $M_2(ds,du)$ *is a Gaussian random measure with control measure*

$$ds\, u^{-(1+\gamma)} du.$$

(ii) *If the reward distribution has infinite variance with a lighter tail than that of the durations,*

$$1 < \gamma < \delta < 2,$$

then the limit of (23) *is the Telecom process*

$$\begin{aligned} Z_{\gamma,\delta}(t) & \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} K_t(s,u) r \left(N(ds,du,dr) - ds\, u^{-(1+\gamma)} du\, r^{-(1+\delta)} dr\right) && (40) \\ &= \sigma_\delta \int_{-\infty}^{\infty} \int_0^{\infty} K_t(s,u)\, M_\delta(ds,du), && (41) \end{aligned}$$

where the random measure $M_\delta(ds,du)$ *is* δ*-stable and has the control measure*

$$ds\, u^{-(\gamma+1)} du.$$

The process $Z_{\gamma,\delta}(t)$ *is a* δ*-stable process, which is* H*-self-similar with*

$$H = \frac{\delta + 1 - \gamma}{\delta} \in (1/\delta, 1).$$

The factor σ_δ *is given in* (30) (*with* $\alpha = \delta$).

(iii) *If we replace* W^*_λ *by* $W^*_{\sharp,\lambda}$*, then for arbitrary parameters*

$$1 < \gamma < 2,\ 1 < \delta \le 2,$$

the limit process of (24) *is the fractional Brownian motion*

$$\big(E(R)\, \sigma\big)\, B_H(t), \quad t \ge 0.$$

Remark 10. The symmetric δ-stable version of the Telecom process appeared in Pipiras and Taqqu (2002). The Telecom process reduces to $CB_H(t)$ when $\delta = 2$. The easiest way to see this is to note that the random measure M_δ is Gaussian when $\delta = 2$ and hence the process $Z_{\gamma,2}$ is Gaussian, has stationary increments and is H-self-similar with $H = (3-\gamma)/2$.

Remark 11. The kernel $K_t(s,u)$ appears both in the representations (40) of the Telecom process and in the representation (35) of the intermediate Telecom process. In (40), the control measure involving r in the stable density $r^{-(1+\delta)}dr$ and thus the Telecom process is a δ-stable process. For the intermediate Telecom process (35), however, the part of the control measure involving r is $F_R(dr)$ which has finite variance in the case $\delta = 2$ and while it has infinite variance in the case $\delta < 2$, the process is not necessarily stable.

2.2.3. Slow connection rate (SCR). The remaining case SCR leads to stable Lévy processes in the asymptotic limit. The interpretation of the scaling condition $E(\#(\lambda, a)) \to 0$, $\lambda, a \to \infty$, in (34) is that there are essentially no sessions that survive the scaling whose remaining durations are so long that they could cause long-range dependence in the limit process. Rather, the additive terms that contribute to the cumulative workload are asymptotically independent and belong to a stable domain of attraction.

The multiplicative constants appearing in the limit depend on the local traffic structure during sessions. Again the limit process for the compound Poisson model depends only on the expected reward $E(R)$, which shows that this is a general property valid for all choices of scaling.

Theorem 3. *Consider the scaling regime*

$$a \to \infty, \quad \frac{\lambda}{a^{\gamma-1}} \to 0$$

or, if λ is bounded away from zero, just

$$\frac{\lambda}{a^{\gamma-1}} \to 0,$$

and take

$$b = (\lambda a)^{1/\gamma} \quad \textit{so that} \quad a/b = (a^{\gamma-1}/\lambda)^{1/\gamma} \to \infty.$$

(i) *If*

$$1 < \gamma < \delta \leq 2,$$

or, more generally, if

$$E(R^\gamma) < \infty, 1 < \gamma < 2$$

(*including $\gamma = \delta$ with slowly varying functions such that $E(R^\gamma)$ is finite*), *then the limit for the continuous flow rate model* (23) *is*

$$[E(R^\gamma)]^{1/\gamma}\, \Lambda_\gamma(t),$$

where Λ_γ is a Lévy-stable process with stable index γ. The limit process can be represented as

$$\begin{aligned} &E(R^\gamma)^{1/\gamma}\, \Lambda_\gamma(t) \\ &\quad = \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} 1_{\{0<s<t\}} ur \Big(N(ds, du, dr) - ds\, u^{-(1+\gamma)} du\, F_R(dr)\Big) \\ &\quad \stackrel{d}{=} \sigma_\gamma \int_{-\infty}^{\infty}\int_0^{\infty} 1_{\{0<s<t\}} r M_\gamma(ds, dr) \end{aligned}$$

where σ_γ is defined in (30) (*with $\alpha = \gamma$*) *and $M_\gamma(ds, dr)$ is γ-stable with control measure $ds\, F_R(dr)$, as defined in* (33).

(ii) *For any choice of parameters*

$$1 < \gamma < 2,\ 1 < \delta \le 2,$$

the compound Poisson workload model (24) *has the limit*

$$E(R)\,\Lambda_\gamma(t).$$

2.3. Remaining choices for the parameters γ and δ

For the continuous model we supposed earlier that $1 < \gamma < \delta \le 2$. We will now consider the remaining cases

$$\gamma = \delta = 2, \quad \textit{and} \quad 1 < \delta < \gamma \le 2$$

The first of these, $\gamma = \delta = 2$, remains also for the compound Poisson model. The second, $1 < \gamma < \delta \le 2$ will be applied to the continuous flow model, together with $1 < \gamma = \delta < 2$, given that proper moments exist. The generic choice of normalization is $b = (\lambda a)^{1/\delta}$ in each of the remaining cases. As $\lambda \to \infty$ and $a \to \infty$, and with this b, the convergence results hold regardless of the limit behavior of $\lambda/a^{\gamma-1}$. Hence the distinctions FCR, ICR, SCR are now irrelevant.

Theorem 4. *Set*

$$b = (\lambda a)^{1/\delta} \tag{42}$$

and assume

$$\lambda \to \infty,\ a \to \infty \quad \textit{or} \quad a \to \infty,\ b \to \infty \quad \textit{in any arbitrary way.}$$

(i) *Assume*

$$\gamma = \delta = 2.$$

Here $E(U^2) < \infty$, $E(R^2) < \infty$. *The continuous flow model in* (23) *has the limit*

$$\sqrt{E(U^2)E(R^2)}\,B(t) \quad t \ge 0,$$

and the compound Poisson model in (24) *has the limit*

$$\sqrt{E(U^2)}E(R)\,B(t) \quad t \ge 0,$$

where $B(t)$, $t \ge 0$, *denotes standard Brownian motion.*

(ii) *Assume*

$$1 < \delta < 2,$$

and that either γ *satisfies*

$$\delta < \gamma \le 2$$

or, more generally, that U *satisfies*

$$E(U^\delta) < \infty$$

(*thus including* $\gamma = \delta$ *with slowly varying functions making* U^δ *have finite mean*). *The limit process for the continuous flow model is*

$$[E(U^\delta)]^{1/\delta}\,\Lambda_\delta(t), \quad t \ge 0,$$

where $\Lambda_\delta(t)$ *is a Lévy stable process with index* δ.

Remark 12. For the case of fixed rewards, $R = 1$, higher-dimensional versions of Theorems 1–3 have been obtained in Kaj *et al.* (2007). Spatial versions of the continuous flow reward model are obtained by replacing the collection of sessions on the real line by a family of sets $\{x_j + u_j C\}_j$ on $\mathbf{R}^d$, where C is a fixed bounded set of volume $|C| = 1$ and vanishing boundary $|\partial C| = 0$. The location and size of the sets are given by a Poisson measure $N(dx, du)$ on $\mathbf{R}^d \times \mathbf{R}^+$ with intensity $\lambda dx\, F(du)$ such that the size distribution $F(du)$ is heavy-tailed at infinity. The analog of the workload functional W^*_λ is taken to be a stochastic integral

$$X(\mu) = \int_{\mathbf{R}^d} \int_{\mathbf{R}^+} \mu(x + uC)\, N(dx, du), \quad \mu \in \mathcal{M},$$

where $\mathcal{M}$ is a suitable subset of signed measures on $\mathbf{R}^d$. Here, $X(\mu)$ represents the configuration of mass on R^d of a Poisson germ-grain model with germs uniformly located in space and heavy-tailed grain size. The choice $d = 1$, $C = [0, 1]$ and $\mu_t(dy) = 1_{\{0<y<t\}}\, dy$ yields

$$\mu_t(s + uC) = \int_{[s,s+u]} 1_{\{0<y<t\}}\, dy = K_t(s, u)$$

in view of (15), which shows for this example $X(\mu_t) = W^*_\lambda(t)$.

