Convergence of scaled renewal processes and a packet arrival model

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Abstract

We study the superposition process of a class of independent renewal processes with long-range dependence. It is known that under two different scalings in time and space either fractional Brownian motion or a stable Lévy process may arise in the rescaling asymptotic limit. It is shown here that in a third, intermediate scaling regime a new limit process appears, which is neither Gaussian nor stable. The new limit process is characterized by its cumulant generating function and some of its properties are discussed.

Running head: Convergence of scaled renewal processes
Keywords: long-range dependence, heavy tails, renewal processes, fractional Brownian motion, weak convergence

1 Introduction

This study is concerned with the asymptotic scaling behavior of sums of independent random processes with long-range dependence. Specifically, the long-range dependent process will be a renewal counting process with heavy-tailed inter-renewal times. It is known that the superposition process of independent copies of such processes, suitably scaled, may exhibit rescaling limits. In fact, different limit processes may arise depending on the details of the rescaling scheme.

To explain the framework, consider $\frac{1}{b(m,T)}\sum_{i=1}^{m}(X^i_{Tt} - cTt)$, the summation process of m i.i.d. copies of a centered random process $\{X_t\}$, where $b(m,T)$ is a normalization constant as m and T tends to infinity. Taqqu and Levy (1986), and recently Levy and Taqqu (2000), take X_t to be a renewal-reward process and study the limit under assumptions of heavy tails of both renewal times and rewards, hence implementing the idea of Mandelbrot (1969) to use renewal-reward processes with a heavy-tailed distribution of the inter-renewal times to study long-range dependence phenomena in physics and economics. The scaling limits investigated in Willinger et al. (1997) and Taqqu et al. (1997) refer to the case $X_t = \int_0^t Z_s\, ds$ where $\{Z_t\}$ is an on/off process with heavy-tailed distributions for on-periods, off-periods, or both. This application is related to the nature of network traffic, with $\{Z_t\}$ being a traffic rate process, and the results discussed in Willinger et al. (1997) relate a global self-similarity property, claimed to be intrinsic for Internet traffic, with the local property of heavy-tailed distributions of on or off periods. The main result for both of these models is that if m tends to infinity first, followed by T, then the rescaled process converges to *fractional Brownian motion*, whereas if the order of the limit operations is reversed then the limit is a *stable Lévy motion*.

This motivates the search for a simultaneous limit regime in which one of the parameters m or T is a function of the other, with the hope of finding a new limiting process which would provide the "missing link" between fractional Brownian motion and stable Lévy motion. This is the idea behind this paper, as well as of Mikosch et al. (2002) for on/off-processes (also infinite sources Poisson model) and Pipiras et al. (2002) for a class of renewal-rate processes. It is shown by Mikosch et al. (2002) that the limit process depends on the "connection rate", i.e. on the rate of growth of the number of users $m = M(T)$ compared to the time scale $T^{\alpha-1}$, where α is the regular variation exponent of the tail of the distribution of the on-periods. However, the limiting processes are still the same as in the double-limit case. Namely, if the connection rate is fast, meaning that $M(T)$ grows faster than $T^{\alpha-1}$, the limit is fractional Brownian motion. If the connection rate is slow the rescaled process converges to a stable Lévy motion. Analogous results are reported in Pipiras et al. (2002) for the case of renewal-rate processes.

We investigate the third limiting regime corresponding to an "intermediate" connection rate, using standard renewal processes with heavy-tailed inter-renewal time distribution, and derive a new non-Gaussian and non-stable limit process. Following the introduction of the model, Section 2 contains the main convergence result. Section 3 studies various properties of the limit process. In Section 4 we have collected results for the underlying heavy-tailed renewal process required for the proof, some of which may also be of independent interest. Finally, in Section 5 we give a proof of the theorem based on cumulant generating functions.

1.1 A renewal-based model for the arrival process

Consider the renewal process $\{S_n\}$ generated by a sequence of independent non-negative random variables $\{U_k\}_{k\geq 1}$, i.e. $S_n = \sum_{k=1}^n U_k$. The inter-renewal times $\{U_k\}_{k\geq 2}$ are supposed to be identically distributed with distribution function $F(t)$, while the distribution of the first inter-renewal time U_1 can be different. Our basic assumption is that the inter-renewal distribution $F(t)$ has a regularly varying tail with exponent $1+\beta$, $0<\beta<1$, i.e. there exist a slowly varying function $L(t)$ such that

$$1 - F(t) \sim t^{-(1+\beta)} L(t), \quad \text{as } t \to \infty. \tag{1}$$

Thus the variables U_k, $k \geq 2$ possess finite expectation μ but the variance is infinite. As for the distribution of the first renewal U_1, in the context of our applications it is natural to choose it to be equal to the equilibrium distribution

$$F_1(t) = \frac{1}{\mu} \int_0^t (1 - F(s))\, ds,$$

so that the resulting renewal point process becomes stationary. In particular, this implies that the corresponding renewal counting process

$$N_t = \max\{n : S_n \leq t\}$$

has stationary increments. The usual *pure* renewal processes with the same distribution $F(t)$ for all renewal intervals also emerge as we proceed. Henceforth, by N_t we will always denote the stationary renewal counting process, while $\widetilde{N}_t$ will correspond to a pure renewal process.

Let $\{N_t^{(i)}\}, i = 1, \dots, m$ be m independent copies of the stationary renewal counting process $\{N_t\}$. We are interested in the asymptotic properties of the superposition process

$$W(m, t) = \sum_{i=1}^m N_t^{(i)},$$

which counts the total number of renewal events occurring in corresponding renewal sequences $\{S_n^{(i)}\}$, $i = 1 \dots m$, up to time $t \geq 0$.