By choosing properly the spatial scale, or equivalently, the size of the grains, in relation to the Poisson intensity and taking a limit the fluctuations of $X(\mu)$ again exhibit three different scaling regimes. The limiting operations are carried out with the use of generalized random fields based on a careful choice of the space of measures $\mathcal{M}$. The results in Kaj *et al.* (2007) generalize the Gaussian, stable and intermediate limits obtained here to a spatial setting and are in complete analogy to those of Theorems 1, 2 and 3, for the case of fixed rewards.

Biermé et al. (2006, 2007), extend the Gaussian and the intermediate scaling limit results further for an analogous model where the intensity of the size of grains has a specified power law behavior close to zero. It turns out that for such models one can obtain in the scaling limit, for example, the family of fractional Brownian fields $\{B_H(x),\, x \in \mathbf{R}^d\}$ with Hurst index H, $0 < H < 1/2$. Here, $B_H(x)$, $x \in \mathbf{R}^d$, are zero mean Gaussian random variables such that

$$\mathrm{Cov}(B_H(x), B_H(y)) = \frac{1}{2}\left(|x|^{2H} + |y|^{2H} - |x - y|^{2H}\right).$$

3. Proof of the theorems

The proofs in our setting provide an intuitive feeling for why the various limits appear. We will focus on the characteristic functions of the scaled and normalized workload process. By performing the appropriate limit operation for each choice of limiting scheme and deriving the limiting characteristic functions, we are able to identify the limit processes. We begin by stating characteristic functions for the processes $W^*_\lambda(t)$ and $W^*_{\sharp,\lambda}(t)$ centered at their expected values. We will then consider each case separately. This includes the intermediate, fast, and slow connection rates when the tails of the durations are heavier than the tails of the rewards. Further cases arise when the reward tails are heavier. We will have to consider separately the continuous flow model and the compound Poisson model.

3.1. Characteristic functions

The formulas given in the next two lemmas, which will be used repeatedly in the sequel, are consequences of a general property of Poisson integrals $\int f(x)(N(dx)-n(dx))$, namely that

$$\ln E \exp\Big\{i \int f(x)(N(dx)-n(dx))\Big\} = \int (e^{i\,f(x)} - 1 - if(x))\, n(dx),$$

which is well defined if

$$\int (f(x)^2 \wedge |f(x)|)\, n(dx) < \infty,$$

and in particular, if either $\int f^2(x)\, n(dx) < \infty$ or $\int |f(x)|\, n(dx) < \infty$.

Lemma 3. *The characteristic function for the finite-dimensional distributions of the centered continuous flow workload process $W^*_\lambda(t) - \lambda\nu E(R)t$, $t \geq 0$, is given for arbitrary $n \geq 1$, $0 \leq t_1 \leq \cdots \leq t_n$, and real $\theta_1, \ldots, \theta_n$, by the relation*

$$\ln E\, \exp\Big\{i \sum_{j=1}^n \theta_j (W^*_\lambda(t_j) - \lambda\nu E(R)t_j)\Big\} = \int_{-\infty}^\infty \int_0^\infty \int_0^\infty h(s,u,r)\, n(ds,du,dr),$$

where

$$h(s,u,r) = \exp\Big\{i \sum_{j=1}^n \theta_j K_{t_j}(s,u) r\Big\} - 1 - i \sum_{j=1}^n \theta_j K_{t_j}(s,u) r \tag{43}$$

and $n(ds,du,dr) = \lambda ds\, F_U(du)\, F_R(dr)$ is the intensity measure defined in (1).

Lemma 4. *The characteristic function for the finite-dimensional distributions of the centered compound Poisson workload process $W^*_{\sharp,\lambda}(t) - \lambda\nu E(R)t$, $t \geq 0$, is given for arbitrary $n \geq 1$, $0 \leq t_1 \leq \cdots \leq t_n$, and real $\theta_1, \ldots, \theta_n$, by*

$$\ln E\, \exp\Big\{i \sum_{j=1}^n \theta_j (W^*_{\sharp,\lambda}(t_j) - \lambda\nu E(R)t_j)\Big\} = \int_{-\infty}^\infty \int_0^\infty g(s,u)\, n(ds,du),$$

where

$$n(ds,du) = \lambda ds\, F_U(du)$$

and

$$\begin{aligned} g(s,u) &= E\Big(\exp\Big\{i\sum_{i=1}^{n}\theta_j\Xi(K_{t_j}(s,u))\Big\} - 1 - i\sum_{i=1}^{n}\theta_j\Xi(K_{t_j}(s,u))\Big) \\ &= \exp\Big\{\int_0^\infty\int_0^\infty\Big(\exp\Big\{i\sum_{j=1}^{n}\theta_j 1_{\{w\le K_{t_j}(s,u)\}}r\Big\} - 1\Big)dw\,F_R(dr)\Big\} \\ &\qquad -1 - i\sum_{j=1}^{n}\theta_j K_{t_j}(s,u)E(R). \end{aligned}$$

Observe that the expressions for the logarithmic characteristic functions stated in Lemmas 3 and 4 above are well defined, because the inequality

$$\Big|e^{iu} - 1 - iu\Big| \le 2|u|, \quad u \in R, \tag{44}$$

and the relation

$$\int_{-\infty}^{\infty} K_t(s,u)\,ds = ut, \tag{45}$$

which is readily derived from (15), imply

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_0^\infty\int_0^\infty |h(s,u,r)|\,n(ds,du,dr) \\ &\qquad \le 2\sum_{i=1}^{n}|\theta_j|\int_{-\infty}^{\infty}\int_0^\infty\int_0^\infty K_t(s,u)r\,n(ds,du,dr) = 2\sum_{i=1}^{n}|\theta_j|\,\lambda\nu E(R)t \end{aligned}$$

and

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_0^\infty |g(s,u)|\,n(ds,du) \\ &\qquad \le 2\sum_{i=1}^{n}|\theta_j|E(R)\int_{-\infty}^{\infty}\int_0^\infty K_t(s,u)\,n(ds,du) = 2\sum_{i=1}^{n}|\theta_j|\,\lambda\nu E(R)t. \end{aligned}$$

More refined estimates will be needed to carry out the various scaling limit operations.

3.2. Proof of Theorem 1 (ICR)

We can lump together the finite and infinite variance cases but we will need to distinguish between the continuous flow model and the compound Poisson model.

3.2.1. The continuous flow model. Applying (26) with $b = a$ and Lemma 3, we have

$$\begin{aligned} &\ln E\,\exp\Big\{i\sum_{j=1}^{n}\theta_j(W_\lambda^*(at_j) - \lambda\nu E(R)at_j)/a\Big\} \\ &\qquad = \int_{-\infty}^{\infty}\int_0^\infty\int_0^\infty h(s,u,r)\,EN(ads,adu,dr), \end{aligned} \tag{46}$$

where h is defined in (43). Under the ICR assumption $\lambda, a \to \infty$ with $\lambda/a^{\gamma-1} \to c^{\gamma-1}$, the scaled intensity measure has the asymptotic form

$$EN(ads, adu, dr) = \lambda ads\, F_U(adu)\, F_R(dr) \sim c^{\gamma-1}\, dsu^{-(1+\gamma)}du\, F_R(dr).$$

The logarithmic characteristic function of the process $Y_{\gamma,R}$ defined by the Poisson integral expression (35) is given by

$$\ln E\, \exp\Big\{i \sum_{j=1}^{n} \theta_j\, Y_{\gamma,R}(t_j)\Big\} = \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} h(s,u,r)\, dsu^{-(1+\gamma)}du\, F_R(dr),$$

in complete analogy to the result of Lemma 3. Thus,

$$\begin{aligned} &\ln E\, \exp\Big\{i \sum_{j=1}^{n} \theta_j\, cY_{\gamma,R}(t_j/c)\Big\} \\ &\quad= \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} h(cs,cu,r)\, dsu^{-(1+\gamma)}du\, F_R(dr) &(47)\\ &\quad= c^{\gamma-1}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} h(s,u,r)\, dsu^{-(1+\gamma)}du\, F_R(dr), &(48)\end{aligned}$$

where (47) follows from (28) expressed as $cK_{t/c}(s,u) = K_t(cu, cs)$. Hence to prove Theorem 1 i), it is enough to verify that (46) converges to (48) under the ICR scaling.

Integration by parts in the variable u shows that the right-hand side of (46) equals

$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{\partial}{\partial u} h(s,u,r)\, ds\, \lambda a P(U > au) du\, F_R(dr), \tag{49}$$

where U, which has the distribution $F_U(du)$, satisfies by assumption

$$\lambda a P(U > au) \to c^{\gamma-1} u^{-\gamma}/\gamma.$$

If we are allowed to take this limit inside the integral in (49), then another integration by parts will revert the resulting integral into the form (48) and hence conclude the proof. To justify the last steps it remains to demonstrate that the integrand in (49) is appropriately dominated. The proofs of the required estimates simplify somewhat if we first make the change of variable $s \to s + u$. Hence we agree to consider instead of (49) the integral

$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{\partial}{\partial u} h(s-u,u,r)\, ds\, \lambda a P(U > au) du\, F_R(dr). \tag{50}$$

(Note that the function in the integrand is the derivative of h with respect to its second argument u, evaluated in the point $(s-u, u, r)$.) We will use the Potter bounds, see Bingham *et al.* (1987). Since the function $P(U > u)$ is regularly varying at $u \to \infty$ with tail behavior $u^{-\gamma}$, the Potter bound yields for any $\epsilon > 0$ a number $a_0 > 0$ such that

$$\frac{P(U > au)}{P(U > a)} \le 2\, u^{-\gamma} \max(u^{-\epsilon}, u^{\epsilon})$$

for $a \geq a_0$ and $au \geq a_0$. Moreover, since $\lambda a P(U > a) \to c^{\gamma-1}/\gamma$, using possibly a larger a_0 we have

$$\lambda a P(U > au) \leq 2(c^{\gamma-1}/\gamma + \epsilon) u^{-\gamma} \max(u^{-\epsilon}, u^{\epsilon}), \quad a \geq a_0, \; au \geq a_0. \tag{51}$$

Since $\frac{\partial}{\partial u} K_t(s,u) = 1_{\{0<s+u<t\}}$ by (17),

$$\frac{\partial}{\partial u} h(s-u,u,r) = i\Big(\exp\Big\{i \sum_{j=1}^{n} \theta_j K_{t_j}(s-u,u) r\Big\} - 1\Big) \sum_{k=1}^{n} \theta_k 1_{\{0<s<t_k\}} r.$$

For any $0 \leq \kappa \leq 1$, we have $|e^{ix} - 1| \leq 2^{1-\kappa}|x|^{\kappa}$ and $(\sum_{i=1}^{n} |x_i|)^{\kappa} \leq \sum_{i=1}^{n} |x_i|^{\kappa}$. Since (16) implies $0 \leq K_t(s,u) \leq u$,

$$\Big|\exp\Big\{i \sum_{j=1}^{n} \theta_j K_{t_j}(s,u) r\Big\} - 1\Big| \leq \Big(2^{1-\kappa} \sum_{j=1}^{n} |\theta_j|^{\kappa} u^{\kappa}\, r^{\kappa}\Big) \wedge 2 \tag{52}$$

and so

$$\Big|\frac{\partial}{\partial u} h(s-u,u,r)\Big| \leq 2 \min\Big(\sum_{j=1}^{n} |\theta_j|^{\kappa} u^{\kappa} r^{\kappa}, 1\Big) \sum_{k=1}^{n} |\theta_k| 1_{\{0<s<t_k\}}\, r. \tag{53}$$

We may assume that $t_1 > 0$ and for convenience that $a \geq a_0$ is so large that $a_0/a \leq t_1$. Relations (51) and (53) now imply that the integrand in (49) is bounded on $\{u \geq a_0/a\}$,

$$\Big|\frac{\partial}{\partial u} h(s,u,r)\Big| \lambda a P(U > au)\, 1_{\{u \geq a_0/a\}} \leq B_1(s,u,r),$$

where

$$B_1(s,u,r) = C_{\epsilon,\kappa} u^{-\gamma} \max(u^{-\epsilon}, u^{\epsilon}) \min\Big(\sum_{j=1}^{n} |\theta_j|^{\kappa} u^{\kappa} r^{\kappa}, 1\Big) \sum_{k=1}^{n} |\theta_k| 1_{\{0<s<t_k\}}\, r$$

and

$$C_{\epsilon,\kappa} = 4(c^{\gamma-1}/\gamma + \epsilon).$$

Now

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} B_1(s,u,r)\, ds du F_R(dr) \\ &\quad \leq C_{\epsilon,\kappa} \sum_{k=1}^{n} |\theta_k| t_k \Big(E(R^{1+\kappa}) \int_0^{t_1} \sum_{j=1}^{n} |\theta_j|^{\kappa} u^{\kappa-\gamma} \max(u^{-\epsilon}, u^{\epsilon})\, du \\ &\qquad + E(R) \int_{t_1}^{\infty} u^{-\gamma} \max(u^{-\epsilon}, u^{\epsilon})\, du\Big) \end{aligned}$$

Since $1 < \gamma < \delta \leq 2$ we may choose ϵ and κ such that

$$1 + \epsilon < \gamma, \quad \gamma + \epsilon < 1 + \kappa < \delta.$$

Then $E(R^{1+\kappa}) < \infty$ and the du-integrals are finite.

Next we must find a dominating function which applies to $0 < u \le a_0/a \le 1$. By (53),

$$\Big|\frac{\partial}{\partial u} h(s-u,u,r)\Big| \le 2\sum_{j=1}^n |\theta_j|^\kappa \sum_{k=1}^n |\theta_k| 1_{\{0<s<t_k\}}\, u^\kappa r^{1+\kappa}.$$

Fix $\epsilon > 0$. Using

$$u^\kappa \le (a_0/a)^{\gamma-1+\epsilon} u^{1+\kappa-\gamma-\epsilon}, \quad u \le a_0/a,$$

it follows from Markov's inequality,

$$\lambda a P(U > au) \le \lambda u^{-1} E(U),$$

that

$$u^\kappa \lambda a P(U > au) \le a_0^{\gamma-1+\epsilon} \nu\, u^{\kappa-\gamma-\epsilon} \lambda / a^{\gamma-1+\epsilon}.$$

Recall that the general scaling assumption for ICR is $\lambda L_U(a)/a^{\gamma-1} \to c^{\gamma-1}$, where L_U is a slowly varying function related to the asymptotic form of the duration U. By a general property of slowly varying functions, $a^{-\epsilon} \le L(a)$ for a sufficiently large. Hence we end up with

$$\Big|\frac{\partial}{\partial u} h(s-u,u,r)\Big| \lambda a P(U > au)\, 1_{\{u \le a_0/a\}} \le B_2(s,u,r),$$

where

$$B_2(s,u,r) = C_{\epsilon,\kappa} \sum_{k=1}^n |\theta_k| 1_{\{0<s<t_k\}}\, u^{\kappa-\gamma-\epsilon} r^{1+\kappa} 1_{\{0<u\le 1\}}$$

and

$$C_{\epsilon,\kappa} = 2a_0^{\gamma-1+\epsilon}\, \nu (c^{\gamma-1} + \epsilon) \sum_{j=1}^n |\theta_j|^\kappa.$$

The bound B_2 is integrable with respect to $ds\, du\, F_R(dr)$ for $\gamma + \epsilon < 1 + \kappa < \delta$. Since ϵ is arbitrary, this allows us to apply the dominated convergence theorem and complete the proof of part i) of Theorem 1.

3.2.2. The compound Poisson workload model. We turn to part ii) of Theorem 1. By Lemma 4,

$$\ln E \exp\Big\{i \sum_{j=1}^n \theta_j (W^*_{\sharp,\lambda}(at_j) - \lambda \nu E(R) a t_j)/a\Big\} = \int_{-\infty}^\infty \int_0^\infty g_a(s,u)\, n(ads, adu),$$

where

$$\begin{aligned} g_a(s,u) \quad = \quad & \exp\Big\{\int_0^\infty \int_0^\infty a\Big(\exp\Big\{i \sum_{j=1}^n \theta_j 1_{\{w \le K_{t_j}(s,u)\}} r/a\Big\} - 1\Big)\, dw F_R(dr)\Big\} \\ & -1 - i \sum_{j=1}^n \theta_j K_{t_j}(s,u) E(R) \end{aligned}$$

after making the change of variables $w \to aw$, $s \to as$, $u \to au$ and using (28). Since

$$g_a(s,u) \sim \exp\Big\{i\sum_{j=1}^n \theta_j K_{t_j}(s,u)\,E(R)\Big\} - 1 - i\sum_{j=1}^n \theta_j K_{t_j}(s,u)E(R), \quad a\to\infty,$$

we can complete the proof in much the same way as in the previous part i), noticing that this case is in fact simpler in the sense that only the expected reward $E(R)$ and not the full distribution $F_R(dr)$ enters the limiting characteristic function. Since, as $a \to \infty$,