The summation processes discussed in the introduction have been suggested as models for the total workload at a network node arising from m independent sources sharing a common medium, such as a local area network. In the light of such applications the process studied here can be thought of as a rudimentary packet arrival model with heavy tails, which in a sense acts as a skeleton for more detailed models designed to capture the workload behavior in real systems. In this simple model source i generates one packet at each time epoch $\{S_n^{(i)}\}_{n \geq 1}$. The amount of work from source i up to time $t \geq 0$ is $N_t^{(i)}$, and $W(m, t)$ represents the cumulative arrival process of the system counting the accumulated work generated by all users. We will investigate the behavior of the (rescaled) process $W(m, t)$ as the number of sources m grows to infinity and time is suitably rescaled, under the assumption that the distribution of packet inter-arrival times is heavy-tailed.

2 Convergence result

Our object of interest is the centered and rescaled process

$$Y^{(m)}(t) = \frac{1}{b_m} \big(W(m, a_m t) - \frac{m a_m t}{\mu} \big), \tag{2}$$

where the sequence a_m governs the rescaling of time and b_m is the corresponding normalization of "space". The centering sequence $ma_m t/\mu$ corresponds to the expected value $EW(m,t) = mt/\mu$.

We assume that the scaling sequence a_m is any sequence such that $a_m \to \infty$. The rate of growth of a_m relative to m determines the asymptotic behavior of $Y^{(m)}$ as $m \to \infty$ and it is known that the choice of such growth rate conditions results in a fundamental dichotomy in the asymptotic limit under the rescaling scheme (2). Essentially, if a_m grows slowly compared to m then the limit is fractional Brownian motion and if a_m grows fast relative to m the limit is a stable Lévy process. The contribution in this work is the derivation of a new limit process under a third, intermediate, limit regime. Thus we will discuss the following three options:

- Fast connection rate (number of users grows faster than rescaling of time):
$$\frac{mL(a_m)}{a_m^\beta} \to \infty, \quad \text{(FCR)}$$
- Slow connection rate (time is rescaled faster than the number of users grows):
$$\frac{mL(a_m)}{a_m^\beta} \to 0, \quad \text{(SCR)}$$
- Intermediate connection rate (time is rescaled proportional to the growth of the number of users):
$$\frac{mL(a_m)}{a_m^\beta} \to \mu. \quad \text{(ICR)}$$

The conditions (FCR) and (SCR) are equivalent to the fast- and slow-growth conditions used in Mikosch et al. (2002), as can be verified similarly as in their Lemma 1.

Some further remarks are in order concerning the choice of the rescaling scheme (2). In this paper the time scale a_m is taken to be a function of the number of users m, and the rescaled process $W(m, a_m t)$ is studied as m tends to infinity. In contrast, Mikosch et al. (2002) consider the number of users $M(T)$ in the on/off-model and the connection intensity $\lambda(T)$ in the M/G/∞ model to be functions of a time parameter T, and let T tend to infinity. The latter limiting scheme is also used by Taqqu and Levy (1986) and Pipiras et al. (2002) in the renewal-reward process setting. This is in effect an inverse scaling as compared to (2), which in our case would correspond to the process $W(m_T, Tt)$, where $T \to \infty$. In the limiting scheme (2) the sequence a_m can be regarded as an inverse function of $M(T)$.

The term "connection rate" is used above in a descriptive sense, only indicating the relationship between the number of sources and the time interval over which the sources are active, and should not be thought of as a rate in the sense of number of connected users per unit of time.

We begin by stating the limit results for $Y^{(m)}$ under fast and slow connection rate scaling. The results are in complete analogy with those found by Mikosch et al. (2002) and by Pipiras et al. (2002) for the more complex but related models studied in these papers, in the case when the heavy tails are parametrized by a single parameter. In our case of simple renewal processes the proof of convergence under fast connection rate, part A), can be carried out by modifying selected parts of the proof of Theorem 1 below, see section 5.5. The convergence result under slow connection rate, part B), will be discussed in a more general framework elsewhere.

Theorem. *Under assumption (1) and one of the assumptions (FCR) or (SCR) the following limiting relations hold:*

A) Under condition (FCR), as $m \to \infty$,

$$\frac{W(m, a_m t) - \frac{m a_m t}{\mu}}{m^{\frac{1}{2}} a_m^{1-\frac{\beta}{2}} L(a_m)^{\frac{1}{2}}} \Rightarrow \mu^{-3/2} \sigma_\beta \, B_H(t), \quad \sigma_\beta^2 = \frac{2}{\beta(1-\beta)(2-\beta)} \tag{3}$$

where $\Rightarrow$ *denotes weak convergence of processes in the space of cadlag functions and* $B_H(t)$ *is standard fractional Brownian motion of index* $H = 1 - \beta/2$.

B) Under condition (SCR), let $L^*(u)$ *be a slowly varying function at infinity such that* $L(u^{\frac{1}{1+\beta}} L^*(u))/L^*(u)^{1+\beta} \to 1$ *as* $u \to \infty$. *Then as* $m \to \infty$,

$$\frac{W(m, a_m t) - \frac{m a_m t}{\mu}}{(m a_m)^{\frac{1}{1+\beta}} L^*(m a_m)} \xrightarrow{fdd} -\mu^{-(2+\beta)/(1+\beta)} \Lambda_\alpha(t), \tag{4}$$

where $\xrightarrow{fdd}$ *denotes convergence of the finite-dimensional distributions and* $\Lambda_\alpha(t)$ *is* α*-stable Lévy motion of index* $\alpha = 1 + \beta$, *such that* $\Lambda_\alpha(t) \sim S_\alpha(c_\alpha^{-1/\alpha} t^{1/\alpha}, 1, 0)$, $c_\alpha = (1-\alpha)/(\Gamma(2-\alpha)\cos(\pi\alpha/2))$.

Our main result establishes a new limit process in the asymptotic regime between A) and B) of the above, corresponding to the condition (ICR), where the appropriate norming sequence in (2) is seen to be simply $b_m = a_m$.

Theorem 1. *Under assumption (1) and the intermediate growth condition (ICR), as* $m \to \infty$ *the weak convergence of processes*

$$Y^{(m)}(t) = \frac{1}{a_m} \sum_{i=1}^{m} (N_{a_m t}^{(i)} - \frac{a_m t}{\mu}) = \frac{W(m, a_m t) - \frac{m a_m t}{\mu}}{a_m} \Rightarrow -\mu^{-1} Y_\beta(t) \tag{5}$$

holds, where $Y_\beta(t)$ *is a zero mean, non-Gaussian and non-stable process with stationary increments. The limit process is not self-similar, it is continuous and has finite moments of all orders. The finite-dimensional distributions* *of the increments of $Y_\beta(t)$ are characterized by the following cumulant generating function:*

$$\log E \exp\Big\{\sum_{i=1}^{n}\theta_i\big(Y_\beta(t_i)-Y_\beta(t_{i-1})\big)\Big\}$$
$$=\frac{1}{\beta}\sum_{i=1}^{n}\theta_i^2\int_0^{t_i-t_{i-1}}\int_0^{v}e^{\theta_i u}u^{-\beta}\,du dv$$
$$+\frac{1}{\beta}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\theta_i\theta_j e^{\sum_{k=i+1}^{j-1}\theta_k(t_k-t_{k-1})}$$
$$\times\int_0^{t_i-t_{i-1}}dv\int_0^{t_j-t_{j-1}}e^{\theta_j u}e^{\theta_i v}(t_{j-1}-t_i+u+v)^{-\beta}\,du, \qquad (6)$$

where $0=t_0<t_1<\cdots<t_n$.

Remark. We have chosen the constant in front of the process $Y_\beta(t)$ in (5) to be negative since it turns out that the limit process of $Y^{(m)}(t)$ has negatively skewed marginal distributions. Hence $Y_\beta(t)$ is "positively skewed".

3 Properties of the limit process

3.1 Elementary properties

Marginal distributions. The cumulant generating function of the marginal distributions of $Y_\beta(t)$ is

$$\log Ee^{\theta Y_\beta(t)}=\frac{\theta^2}{\beta}\int_0^t\int_0^v e^{\theta u}u^{-\beta}\,du dv. \qquad (7)$$

This also can be written as

$$\log Ee^{\theta Y_\beta(t)}=\frac{\theta^2 t^{2-\beta}}{\beta(1-\beta)(2-\beta)}M(1-\beta,3-\beta;\theta t), \qquad (8)$$

where

$$M(a,b;z)=\frac{\Gamma(b)}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)z^k}{\Gamma(b+k)k!}$$

is the *Kummer's special function* from the family of confluent hypergeometric functions.

Proof. The expression (7) follows from (6) with $n=1$. To prove (8), observe that for parameter values with $\operatorname{Re} b>\operatorname{Re} a>0$ Kummer's function $M(a,b;z)$ possesses the following integral representation (Abramowitz and Stegun (1992, 13.2)):

$$M(a,b;z)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{zu}u^{a-1}(1-u)^{b-a-1}\,du.$$

On the other hand, the change of the order of integration in the double integral (7) together with the change of variables $u' = u/t$ gives

$$\int_0^t \int_0^v e^{\theta u} u^{-\beta}\, du dv = t^{2-\beta} \int_0^1 e^{\theta t u} u^{-\beta}(1-u)\, du,$$

and (8) follows. $\square$

Moments. The process $Y_\beta(t)$ has finite moments of all orders. In particular,

$$\begin{aligned} EY_\beta(t) &= 0, \\ EY_\beta^2(t) &= \frac{2}{\beta(1-\beta)(2-\beta)}\, t^{2-\beta}, \\ \text{Skewness} &= \frac{E[Y_\beta^3(t)]}{E[Y_\beta^2(t)]^{3/2}} = \frac{3\sqrt{\beta(2-\beta)}(1-\beta)^{3/2}}{\sqrt{2}(3-\beta)}\, t^{\beta/2}, \\ \text{Kurtosis} &= \frac{E[Y_\beta^4(t)]}{E[Y_\beta^2(t)]^2} = 3\Big(1 + \frac{\beta(1-\beta)^2(2-\beta)^2}{(3-\beta)(4-\beta)}\, t^\beta\Big). \end{aligned} \tag{9}$$

In general, for $k \geq 1$,

$$EY_\beta^k(t) \sim \frac{(k-1)k}{\beta(k-1-\beta)(k-\beta)}\, t^{k-\beta}, \quad \text{as } t \to \infty. \tag{10}$$

Proof. Since the moment generating function of $Y_\beta(t)$ exists for each real θ, all moments are finite. Relation (8) yields

$$\log E e^{\theta Y_\beta(t)} = \frac{1}{\beta} \sum_{k=2}^\infty \frac{(k-1)k\, t^{k-\beta} \theta^k}{(k-1-\beta)(k-\beta)k!}.$$

The cumulants of the marginal distribution are now obtained by differentiation with respect to θ:

$$C_k(t) = \frac{d^k}{d\theta^k} \log E e^{\theta Y_\beta(t)}\Big|_{\theta=0} = \frac{(k-1)k}{\beta(k-1-\beta)(k-\beta)}\, t^{k-\beta}. \tag{11}$$

In our case, since $EY_\beta(t) = C_1(t) = 0$, the first three moments of $Y_\beta(t)$ are equal to the cumulants. Computations for any higher moment can also be carried out but do not lead to simple expressions. However, the moments have the same asymptotic behavior when $t \to \infty$ as the cumulants, and thus (10) follows from (11). $\square$

Covariance. The covariance function of $Y_\beta(t)$ equals

$$EY_\beta(t)Y_\beta(s) = \sigma_\beta^2 \left(t^{2-\beta} + s^{2-\beta} - |t-s|^{2-\beta}\right), \tag{12}$$

where the constant σ_β^2 is defined in (3).