$$n(ads, adu) = \lambda ads\, F_U(adu) \sim c^{\gamma-1}\, ds\, u^{-(\gamma+1)}du,$$

the result in this case is

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_0^{\infty} g_a(s,u)\, n(ads, adu)\\
\to\quad & c^{\gamma-1}\int_{-\infty}^{\infty}\int_0^{\infty}\Big(\exp\Big\{i\sum_{j=1}^n \theta_j K_{t_j}(s,u)\,E(R)\Big\}\\
&\qquad\qquad -1 - i\sum_{j=1}^n \theta_j K_{t_j}(s,u)E(R)\Big)\, ds\, u^{-(\gamma+1)}du\\
=\quad & \int_{-\infty}^{\infty}\int_0^{\infty}\Big(\exp\Big\{i\sum_{j=1}^n \theta_j cK_{t_j/c}(s,u)\,E(R)\Big\}\\
&\qquad\qquad -1 - i\sum_{j=1}^n \theta_j cK_{t_j/c}(s,u)E(R)\Big)\, ds\, u^{-(\gamma+1)}du\\
&= \ln E\,\exp\Big\{i\sum_{j=1}^n \theta_j\, cY_\gamma(t_j/c)\,E(R)\Big\},
\end{aligned}$$

where we used (28) and where the process Y_γ is defined in (36).

3.3. Proof of Theorem 2 (FCR)

In the asymptotic regime of fast connection rate and for the continuous flow model it is necessary to study the cases $\delta = 2$ and $\delta < 2$ separately.

3.3.1. The continuous flow model, finite variance rewards. We start with the case $\delta = 2$ of finite second moment rewards and we use representation (26) of the workload process, to avoid scaling in the variable r. Observe first that

$$\begin{aligned}
EN(ads, adu, dr) &= \lambda ads\, F_U(adu)\, F_R(dr)\\
&\sim \lambda a a^{-\gamma} ds\, u^{-\gamma-1}du\, F_R(dr)\\
&= \left(\frac{b}{a}\right)^2 ds\, u^{-\gamma-1}du\, F_R(dr).
\end{aligned}$$

Hence, setting $\zeta = b/a$, by (26) and Lemma 3,

$$\begin{aligned}
\ln E\,\exp\Big\{i\sum_{j=1}^{n}\theta_j(W_\lambda^*(at_j)-\lambda\nu E(R)at_j)/b\Big\} \\
&= \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} h(s,u,\zeta^{-1}r)\,EN(ads,adu,dr) \\
&\sim \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} h(s,u,\zeta^{-1}r)\,\zeta^2\,ds\,u^{-(\gamma+1)}du\,F_R(dr).
\end{aligned}$$

To justify taking the limit inside of the integral a similar argument applies as in the proof of Theorem 1. The task is to dominate the integrand in

$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{\partial}{\partial u}h(s-u,u,\zeta^{-1}r)\,\lambda a P(U>au)\,ds\,du\,F_R(dr), \tag{54}$$

just as we did earlier for ICR in (50). Because of the finite variance condition $E(R^2)<\infty$, this case is simpler and we can use (53) with $\kappa = 1$. Potter's theorem and the Markov inequality apply again to obtain bounds for the tail probability $P(U>au)$. The resulting estimates together justify using the dominated convergence theorem. Hence the Taylor expansion

$$h(s,u,\zeta^{-1}r) = -\zeta^{-2}\frac{1}{2}\Big(\sum_{j=1}^{n}\theta_j K_{t_j}(s,u)r\Big)^2 + o(\zeta^{-2}), \quad \zeta\to\infty,$$

shows that

$$\begin{aligned}
\ln E\,\exp\Big\{i\sum_{j=1}^{n}\theta_j(W_\lambda^*(at_j)-\lambda\nu E(R)at_j)/b\Big\} \\
&\to -\frac{1}{2}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}\Big(\sum_{j=1}^{n}\theta_j K_{t_j}(s,u)r\Big)^2 ds\,u^{-(\gamma+1)}du\,F_R(dr) \\
&= -\frac{1}{2}E(R^2)\sum_{i=1}^{n}\sum_{j=1}^{n}\theta_i\theta_j\int_{-\infty}^{\infty}\int_0^{\infty} K_{t_i}(s,u)K_{t_j}(s,u)\,ds\,u^{-\gamma-1}du.
\end{aligned}$$

One has

$$\int_{-\infty}^{\infty}\int_0^{\infty} K_{t_i}(s,u)K_{t_j}(s,u)\,ds\,u^{-\gamma-1}du = \frac{\sigma^2}{2}(t_i^{2H}+t_j^{2H}-|t_i-t_j|^{2H}),$$

where $H=(3-\gamma)/2$ and σ is given by (38), and therefore the limit process is the fractional Brownian motion

$$E(R^2)^{1/2}\,\sigma\,B_H(t).$$

An alternative way to see that the limit is fractional Brownian motion is to observe that the process (39) is Gaussian, H-self-similar and has stationary increments.

3.3.2. Continuous flow model, infinite variance rewards ($\delta < 2$). In the case $1 < \gamma < \delta < 2$ of infinite variance rewards, we have $F_R(dr) \sim r^{-\delta-1}\,dr$ as $r \to \infty$. Lemma 3 and the scaling representation (27) yield

$$\begin{aligned} &\ln E\,\exp\Big\{i\sum_{j=1}^{n}\theta_j(W_\lambda^*(at_j) - \lambda\nu E(R)at_j)/b\Big\} \\ &\quad= \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} h(s,u,r)\,EN(ads,adu,(b/a)dr) \end{aligned}$$

where h is defined in (43). Because of the choice of the normalization factor b,

$$\begin{aligned} EN(ads,adu,(b/a)dr) &= \lambda ads\,F_U(adu)\,F_R((b/a)dr) \\ &\sim ds\,u^{-\gamma-1}du\,r^{-\delta-1}dr, \quad \frac{b}{a} \to \infty. \end{aligned}$$

We need to verify that the limiting log-characteristic function is given by

$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} h(s,u,r)\,ds\,u^{-\gamma-1}du\,r^{-\delta-1}dr,$$

which is the logarithm of the characteristic function of the Telecom process as defined in (40). In view of (29) this also yields the representation (41). The corresponding δ-stable form of the characteristic function is obtained by integrating over r (Samorodnitsky and Taqqu (1994), Exercise 3.24):

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} h(s,u,r)\,ds\,u^{-\gamma-1}du\,r^{-\delta-1}dr \\ &\quad= -\frac{1}{2}(\sigma_\delta)^\delta\int_{-\infty}^{\infty}\int_0^{\infty}\Big|\sum_{j=1}^{n}\theta_j K_{t_j}(s,u)\Big|^\delta k_\delta\Big(\sum_{j=1}^{n}\theta_j K_{t_j}(s,u)\Big)\,ds\,u^{-\gamma-1}du, \end{aligned}$$

where σ_δ is given by (30) and $k_\delta(\theta)$ by (32), with $\alpha = \delta$.

To establish the limit result we fix $\epsilon > 0$ and split the integral in three parts,

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} h(s,u,r)(\lambda ads\,F_U(adu)\,F_R((b/a)dr) \\ &\qquad\qquad -ds\,u^{-\gamma-1}du\,r^{-\delta-1}dr) = I_\epsilon^1 + I_\epsilon^2 + I_\epsilon^3, \end{aligned}$$

corresponding to the three domains of integration $A_\epsilon^1 = \{u > \epsilon, r > \epsilon\}$, $A_\epsilon^2 = \{u < \epsilon < r\}$ and $A_\epsilon^3 = \{r < \epsilon\}$, not involving the integration over s.

Writing

$$\begin{aligned} \mu_\lambda(du,dr) &= \lambda a\,uF_U(adu)\,rF_R((b/a)dr), \\ \mu(du,dr) &= u^{-\gamma}du\,r^{-\delta}dr, \end{aligned}$$

and

$$H(u,r) = \frac{1}{ur}\int_{-\infty}^{\infty} h(s,u,r)\,ds,$$

we have

$$I_\epsilon^1 = \int_\epsilon^\infty \int_\epsilon^\infty H(u,r)(\mu_\lambda(du,dr) - \mu(du,dr)).$$

Here, $|H(u,r)| \le 2\sum_{j=1}^n \theta_j t_j < \infty$ in view of (44) and (45). It follows similarly that $H(u,r)$ is jointly continuous in A_ϵ^1. Since

$$\int\int_{A_\epsilon^1} \mu_\lambda(du,dr) < \infty, \quad \int\int_{A_\epsilon^1} \mu(du,dr) < \infty,$$

and the measure μ_λ converges weakly to μ, we obtain $I_\epsilon^1 \to 0$ by weak convergence.