Proof. Since the process $Y_\beta(t)$ has stationary increments, the covariance follows from (9):

$$EY_\beta(t)Y_\beta(s) = \frac{1}{2}\big(\operatorname{Var}[Y_\beta(t)] + \operatorname{Var}[Y_\beta(s)] - \operatorname{Var}[Y_\beta(t-s)]\big). \quad \square$$

Regularity. The trajectories of the process $Y_\beta(t)$ are Hölder continuous of order γ for any $0 < \gamma < 1$.

Proof. This property follows from the general asymptotic form (10) of the moments by applying the Kolmogorov-Čentsov criteria. $\square$

3.2 Relation to fractional Brownian motion

We have seen above that the first two moments of $Y_\beta(t)$ coincide with the corresponding moments of the (multiple of) fractional Brownian motion $\sigma_\beta B_H(t)$ of index $H = 1 - \beta/2$, while higher order cumulants and moments are different. For comparison with (10),

$$EB_H^k(t) \sim \text{const } t^{k(1-\beta/2)}, \quad \text{as } t \to \infty.$$

The Kolmogorov-Čentsov criteria applied to this case yields that fractional Brownian motion of index H is Hölder continuous of order γ only for $0 < \gamma < H$. Consequently, the process Y_β is more regular than FBM.

Note also that the process $Y_\beta(t)$ has the same covariance function as the (multiple of) fractional Brownian motion $\sigma_\beta\, B_H(t)$. Since this function is self-similar in the sense of self-similarity of deterministic functions, the process $Y_\beta(t)$ is *second-order self-similar*. However, it is not self-similar in general. The relationship of the limit process Y_β to FBM as well as its scaling properties are further clarified in the next result.

Corollary 1. *The process $Y_\beta(t)$ obeys the scaling limit relation*

$$c^H\, Y_\beta(t/c) \;\Rightarrow\; \sigma_\beta\, B_H(t), \quad H = 1 - \beta/2, \qquad c \to \infty, \tag{13}$$

in the sense of weak convergence of continuous processes.

Proof. This result is easily derived from (6), by observing that the cumulant generating function of the finite-dimensional distributions of increments of fractional Brownian motion with index $H = 1 - \beta/2$ can be written

$$\begin{aligned}
&\log E \exp\Big\{ \sum_{i=1}^n \theta_i \big(B_H(t_i) - B_H(t_{i-1})\big)\Big\} \\
&= \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \theta_i\theta_j \operatorname{Cov}\Big(B_H(t_i) - B_H(t_{i-1}), B_H(t_j) - B_H(t_{j-1})\Big) \\
&= \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \theta_i\theta_j \Big(|t_{i-1} - t_j|^{2-\beta} - |t_i - t_j|^{2-\beta} \\
&\qquad\qquad + |t_i - t_{j-1}|^{2-\beta} - |t_{i-1} - t_{j-1}|^{2-\beta}\Big) \\
&= \sigma_1 \sum_{i=1}^n \theta_i^2 \int_0^{t_i - t_{i-1}} \int_0^v u^{-\beta}\, du dv
\end{aligned}$$

$$+\sigma_1 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \theta_i\theta_j \int_0^{t_i-t_{i-1}} \int_0^{t_j-t_{j-1}} (t_{j-1}-t_i+u+v)^{-\beta}\, dudv,$$

where $\sigma_1 = (1-\beta)(2-\beta)$. By comparing this representation with (6), it is seen that under the scaling (13) all the exponential factors appearing in (6) are wiped out leaving only the Gaussian distribution of the FBM. $\Box$

3.3 Relation to stable Lévy motion

In order to indicate the relationship of $Y_\beta(t)$ to a stable Lévy motion, consider the stable Lévy motion $\Lambda_\alpha(t)$ with marginal distributions totally skewed to the right, which appears in (4). The cumulant generating function of $(\Lambda_\alpha(t_1), \ldots, \Lambda_\alpha(t_n))$ exists for $(\theta_1, \ldots, \theta_n)$ such that $\theta_i \le 0$, $i = 1, \ldots, n$ (Samorodnitsky and Taqqu (1994, Proposition 1.2.12)), even if it does not characterize the distribution. Since the increments of the process $\Lambda_\alpha(t)$ are independent and $\Lambda_\alpha(t_i) - \Lambda_\alpha(t_{i-1}) \sim S_\alpha(c_\alpha^{-1/\alpha}(t_i-t_{i-1})^{1/\alpha}, 1, 0)$, where c_α is introduced in (4), the same proposition yields

$$\begin{aligned}
&\log E \exp\Big\{ \sum_{i=1}^n \theta_i\big(\Lambda_\alpha(t_i) - \Lambda_\alpha(t_{i-1})\big)\Big\} \\
&\quad = \sum_{i=1}^n \log E \exp\Big\{\theta_i\big(\Lambda_\alpha(t_i) - \Lambda_\alpha(t_{i-1})\big)\Big\} \\
&\quad = -(c_\alpha \cos\frac{\pi\alpha}{2})^{-1} \sum_{i=1}^n (-\theta_i)^\alpha (t_i - t_{i-1}) \\
&\quad = -\frac{1}{\beta} \sum_{i=1}^n \theta_i^2 \int_0^{t_i-t_{i-1}} \int_0^\infty e^{\theta_i u} u^{-\beta}\, dudv.
\end{aligned}$$

This expression may now be compared with (6).