We now consider I_ϵ^2. Using the more general inequality $|e^{ix} - 1 - ix| \le c_\kappa |x|^{1+\kappa}$ where c_κ is a constant and $\kappa \in [0,1]$, we obtain

$$\Big| \int_{-\infty}^\infty \int_0^\epsilon \int_\epsilon^\infty h(s,u,r)\, ds\, u^{-\gamma-1} du\, r^{-\delta-1} dr \Big|$$
$$\le c_\kappa 2^\kappa \sum_{j=1}^n |\theta_j|^{1+\kappa} \int_\epsilon^\infty r^{\kappa-\delta}\, dr \int_{-\infty}^\infty \int_0^\epsilon K_{t_j}(s,u)^{1+\kappa} ds\, u^{-\gamma-1} du. \quad (55)$$

Using (16) in the form $0 \le K_t(s,u) \le u$ together with (45) we may continue with

$$\int_{-\infty}^\infty K_{t_j}(s,u)^{1+\kappa}\, ds \le u^\kappa \int_{-\infty}^\infty K_{t_j}(s,u)\, ds = u^{1+\kappa} t_j,$$

and then compute the remaining integrals on the right-hand side of (55). Under the assumption $\gamma < 1+\kappa < \delta$, this yields a constant c'_κ such that

$$\Big| \int_{-\infty}^\infty \int_0^\epsilon \int_\epsilon^\infty h(s,u,r)\, ds\, u^{-\gamma-1} du\, r^{-\delta-1} dr \Big| \le c'_\kappa \epsilon^{2(1+\kappa)-\gamma-\delta}.$$

Similarly,

$$\Big| \int_{-\infty}^\infty \int_0^\epsilon \int_\epsilon^\infty h(s,u,r)\lambda a ds\, F_U(adu)\, F_R((b/a)dr) \Big|$$
$$\le d_\kappa \lambda a \int_\epsilon^\infty r^{1+\kappa}\, F_R((b/a)dr) \int_0^\epsilon u^{1+\kappa}\, F_U(adu)$$

for a suitable constant d_κ. By the properties of regularly varying functions, we have

$$(b/a)^\delta \int_\epsilon^\infty r^{1+\kappa}\, F_R((b/a)dr) \to \int_\epsilon^\infty r^{\kappa-\delta}\, dr, \quad b/a \to \infty$$

if $1+\kappa < \delta$, and

$$a^\gamma \int_0^\epsilon u^{1+\kappa}\, F_U(adu) \to \int_0^\epsilon u^{\kappa-\gamma}\, du, \quad a \to \infty$$

if $\gamma < 1+\kappa$. Hence we can find d'_κ such that

$$\Big| \int_{-\infty}^\infty \int_0^\epsilon \int_\epsilon^\infty h(s,u,r)\lambda a ds\, F_U(adu)\, F_R((b/a)dr) \Big|$$
$$\le d'_\kappa \epsilon^{2(1+\kappa)-\gamma-\delta}, \quad \gamma < 1+\kappa < \delta.$$

By taking in addition κ such that $(\gamma+\delta)/2 < 1+\kappa < \delta$ this shows

$$I_\epsilon^2 \le (c'_\kappa + d'_\kappa)\, \epsilon^{2(1+\kappa)-\gamma-\delta} \to 0 \quad \epsilon \to 0.$$

Finally,

$$\begin{aligned}
&\Big| \int_{-\infty}^\infty \int_0^\infty \int_0^\epsilon h(s,u,r)\, ds\, u^{-\gamma-1} du\, r^{-\delta-1} dr \Big| \\
&\qquad \le c_2 \sum_{i,j=1}^n \theta_i\theta_j \int_{-\infty}^\infty \int_0^\infty K_{t_i}(s,u) K_{t_j}(s,u)\, ds\, u^{-\gamma-1} du \int_0^\epsilon r^{1-\delta}\, dr.
\end{aligned}$$

Since the $dsdu$-integral is the finite covariance function $\mathrm{Cov}(B_H(t_i), B_H(t_j))$ of fractional Brownian motion with $H = (3-\gamma)/2$, the right-hand side takes the form const $\epsilon^{2-\delta} \to 0$, $\epsilon \to 0$. Similarly, for λ and a sufficiently large, we obtain

$$\Big| \int_{-\infty}^\infty \int_0^\infty \int_0^\epsilon h(s,u,r) \lambda a ds\, F_U(adu)\, F_R((b/a)dr) \Big| \le \text{const}\, \epsilon^{2-\delta}.$$

Thus $I_\epsilon^3 \to 0$ as $\epsilon \to 0$, which concludes the proof of the desired convergence of characteristic functions for this case.

3.3.3. The compound Poisson model. For the compound Poisson model the limit process is the same for all parameters in the range $1 < \gamma < 2$, $1 < \delta \le 2$. By Lemma 4 and (28),

$$\ln E \exp\Big\{ i \frac{1}{b} \sum_{j=1}^n \theta_j (W^*_{\sharp,\lambda}(at_j) - \lambda \nu E(R) a t_j) \Big\} = \int_{-\infty}^\infty \int_0^\infty g_{a,b}(s,u)\, \lambda a ds\, F_U(adu),$$

where

$$\begin{aligned}
g_{a,b}(s,u) &= \exp\Big\{ \int_0^\infty \int_0^\infty \Big(\exp\Big\{ i \sum_{j=1}^n \theta_j 1_{\{w \le a K_{t_j}(s,u)\}} r/b \Big\} - 1 \Big) dw\, F_R(dr) \Big\} \\
&\qquad\qquad -1 - i \sum_{j=1}^n \theta_j a K_{t_j}(s,u) E(R)/b \\
&\sim \exp\Big\{ \int_0^\infty \int_0^\infty i \sum_{j=1}^n \theta_j 1_{\{w \le K_{t_j}(s,u)\}}\, r(a/b)\, dw\, F_R(dr) \Big\} \\
&\qquad\qquad -1 - i \sum_{j=1}^n \theta_j a K_{t_j}(s,u) E(R)/b \\
&= \exp\Big\{ i \sum_{j=1}^n \theta_j K_{t_j}(s,u) E(R)(a/b) \Big\} - 1 - i \sum_{j=1}^n \theta_j K_{t_j}(s,u) E(R)(a/b).
\end{aligned}$$

Hence, by Taylor expansion as $a/b \to 0$, the log-characteristic function converges to

$$-\frac{1}{2}\int_{-\infty}^{\infty}\int_0^{\infty}\Big(\sum_{j=1}^{n}\theta_j K_{t_j}(s,u)E(R)\Big)^2\,ds\,u^{-\gamma-1}du$$
$$= -\frac{1}{2}E(R)^2\sum_{i=1}^{n}\sum_{j=1}^{n}\theta_i\theta_j\int_{-\infty}^{\infty}\int_0^{\infty}K_{t_i}(s,u)K_{t_j}(s,u)\,ds\,u^{-\gamma-1}du.$$

The limit is therefore the fractional Brownian motion $E(R)\,\sigma\,B_H(t)$, $t \geq 0$.

3.4. Proof of Theorem 3 (SCR)

The proofs in the regime of slow connection rate are similar to the previous ones. To see which limit to expect we shall scale directly the integral representations instead of the characteristic functions.

3.4.1. The continuous flow model. The relevant scaling for any choice of parameters $1 < \gamma < \delta \leq 2$, is

$$\frac{1}{b}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}K_{at}(s,u)r\,\widetilde{N}(ds,du,dr)$$
$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}K_{(a/b)t}(s/b,u/b)r\,\widetilde{N}(ds,du,dr)$$
$$= \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}K_{(a/b)t}((a/b)s,u)r\,\widetilde{N}(a\,ds,b\,du,dr),$$

where $\widetilde{N}$ is defined by (6). Here, the compensator n scales as

$$\begin{aligned} n(a\,ds,b\,du,dr) &= \lambda a ds\,F_U(b\,du)\,F_R(dr) \\ &\sim ds\,u^{-1-\gamma}du\,F_R(dr), \end{aligned}$$

since $b = (\lambda a)^{1/\gamma} \to \infty$. Moreover, if we write $z = a/b$ then $z \to \infty$ and, using (17),

$$\begin{aligned} K_{zt}(zs,u) &= \int_0^u 1_{\{0<y+zs<zt\}}\,dy \\ &\to \int_0^u 1_{\{0<s<t\}}\,dy = u1_{\{0<s<t\}}, \quad z \to \infty. \end{aligned} \tag{56}$$

This suggests that the limit process is given by

$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}u1_{\{0<s<t\}}\,r\,\left(N(ds,du,dr) - ds\,u^{-1-\gamma}\,du\,F_R(dr)\right)$$
$$\stackrel{d}{=} \sigma_\gamma\int_0^t\int_0^{\infty}r\,M_\gamma(ds,dr) \stackrel{d}{=} E(R^\gamma)^{1/\gamma}\sigma_\gamma\int_0^t M_\gamma(ds) \tag{57}$$
$$\stackrel{d}{=} E(R^\gamma)^{1/\gamma}\,\Lambda_\gamma(t), \tag{58}$$

where σ_γ is defined in (30) and $M_\gamma(ds, dr)$ and $M_\gamma(ds)$ are γ-stable random measures with control measures $ds\, F_R(dr)$ and ds respectively and where $\Lambda_\gamma(t)$ is a Lévy-stable process with index γ. The limit process is well defined for any distribution F_R with $E(R^\gamma) < \infty$, in particular if we keep our assumption on R being regularly varying with a tail of index δ, such that $\gamma < \delta \le 2$.