3.4 Interpretation as packet arrival model

Recall that an application we have in mind of the summation scheme in (2) is that $W(m,t)$ counts the accumulated number of packets generated by m independent users sharing a common medium, when the arrival stream from each source is characterized by a heavy-tailed interarrival distribution. It follows from Theorem 1, applying (ICR), that for large m

$$W(m,t) \approx \frac{mt}{\mu} - \frac{a_m}{\mu} Y_\beta(t/a_m) \sim \frac{mt}{\mu} - \frac{1}{\mu^{3/2}} \sqrt{mL(a_m)}\, a_m^{1-\beta/2} Y_\beta(t/a_m).$$

Invoking also Corollary 1 gives the coarser approximative representation

$$W(m,t) \approx \frac{1}{\mu} mt - \frac{\sigma_\beta}{\mu^{3/2}} \sqrt{mL(a_m)}\, B_{1-\beta/2}(t).$$

This provides a verification of the model for Ethernet type traffic proposed by Norros (1995). A more comprehensive discussion of arrival process modeling with long-range dependence and further references can be found in Kaj (2002).

4 Some properties of renewal processes

To prepare for the proof of Theorem 1 we need structure results for the functionals $E[e^{\sum_{i=1}^n \theta_i N_{t_i}}]$ as well as the precise asymptotics of high order moments $E(N_t - t/\mu)^k$, and the analogous results for the pure renewal process $\widetilde{N}_t$. The technical key to our proof of Theorem 1 is Proposition 1 below, which is somewhat related to an integral equation in Kaj and Sagitov (1998, Lemma 3), derived in a different context.

4.1 Moment generating function for the n-dimensional distributions

In this section we give two results for the multivariate moment generating functions of general renewal processes, not necessarily subject to a tail condition such as (1). It is assumed only that $\{N_t\}$ is a stationary renewal process with inter-renewal times $\{U_n\}_{n\geq 2}$ with distribution function $F(t)$ and finite mean value $\mu = E(U_n)$, and that the first renewal time have the equilibrium distribution $F_1(t)$. The notation $\{\widetilde{N}_t\}$ is used for the corresponding pure renewal process with all interrenewal times having the same distribution function $F(t)$.

Proposition 1. *Fix $n \geq 2$ and a sequence of time points $0 \leq t_1 \leq \cdots \leq t_n$. The moment generating function of the finite-dimensional distributions of the stationary renewal counting process $\{N_t\}$ satisfies the recurrence relation*

$$\begin{aligned} E[e^{\sum_{i=1}^n \theta_i N_{t_i}}] &= E[e^{\sum_{i=2}^n \theta_i N_{t_i}}] \\ &\quad + \frac{1-e^{-\theta_1}}{1-e^{-\sum_{i=1}^n \theta_i}} \int_0^{t_1} E[e^{\sum_{i=2}^n \theta_i \widetilde{N}_{t_i - u}}]\, dE[e^{N_u \sum_{i=1}^n \theta_i}], \end{aligned} \tag{14}$$

where $\widetilde{N}_t$ is the corresponding pure renewal counting process.

Proof. We have

$$E[e^{\sum_{i=1}^n \theta_i N_{t_i}}] - E[e^{\sum_{i=2}^n \theta_i N_{t_i}}] = E[e^{\sum_{i=2}^n \theta_i N_{t_i}}(e^{\theta_1 N_{t_1}} - 1)].$$

By summing over all jumps in $(0, t_1]$ the term on the right side can be written

$$\begin{aligned} &E[e^{\sum_{i=2}^n \theta_i N_{t_i}}(e^{\theta_1 N_{t_1}} - 1)] \\ &\quad = E\Big[\sum_{j=1}^\infty 1_{\{S_j \leq t_1\}} e^{\sum_{i=2}^n \theta_i N_{t_i}}(e^{\theta_1 N_{S_j}} - e^{\theta_1 N_{S_j -}})\Big] \end{aligned}$$

$$= E\Big[\sum_{j=1}^{\infty} 1_{\{S_j \le t_1\}} e^{\sum_{i=2}^n \theta_i (N_{t_i} - N_{S_j})} e^{\sum_{i=1}^n \theta_i N_{S_j-}} e^{\sum_{i=2}^n \theta_i}(e^{\theta_1} - 1)\Big].$$

For any j and $i \ge 2$, on the set $\{S_j \le t_1\}$ the increment $N_{t_i} - N_{S_j}$ has the same distribution as $N_{t_i - S_j}$, by stationarity. Now conditional on $\{S_j = u\}$, N_{t-u}, $t \ge u$, is the pure renewal process associated with the sequence $\{S_n\}$, $n \ge 2$. It follows that

$$\begin{aligned} &E[e^{\sum_{i=1}^n \theta_i N_{t_i}}] - E[e^{\sum_{i=2}^n \theta_i N_{t_i}}] \\ &\quad = e^{\sum_{i=2}^n \theta_i}(e^{\theta_1} - 1) E\Big[\sum_{j=1}^{\infty} 1_{\{S_j \le t_1\}} E[e^{\sum_{i=2}^n \theta_i \widetilde{N}_{t_i - S_j}} | \mathcal{F}_{S_j}] \, e^{\sum_{i=1}^n \theta_i N_{S_j-}}\Big]. \end{aligned}$$

Moreover,

$$\begin{aligned} e^{\sum_{i=1}^n \theta_i N_{S_j-}} &= (e^{\sum_{i=1}^n \theta_i N_{S_j}} - e^{\sum_{i=1}^n \theta_i N_{S_j-}})/(e^{\sum_{i=1}^n \theta_i} - 1) \\ &=: \Delta_{S_j}(e^{N_u \sum_{i=1}^n \theta_i})/(e^{\sum_{i=1}^n \theta_i} - 1) \end{aligned}$$

and so

$$\begin{aligned} &E[e^{\sum_{i=1}^n \theta_i N_{t_i}}] - E[e^{\sum_{i=2}^n \theta_i N_{t_i}}] \\ &\quad = \frac{e^{\sum_{i=2}^n \theta_i}(e^{\theta_1} - 1)}{e^{\sum_{i=1}^n \theta_i} - 1} E\Big[\sum_{j=1}^{\infty} 1_{\{S_j \le t_1\}} E[e^{\sum_{i=2}^n \theta_i \widetilde{N}_{t_i - S_j}} | \mathcal{F}_{S_j}] \, \Delta_{S_j}(e^{N_u \sum_{i=1}^n \theta_i})\Big] \\ &\quad = \frac{1 - e^{-\theta_1}}{1 - e^{-\sum_{i=1}^n \theta_i}} E\Big[\int_0^{t_1} E[e^{\sum_{i=2}^n \theta_i \widetilde{N}_{t_i - u}}] \, d(e^{N_u \sum_{i=1}^n \theta_i})\Big] \\ &\quad = \frac{1 - e^{-\theta_1}}{1 - e^{-\sum_{i=1}^n \theta_i}} \int_0^{t_1} E[e^{\sum_{i=2}^n \theta_i \widetilde{N}_{t_i - u}}] \, dE(e^{N_u \sum_{i=1}^n \theta_i}). \end{aligned}$$

□

Lemma 1. *The moment generating function of the finite-dimensional distributions of the process $\{N_t\}$ is differentiable in the time variable and relates to the corresponding function for the pure renewal process $\{\widetilde{N}_t\}$ as follows:*

$$E[e^{\sum_{i=1}^n \theta_i \widetilde{N}_{t_i}}] = \frac{\mu}{e^{\sum_{i=1}^n \theta_i} - 1} \sum_{j=1}^n \frac{\partial}{\partial t_j} E[e^{\sum_{i=1}^n \theta_i N_{t_i}}]. \tag{15}$$

Proof. We will use a counterpart to the one-dimensional renewal theory in higher dimensions developed by Hunter (1974). The author considered the two-dimensional case, but the ideas are based on two-dimensional Laplace transforms and convolutions, and carry over to any higher dimension. We will prove our claim for $n = 2$ as well, the proof for any other n follows the same pattern. Consider the two-dimensional convolution

$$A **B(s,t) = \int_0^s \int_0^t A(s-u, t-v) \, dB(u,v).$$

It is commutative with respect to $A(s,t)$ and $B(s,t)$ and, if $A(s,t)$ is differentiable almost everywhere, then

$$\frac{\partial}{\partial s}\big(A \ast\ast B\big)(s,t) = (\frac{\partial}{\partial s}A) \ast\ast B(s,t) + A(0,\cdot) \ast\ast B(s,t). \tag{16}$$

Proceeding with the proof, we have

$$Ee^{\theta_1 N_s + \theta_2 N_t} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} e^{\theta_1 n + \theta_2 m} P(N_s = n, N_t = m) \tag{17}$$

and the same holds of course for the process $\{\widetilde{N}_t\}$. We will prove that the probability $P(N_s = n, N_t = m)$ is differentiable in both s and t, and

$$\begin{aligned}&\frac{\partial}{\partial s}P(N_s = n, N_t = m) + \frac{\partial}{\partial t}P(N_s = n, N_t = m)\\ &\quad = \begin{cases} \mu^{-1}\big(P(\widetilde{N}_s = n-1, \widetilde{N}_t = m-1) \\ \qquad\qquad -P(\widetilde{N}_s = n, \widetilde{N}_t = m)\big) & \text{if } n \geq 1,\ m \geq 1, \\ -\mu^{-1}P(\widetilde{N}_s = n, \widetilde{N}_t = m), & \text{if } n = 0 \text{ or } m = 0. \end{cases}\end{aligned} \tag{18}$$

In view of these relations, the claim of the lemma will follow by differentiating both sides of (17).

Introduce the bivariate renewal distribution

$$G(s,t) = P(U_n \leq s, U_n \leq t) = P(U_n \leq s \wedge t) = F(s \wedge t).$$

By Corollary 3.1.1 of Hunter (1974), for $n, m \geq 0$,

$$P(\widetilde{N}_s = n, \widetilde{N}_t = m) = H_{nm} \ast\ast G^{\ast\ast n \wedge m}(s,t), \tag{19}$$

where

$$H_{nm}(s,t) = \begin{cases} 1 - \bar{G}(s,t) - \bar{\bar{G}}(s,t) + G(s,t), & \text{if } n = m, \\ \bar{G}_{n-m}(s,t) - \bar{G}_{n-m+1}(s,t) \\ \quad -\bar{G}_{n-m-1} \ast\ast G(s,t) + \bar{G}_{n-m} \ast\ast G(s,t), & \text{if } n > m, \\ \bar{\bar{G}}_{m-n}(s,t) - \bar{\bar{G}}_{m-n+1}(s,t) \\ \quad -\bar{\bar{G}}_{m-n-1} \ast\ast G(s,t) + \bar{\bar{G}}_{m-n} \ast\ast G(s,t), & \text{if } n < m, \end{cases}$$

with the notation $A_k(s,t) = A^{\ast\ast k}(s,t)$, $\bar{A}(s,t) = A(s,\infty)$, and $\bar{\bar{A}}(s,t) = A(\infty,t)$ for any function $A(s,t)$. Observe that $\bar{G}_k(s,t) = G^{\ast\ast k}(s,\infty)$.