In order to establish the convergence, we begin as in (50), with the representation

$$\begin{aligned}
&\ln E\, \exp\Big\{i \sum_{j=1}^{n} \theta_j (W_\lambda^*(t_j) - \lambda\nu E(R) t_j)\Big\} \\
&\qquad = \int_{-\infty}^{\infty}\int_0^\infty\int_0^\infty \frac{\partial}{\partial u} h(s-u,u,r)\, \lambda ds\, P(U>u) du\, F(dr).
\end{aligned}$$

Applying the scaling parameters a and b it follows that

$$\begin{aligned}
&\ln E\, \exp\Big\{i \sum_{j=1}^{n} \theta_j (W_\lambda^*(at_j) - \lambda\nu E(R) at_j)/b\Big\} \\
&= \int_{-\infty}^{\infty}\int_0^\infty\int_0^\infty i\Big(\exp\Big\{i\sum_{j=1}^n \theta_j K_{at_j}(s-u,u) r/b\Big\} - 1\Big) \\
&\qquad\qquad \times \frac{1}{b}\sum_{k=1}^n \theta_k 1_{\{0<s<at_k\}} r\, \lambda ds\, P(U>u) du\, F(dr) \\
&= \int_{-\infty}^{\infty}\int_0^\infty\int_0^\infty i\Big(\exp\Big\{i\sum_{j=1}^n \theta_j K_{zt_j}(zs-u,u) r\Big\} - 1\Big) \\
&\qquad\qquad \times \sum_{k=1}^n \theta_k 1_{\{0<s<t_k\}} r\, ds\, b^\gamma P(U>bu) du\, F(dr),
\end{aligned}$$

where $z = a/b \to \infty$ and we have used the normalization $b^\gamma = \lambda a$ valid under SCR. Now

$$K_{zt}(zs-u,u) \to u 1_{\{0<s<t\}}, \quad z \to \infty$$

and

$$b^\gamma P(U > bu) \to \frac{1}{\gamma u^\gamma}, \quad b \to \infty.$$

This shows that the above integrand with respect to $ds\, du\, F_R(dr)$,

$$f_\lambda(s,u,r) = i\Big(\exp\Big\{i\sum_{j=1}^n \theta_j K_{zt_j}(zs-u,u) r\Big\} - 1\Big)\sum_{k=1}^n \theta_k 1_{\{0<s<t_k\}} r\, b^\gamma P(U>bu),$$

has the pointwise limit

$$f(s,u,r) = i\Big(\exp\Big\{i\sum_{j=1}^n \theta_j 1_{\{0<s<t_j\}}\, ur\Big\} - 1\Big)\sum_{k=1}^n \theta_k 1_{\{0<s<t_k\}} r\, \gamma^{-1} u^{-\gamma}$$

as λ and hence z and b tend to infinity. Since the logarithmic characteristic function of the limit process in Theorem 3 i) is given by

$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} f(s,u,r)\,ds\,du\,F_R(dr),$$

by Lemma 3, it remains to show that $|f_\lambda(s,u,r)|$ is dominated by an integrable function.

Since $b^\gamma P(U>b)\to 1/\gamma$, $b\to\infty$, it follows from the Potter bound as in (51) that for any $\epsilon>0$ there is a number b_0, such that

$$b^\gamma P(U>ub)\le 2(1/\gamma+\epsilon)\,u^{-\gamma}\max(u^{-\epsilon},u^{\epsilon}) \qquad b\ge b_0,\quad ub\ge b_0.$$

There is no restriction to assume $t_1>0$ and that λ and thus b are so large that $b_0/b\le t_1$. The task of estimating $|f_\lambda(s,u,r)|$ will be split accordingly in the three cases $0<u<b_0/b$, $b_0/b\le u<t_1$ and $t_1\le u<\infty$, where Potter's bound is applicable in the two latter but not in the first interval.

As in (52), for any $0<\kappa<1$,

$$\Big|\exp\Big\{i\sum_{j=1}^n\theta_j K_{zt_j}(zs-u,u)r\Big\}-1\Big|\le 2\min\Big(\sum_{j=1}^n|\theta_j|^\kappa u^\kappa r^\kappa,1\Big).$$

This shows

$$|f_\lambda(s,u,r)|\,1_{\{b_0/b\le u\}}\le f_1(s,u,r)$$

where

$$f_1(s,u,r)=4(1/\gamma+\epsilon)\sum_{k=1}^n|\theta_k|1_{\{0<s<t_k\}}\,r\min\Big(\sum_{j=1}^n|\theta_j|^\kappa u^\kappa r^\kappa,1\Big)u^{-\gamma}\max(u^{-\epsilon},u^{\epsilon}).$$

This upper bound is integrable, since

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} f_1(s,u,r)\,ds\,du\,F_R(dr)\\ &\quad\le 4(1/\gamma+\epsilon)\sum_{k=1}^n|\theta_k|t_k\Big(ER^{1+\kappa}\int_0^{t_1}u^{-\gamma+\kappa}\max(u^{-\epsilon},u^{\epsilon})\,du\\ &\qquad\qquad +E(R)\int_{t_1}^{\infty}u^{-\gamma}\max(u^{-\epsilon},u^{\epsilon})\,du\Big)<\infty,\end{aligned}$$

if we choose $\gamma+\epsilon<1+\kappa<\delta$ and $1+\epsilon<\gamma$.

It remains to find a dominating function for small u, that is $0<u<b_0/b$. Again by (52), for $0<\kappa<1$,

$$|f_\lambda(s,u,r)|\le\sum_{j=1}^n|\theta_j|^\kappa\sum_{k=1}^n\theta_k 1_{\{0<s<t_k\}}r^{1+\kappa}\,u^\kappa\,b^\gamma P(U>bu).$$

Moreover, by Markov's inequality

$$\begin{aligned} u^{\kappa} b^{\gamma} P(U > bu) &\leq u^{1+\kappa-\gamma}(b_0/b)^{\gamma-1} b^{\gamma} \frac{1}{bu} E(U) \\ &= \nu b_0^{\gamma-1} u^{\kappa-\gamma}. \end{aligned}$$

With the choice $\gamma < 1+\kappa < \delta$ we obtain the integrable upper bound

$$|f_\lambda(s,u,r)| 1_{\{0<u<b_0/b\}} \leq \nu b_0^{1-\kappa} \sum_{j=1}^{n} |\theta_j| \kappa \sum_{k=1}^{n} \theta_k 1_{\{0<s<t_k\}} r^{1+\kappa} u^{\kappa-\gamma} 1_{\{0<u\leq t_1\}}.$$

3.4.2. The compound Poisson workload model. For the compound Poisson model (11) one has

$$\begin{aligned} &\frac{1}{b} \int_{-\infty}^{\infty} \int_0^{\infty} \int_D \xi(K_{at}(s,u)) \widetilde{N}_\sharp(ds,du,d\xi) \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} \int_D \frac{1}{b} \xi(bK_{(a/b)t}((a/b)s,u)) \widetilde{N}_\sharp(a\,ds, b\,du, d\xi), \end{aligned}$$

where $\widetilde{N}_\sharp$ is defined in (14). Its compensator $n_\sharp$ in (10) is like the compensator n in (1) but with $F_R(dr)$ replaced by $\mu(d\xi)$. Hence as $a, b \to \infty$, we have as in (56)

$$n_\sharp(a\,ds, b\,du, d\xi) \sim ds\, u^{-1-\gamma} du\, \mu(d\xi)$$

and, again observing that $z = a/b \to \infty$,

$$\begin{aligned} &\int_D \frac{1}{b} \xi(bK_{(a/b)t}((a/b)s,u))\, \mu(d\xi) \\ &\quad = E(R)\, K_{zt}(zs,u) \\ &\quad \sim E(R)\, u\, 1_{\{0<s<t\}}, \end{aligned}$$

by (12) and (56). The limit process is therefore

$$\begin{aligned} E(R) \int_{-\infty}^{\infty} \int_0^{\infty} u 1_{\{0<s<t\}} \left(N(ds,du) - ds\, u^{-(1+\gamma)}\, du\right) &\stackrel{d}{=} E(R)\, \sigma_\gamma \int_0^t M_\gamma(ds) \\ &\stackrel{d}{=} E(R)\, \Lambda_\gamma(t), \end{aligned}$$

the formal verification of which rests again on studying the scaled characteristic function, this time using Lemma 4. The processes M_γ and Λ_γ are as in (57) and (58).