In the stationary case the same argument with slight modifications gives for $n \geq 1$, $m \geq 1$,

$$P(N_s = n, N_t = m) = H_{nm} \ast\ast K \ast\ast G^{\ast\ast n \wedge m-1}(s,t), \tag{20}$$

where $K(s,t) = F_1(s \wedge t)$. Cases where $n = 0$ or $m = 0$ are special:

$$P(N_s = n, N_t = m) =$$

$$= \begin{cases} 1 - \bar{K}(s,t) - \bar{\bar{K}}(s,t) + K(s,t) & \text{if } n = m = 0, \\ \bar{L}_{n-1}(s,t) - \bar{L}_n(s,t) & \\ \quad -\bar{G}_{n-1} **K(s,t) + \bar{G}_n **K(s,t) & \text{if } n \geq 1,\ m = 0, \\ \bar{\bar{L}}_{m-1}(s,t) - \bar{\bar{L}}_m(s,t) & \\ \quad -\bar{\bar{G}}_{m-1} **K(s,t) + \bar{\bar{G}}_m **K(s,t) & \text{if } n = 0,\ m \geq 1, \end{cases} \tag{21}$$

where $L_k(s,t) = K **G_k(s,t)$.

To prove (18) for $n, m \geq 1$, consider the probability in (20). Since $F_1(t)$ is differentiable, so is the function $K(s,t)$ for $s \neq t$ and $\frac{\partial}{\partial s}K(s,t) = F_1'(s)1_{\{s<t\}}$. In particular, for $s \neq t$,

$$\frac{\partial}{\partial s}K(s,t) + \frac{\partial}{\partial t}K(s,t) = F_1'(s \wedge t) = (1 - G(s,t))/\mu.$$

Consequently, the convolution (20) is also differentiable and an application of (16) with $A = K$ and $B = H_{nm} **G^{**n\wedge m-1}$ yields

$$\begin{aligned} &\frac{\partial}{\partial s}P(N_s = n, N_t = m) + \frac{\partial}{\partial t}P(N_s = n, N_t = m) \\ &\quad = H_{nm} **(\frac{\partial}{\partial s}K) **G^{**n\wedge m-1}(s,t) \\ &\qquad\qquad + H_{nm} **(\frac{\partial}{\partial t}K) **G^{**n\wedge m-1}(s,t) \\ &\quad = \mu^{-1} H_{nm} **(1 - G) **G^{**n\wedge m-1}(s,t). \end{aligned} \tag{22}$$

The function $H_{nm}(s,t)$ depends only on the difference $|n - m|$, so that $H_{nm}(s,t) = H_{n-1\,m-1}(s,t)$. Hence combining (22) and (19) we get the first part of (18).

In the case when $n = m = 0$, observe that $\bar{K}(s,t) = F_1(s)$, which implies

$$\frac{\partial}{\partial s}\bar{K}(s,t) + \frac{\partial}{\partial t}\bar{K}(s,t) = F_1'(s) = (1 - \bar{G}(s,t))/\mu. \tag{23}$$

Thus, differentiating (21) in this case yields

$$\begin{aligned} &\frac{\partial}{\partial s}P(N_s = 0, N_t = 0) + \frac{\partial}{\partial t}P(N_s = 0, N_t = 0) \\ &\quad = -\mu^{-1}\big(1 - \bar{G}(s,t) - \bar{\bar{G}}(s,t) + G(s,t)\big) \\ &\quad = -\mu^{-1} H_{00}(s,t), \end{aligned}$$

which gives (18) for $n = m = 0$ in view of (19).

It remains to prove (18) in the case when only one of n and m is equal to zero. As earlier, since the function $K(s,t)$ is differentiable almost everythere, so is the convolution $\big(\bar{G}_k **K\big)(s,t)$ and due to (16),

$$\begin{aligned} &\frac{\partial}{\partial s}\big(\bar{G}_k **K\big)(s,t) + \frac{\partial}{\partial t}\big(\bar{G}_k **K\big)(s,t) \\ &\quad = \bar{G}_k **(\frac{\partial}{\partial s}K)(s,t) + \bar{G}_k **(\frac{\partial}{\partial t}K)(s,t) \end{aligned}$$

$$= \mu^{-1}\bar{G}_k **(1-G)(s,t). \tag{24}$$

Further, since $L_k(s,t) = K **G_k(s,t)$ is differentiable, we can interchange the limits:

$$\frac{\partial}{\partial s}\bar{L}_k(s,t) = \frac{\partial}{\partial s}\big(\lim_{t\to\infty}\big(K **G_k\big)(s,t)\big) = \lim_{t\to\infty}\frac{\partial}{\partial s}\big(K **G_k\big)(s,t).$$

Hence

$$\begin{aligned}\frac{\partial}{\partial s}\bar{L}_k(s,t) + \frac{\partial}{\partial t}\bar{L}_k(s,t) &= (\frac{\partial}{\partial s}K) **G_k(s,\infty) + (\frac{\partial}{\partial t}K) **G_k(s,\infty)\\ &= \mu^{-1}(1-G) **G_k(s,\infty)\\ &= \mu^{-1}\big(\bar{G}_k(s,t) - \bar{G}_{k+1}(s,t)\big).\end{aligned} \tag{25}$$

In such a way, differentiatiation of (21) in view of (24) and (25) yields

$$\begin{aligned}&\frac{\partial}{\partial s}P(N_s = n, N_t = m) + \frac{\partial}{\partial t}P(N_s = n, N_t = m)\\ &\quad = -\mu^{-1}\begin{cases}\bar{G}_n(s,t) - \bar{G}_{n+1}(s,t) & \\ \quad -\bar{G}_{n-1} **G(s,t) + \bar{G}_n **G(s,t), & \text{if } n\ge 1,\ m=0,\\ \bar{\bar{G}}_m(s,t) - \bar{\bar{G}}_{m+1}(s,t) & \\ \quad -\bar{\bar{G}}_{m-1} **G(s,t) + \bar{\bar{G}}_m **G(s,t), & \text{if } n=0,\ m\ge 1,\end{cases}\end{aligned}$$