3.5. Proof of Theorem 4

3.5.1. Finite variance durations and rewards, $\gamma = \delta = 2$. Here, $E(U^2) < \infty$ and $E(R^2) < \infty$. To avoid scaling U and R, we use representation (25), i.e.,

$$\frac{1}{b}(W_\lambda^*(at) - \lambda at \nu E(R)) = \frac{1}{b} \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} K_{at}(as,u) r \widetilde{N}(a\,ds, du, dr)$$

where

$$\begin{aligned} EN(ads, du, dr) &= \lambda ads\, F_U(du)\, F_R(dr) \\ &= b^2\, ds\, F_U(du)\, F_R(dr). \end{aligned}$$

By (17),

$$K_{at}(as, u) = \int_0^u 1_{\{0<y+as<at\}}\, dy \to u 1_{\{0<s<t\}} \quad \text{as} \quad a \to \infty.$$

Hence by Lemma 3, as $b \to \infty$,

$$\begin{aligned} &\ln E \exp\Big\{ i \sum_{j=1}^n \theta_j (W_\lambda^*(at_j) - \lambda\nu E(R) at_j)/b \Big\} \\ &= \int_{-\infty}^\infty \int_0^\infty \int_0^\infty h(as, u, r/b)\, EN(ads, du, dr) \\ &\sim -\frac{1}{2} \int_{-\infty}^\infty \int_0^\infty \int_0^\infty \Big(b^{-1} \sum_{j=1}^n \theta_j K_{at_j}(as, u) r \Big)^2 b^2\, ds\, F_U(du)\, F_R(dr) \\ &\sim -\frac{1}{2} \int_{-\infty}^\infty \int_0^\infty F_U(du) \Big(\sum_{j=1}^n \theta_j 1_{\{0<s<t_j\}} u \Big)^2 ds \int_0^\infty r^2 F_R(dr) \\ &= -\frac{1}{2} \int_{-\infty}^\infty \Big(\sum_{j=1}^n \theta_j 1_{\{0<s<t_j\}} \Big)^2 ds\, E(U^2)\, E(R^2), \end{aligned}$$

so the limit is

$$E(U^2)^{1/2}\, E(R^2)^{1/2}\, B(t)$$

where $B(t)$ is Brownian motion.

When we consider instead $W_{\sharp,\lambda}$ and apply Lemma 4, then the resulting expression is slightly different:

$$\begin{aligned} &\ln E \exp\Big\{ i \sum_{j=1}^n \theta_j (W_{\sharp,\lambda}^*(at_j) - \lambda\nu E(R) at_j)/b \Big\} \\ &\quad \sim -\frac{1}{2} (ER)^2 \int_{-\infty}^\infty \int_0^\infty \Big(b^{-1} \sum_{j=1}^n \theta_j K_{at_j}(as, u) \Big)^2 b^2\, ds\, F_U(du) \\ &\quad \sim -\frac{1}{2} \int_{-\infty}^\infty \Big(\sum_{j=1}^n \theta_j 1_{\{0<s<t_j\}} \Big)^2 ds\, E(U^2)\, (ER)^2, \end{aligned}$$

which corresponds to the limit process $E(U^2)^{1/2}\, E(R)\, B(t)$.

3.5.2. Continuous model, rewards have heavier tails than those of durations, $1 < \delta < \gamma \le 2$**.** Take $1 < \delta < 2$, and assume either $\delta < \gamma \le 2$ or that we have an arbitrary distribution F_U with $E(U^\delta) < \infty$. Recall that

$$b = (\lambda a)^{1/\delta}.$$

Using (17) and (42),

$$\begin{aligned}
&\frac{1}{b}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} K_{at}(s,u)r\,\widetilde{N}(ds,du,dr)\\
&=\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}(r/b)\int_{0}^{u} 1_{\{0<y+s<at\}}\,dy\,\widetilde{N}(ds,du,dr)\\
&=\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{u} 1_{\{0<y/a+s<t\}}\,dy\,r\,\widetilde{N}(a\,ds,du,b\,dr)\\
&\sim\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} 1_{\{0<s<t\}}\,ur(N(ds,du,dr)-ds\,F_U(du)\,r^{-1-\delta}\,dr)\\
&\overset{d}{=}\sigma_\delta\int_{-\infty}^{\infty}\int_{0}^{\infty} 1_{\{0<s<t\}}\,uM_\delta(ds,du)\\
&\overset{d}{=}E(U^\delta)^{1/\delta}\,\Lambda_\delta(t),
\end{aligned}$$

where $M_\delta(ds,du)$ is δ-stable with control measure $m(ds,du) = ds\,F_U(du)$ and $\Lambda_\delta(t)$ is a Lévy stable process with index γ.

4. Weak convergence

This section is devoted to extending our previous results on convergence of the finite-dimensional distributions to weak convergence in function space.

Theorem 5. *For the continuous flow model, which has continuous trajectories, the convergence holds in the sense of weak convergence of stochastic processes in the space of continuous functions.*

4.1. Proof of tightness for the continuous flow model

To prove weak convergence in the continuous case, we are going to establish the following tightness criterion. For some $\alpha > 0$ (in our case $1 < \alpha \le 2$) and $\beta > 1$,

$$E\Big|\frac{1}{b}(W^*_\lambda(at_1)-\lambda\nu E(R)at_1)-\frac{1}{b}(W^*_\lambda(at_2)-\lambda\nu E(R)at_2)\Big|^\alpha \le \mathrm{const}\,|t_2-t_1|^\beta,$$

uniformly in λ, a, b. Clearly, because of stationarity of the increments, it suffices to show for any fixed $t > 0$ the uniform bound

$$E\Big|\frac{1}{b}(W^*_\lambda(at)-\lambda\nu E(R)at)\Big|^\alpha \le \mathrm{const}\,t^\beta. \tag{59}$$

Lemma 5. *For the continuous flow model* (2) *and for any $1 < \alpha \le 2$, we have the estimate*

$$E|W^*_\lambda(t)-\lambda\nu E(R)t|^\alpha \le 2E(R^\alpha)\int_{-\infty}^{\infty}\int_{0}^{\infty} K_t(s,u)^\alpha\,\lambda ds\,F_U(du).$$

Proof. Suppose first that $E(R^2) < \infty$. Then we can take $\alpha = 2$. It is readily checked that in this case we have the equality

$$E(W_\lambda^*(t) - \lambda\nu E(R)t))^2 = E(R^2) \int_{-\infty}^{\infty} \int_0^{\infty} K_t(s,u)^2 \, \lambda ds \, F_U(du).$$

For $1 < \alpha < 2$ we will use the estimate

$$E|X|^\alpha \leq A(\alpha) \int_0^\infty (1 - |\Phi_X(\theta)|^2)\theta^{-\alpha-1} \, d\theta, \tag{60}$$

where

$$A(\alpha) = \Big(\int_0^\infty (1 - \cos(x)) x^{-\alpha-1} \, dx \Big)^{-1} < \infty \tag{61}$$

and $\Phi_X(\theta) = E(e^{i\theta X})$ is the characteristic function of the random variable X. This technique goes back to von Bahr and Esseen (1965), and is used in Gaigalas (2004) in a similar context as here. With $X = W_\lambda^*(t) - \lambda\nu E(R)t$ we have

$$\Phi_X(\theta) = \exp\Big\{ \int_{-\infty}^\infty \int_0^\infty \int_0^\infty (e^{i\theta K_t(s,u)r} - 1 - i\theta K_t(s,u)r) \, n(ds, du, dr) \Big\}$$

and

$$\begin{aligned} 1 - |\Phi_X(\theta)|^2 &= 1 - \exp\Big\{ -2 \int_{-\infty}^\infty \int_0^\infty \int_0^\infty (1 - \cos(\theta K_t(s,u)r)) \, n(ds, du, dr) \Big\} \\ &\leq 2 \int_{-\infty}^\infty \int_0^\infty \int_0^\infty (1 - \cos(\theta K_t(s,u)r)) \, n(ds, du, dr). \end{aligned} \tag{62}$$

Since this last relation implies

$$\begin{aligned} &\int_0^\infty (1 - |\Phi_X(\theta)|^2)\theta^{-\alpha-1} \, d\theta \\ &\quad \leq 2 \int_{-\infty}^\infty \int_0^\infty \int_0^\infty E(1 - \cos(\theta K_t(s,u)R))\theta^{-\alpha-1} \, d\theta \, \lambda ds \, F_U(du) \\ &\quad = 2E(R^\alpha) \int_{-\infty}^\infty \int_0^\infty K_t(s,u)^\alpha \, \lambda ds \, F_U(du)/A(\alpha), \end{aligned}$$

we obtain the estimate stated in the lemma by using (60). □

We are now prepared to prove tightness under the scaling of intermediate connection rates. Because of the assumption $\gamma < \delta$ we can apply Lemma 5 with α such that $\gamma < \alpha < \delta$. If $\delta = 2$ we may even take $\alpha = 2$. In all cases $E(R^\alpha) < \infty$ and, using (28) and an integration by parts,

$$\begin{aligned} E\Big|\frac{W_\lambda^*(at) - \lambda\nu E(R)at}{a}\Big|^\alpha &\leq 2E(R^\alpha) \int_{-\infty}^\infty \int_0^\infty K_t(s,u)^\alpha \, \lambda a ds \, F_U(adu) \\ &= 2E(R^\alpha) \int_{-\infty}^\infty \int_0^\infty \alpha K_t(s,u)^{\alpha-1} 1_{\{0<s+u<t\}} \, \lambda a ds \, P(U > au) \, du. \end{aligned}$$

By using (16) and applying the Potter bound and the fact that $\lambda/a^{\gamma-1} \to c^{\gamma-1} \in (0,\infty)$, it follows that the last double integral is bounded by

$$\mathrm{const} \int_{-\infty}^{\infty} \int_{0}^{\infty} \alpha(t \wedge u)^{\alpha-1}\, 1_{\{0<s+u<t\}} \max(u^{-\gamma-\epsilon}, u^{-\gamma+\epsilon})\, dsdu < \infty,$$

where the integral is finite since we can take $\epsilon > 0$ such that $\alpha - \epsilon < \gamma < \alpha$. Hence the dominated convergence theorem applies, and we have

$$E\Big|\frac{W_\lambda^*(at) - \lambda\nu E(R)at}{a}\Big|^{\alpha} \le 3\, c^{\gamma-1}\, E(R^{\alpha}) \int_{-\infty}^{\infty}\int_{0}^{\infty} K_t(s,u)^{\alpha}\, ds\, u^{-\gamma-1} du,$$

say, for sufficiently large λ and a. Using once again (16) and (45),

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{0}^{\infty} K_t(s,u)^{\alpha}\, ds\, u^{-\gamma-1} du &\le \int_{0}^{\infty} (u \wedge t)^{\alpha-1} u^{-\gamma} t\, du \\ &= \frac{\alpha-1}{(\alpha-\gamma)(\gamma-1)}\, t^{1+\alpha-\gamma}. \qquad (63)\end{aligned}$$

Thus we have found α and $\beta = 1 + \alpha - \gamma > 1$, such that (59) holds uniformly in λ and a. This completes the proof of weak convergence for the intermediate Telecom process in Theorem 1 i).

The proof of tightness for the case of fast connection rate scaling and finite variance rewards, that is Theorem 2 i) where the fractional Brownian motion arises in the limit, is very similar to that of the preceding case. When we apply (28) and use the parameters $\gamma < \alpha = \delta = 2$ and $b^2 = \lambda a^{3-\gamma}$ under the scaling FCR, then Lemma 5 yields the estimate

$$\begin{aligned}E\Big[\Big(\frac{W_\lambda^*(at) - \lambda\nu E(R)at}{b}\Big)^2\Big] &\le 2E(R^2)\frac{1}{a^{3-\gamma}} \int_{-\infty}^{\infty}\int_{0}^{\infty} K_{at}(s,u)^2\, ds\, F_U(du) \\ &= 2E(R^2) \int_{-\infty}^{\infty}\int_{0}^{\infty} K_t(s,u)^2\, ds\, a^{\gamma} F_U(adu).\end{aligned}$$

The same arguments as above lead to the uniform bound $\mathrm{const}\, t^{3-\gamma}$, which verifies the tightness criterion (59) with $\alpha = 2$ and $\beta = 3 - \gamma > 1$.

The final case for the continuous flow model is tight convergence to the Telecom process in Theorem 2 ii). In this case we will need the following version of the previous Lemma 5. This is simply the inequality (60), expressed in terms of (62).

Lemma 6. *For the continuous flow model* (2) *and for any* $1 < \alpha < 2$, *we have the estimate*

$$\begin{aligned}&E|W_\lambda^*(t) - \lambda\nu E(R)t|^{\alpha} \le A(\alpha) \\ &\times \int_0^{\infty} \Big(1 - \exp\Big\{-2\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} (1-\cos(\theta K_t(s,u)r))\, n(ds,du,dr)\Big\}\Big)\theta^{-\alpha-1}\, d\theta,\end{aligned}$$

with $A(\alpha)$ *defined in* (61).

For any $\gamma < \alpha < \delta$, it follows from the lemma that

$$E\Big|\frac{W^*_\lambda(at) - \lambda\nu E(R)at}{b}\Big|^\alpha \le A(\alpha)\int_0^\infty (1 - |\Phi_{\lambda,a,b}(\theta)|^2)\,\theta^{-\alpha-1}\,d\theta,$$

where

$$\begin{aligned}|\Phi_{\lambda,a,b}(\theta)|^2 &= \exp\Big\{-2\int_{-\infty}^\infty\int_0^\infty\int_0^\infty (1-\cos(\theta K_t(s,u)r))\,n(ads,adu,(b/a)dr)\Big\}\\ &\sim \exp\Big\{-2\int_{-\infty}^\infty\int_0^\infty\int_0^\infty (1-\cos(\theta K_t(s,u)r))\,ds\,u^{-(1+\gamma)}du\,r^{-(1+\delta)}dr\Big\}\\ &= \exp\{-2\theta^\delta J_t(\gamma,\delta)\},\end{aligned}$$

where

$$J_t(\gamma,\delta) = A(\delta)^{-1}\int_{-\infty}^\infty\int_0^\infty K_t(s,u)^\delta\,ds\,u^{-(1+\gamma)}du.$$

By the method based on Potter bounds, used repeatedly above, it follows that we can find a constant $C_{\alpha,\gamma,\delta}$ (changing each time it occurs below), such that the inequality

$$E\Big|\frac{W^*_\lambda(at) - \lambda\nu E(R)at}{b}\Big|^\alpha \le C_{\alpha,\gamma,\delta}\int_0^\infty (1-\exp\{-2J_t(\gamma,\delta)\theta^\delta\})\,\theta^{-\alpha-1}\,d\theta$$

holds uniformly in λ, a and b. Since, for $\alpha < \delta$, the integral $\int_0^\infty (1-e^{-2\theta^\delta})\theta^{-1-\alpha}\,d\theta$ is finite, and since it was shown in (63) that $J_t(\gamma,\delta) \le \mathrm{const}\, t^{1+\delta-\gamma}$, this yields the final estimate

$$E\Big|\frac{W^*_\lambda(at) - \lambda\nu E(R)at}{b}\Big|^\alpha \le C_{\alpha,\gamma,\delta}\,J_t(\gamma,\delta)^{\alpha/\delta} \le C_{\alpha,\gamma,\delta}\,t^{(1+\delta-\gamma)\alpha/\delta}.$$

Now, $1 < \gamma < \alpha < \delta$ implies that $(1+\delta-\gamma)\alpha/\delta > 1$, and hence the growth criterion (59) is fulfilled. This ends the proof of weak convergence of the scaled infinite Poisson process towards the Telecom process in Theorem 2 ii).

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Ingemar Kaj
Department of Mathematics
Uppsala University
Box 480
SE 751 06 Uppsala, Sweden
e-mail: ikaj@math.uu.se

Murad S. Taqqu
Department of Mathematics
Boston University
Boston, MA 02215-2411, USA
e-mail: murad@math.bu.edu