The right-hand side is the function $-\mu^{-1}H_{nm}$, where one of n or m is equal to zero. But in this case the right-hand side of (19) is equal to H_{nm}, and the second part of (18) follows. □

4.2 Moments of the renewal counting process

As we shall see, the limit of the one-dimensional distributions is determined by the asymptotic behavior of the moments EN_t^k of the counting process. Introduce as usual the renewal function $U(t) = \sum_{n=1}^{\infty} F^{*n}(t)$. Observe that for a pure renewal process $E\tilde{N}_t = U(t)$, while in the stationary case we have $EN_t = F_1(t) + F_1 * U(t) = t/\mu$. The known result by Teugels (1968) states that if the renewal distribution $F(t)$ has a regularly varying tail with index α, $1<\alpha<2$, then for the pure renewal process, as $t\to\infty$,

$$U(t) - t/\mu \sim \frac{1}{(\alpha-1)(2-\alpha)}\frac{t^{2-\alpha}}{\mu^2}\,L(t) \tag{26}$$

and

$$\mathrm{Var}\,\widetilde{N}_t \sim \frac{2}{(2-\alpha)(3-\alpha)}\frac{t^{3-\alpha}}{\mu^3}\,L(t).$$

A well-known fact is that all moments of the renewal counting process exist and are finite (Asmussen (1987)). The following proposition extends (26) to cover arbitrary moments EN_t^k and stationary renewal processes.

Proposition 2. *If the renewal distribution $F(t)$ has a regularly varying tail with index α, $1 < \alpha < 2$, then for any integer $k \geq 1$, as $t \to \infty$*

a) for the pure renewal process

$$E\widetilde{N}_t^k - \big(t/\mu\big)^k \sim \frac{k \cdot k!\,\Gamma(2-\alpha)}{(\alpha-1)\,\Gamma(k+2-\alpha)} \frac{t^{k+1-\alpha}}{\mu^{k+1}} L(t), \tag{27}$$

$$E\big(\widetilde{N}_t - E\widetilde{N}_t\big)^k \sim \frac{(-1)^k k}{(k-\alpha)(k+1-\alpha)} \frac{t^{k+1-\alpha}}{\mu^{k+1}} L(t). \tag{28}$$

b) for the stationary renewal process

$$EN_t^k - \big(t/\mu\big)^k \sim \frac{(k-1) \cdot k!\,\Gamma(2-\alpha)}{(\alpha-1)\,\Gamma(k+2-\alpha)} \frac{t^{k+1-\alpha}}{\mu^{k+1}} L(t), \tag{29}$$

$$E(N_t - t/\mu)^k \sim \frac{(-1)^k (k-1) k}{(\alpha-1)(k-\alpha)(k+1-\alpha)} \frac{t^{k+1-\alpha}}{\mu^{k+1}} L(t). \tag{30}$$

In the proof we need some properties of the class R_ρ of regularly varying functions with index ρ. The following are stated in Bingham et al. (1987, Proposition 1.5.7):

(i) If $f \in R_\rho$, then $f^\alpha \in R_{\alpha\rho}$.

(ii) If $f_i \in R_{\rho_i}$, $i = 1, 2$, then $f_1 + f_2 \in R_\rho$, where $\rho = \max\{\rho_1, \rho_2\}$.

Lemma 2. *Further properties of regularly varying functions are:*

(iii) *If $f_i \in R_{\rho_i}$, $i = 1, \ldots, n$, $\rho_i \neq \rho_j$ for $i \neq j$, and $c_i \in \mathbb{R}$, then $\sum_{i=1}^n c_i f_i(x) \sim c_k f_k(x)$, as $x \to \infty$, where k is the index of the largest ρ_i, $i = 1, \ldots, n$.*

(iv) *If $f_i(x) \sim a_i x^\rho L(x)$, as $x \to \infty$, with $a_i \in \mathbb{R}$, $L(x)$ slowly varying and $c_i \in \mathbb{R}$, then $\sum_{i=1}^n c_i f_i(x) \sim \sum_{i=1}^n c_i a_i\, x^\rho L(x)$, $x \to \infty$.*

Proof. (iii): Property (ii) yields that $\sum_{i=1}^n c_i f_i(x) \in R_{\rho_k}$, where $\rho_k = \max\{\rho_1, \ldots, \rho_n\}$ and hence $\sum_{i=1}^n c_i f_i(x) \sim c_k f_k(x) \widetilde{L}(x)$, for some slowly varying $\widetilde{L}(x)$. We claim that if $\rho_i \neq \rho_j$ for $i \neq j$, even more is true: $\widetilde{L}(x) = 1$. This follows trivially from the representation $\sum_{i=1}^n c_i f_i(x) = \sum_{i=1}^n c_i x^{\rho_i} L_i(x)$ by dividing with $c_k f_k(x) = c_k x^{\rho_k} L_k(x)$ and taking $x \to \infty$. $\square$

The idea of the proof of Proposition 2 is that arbitrary moments EN_t^k can be expressed as "polynomials" with respect to convolutions of the renewal function, hence the asymptotics for them can be obtained from Teugels' result (26). Standard techniques when dealing with convolutions involve Laplace-Stieltjes (LS) transforms and Karamata's Tauberian theorem (Bingham et al. (1987, Theorem 1.7.1)), which states that the behavior of a function at infinity is determined by the behavior of its LS-transform at zero.

Denote $V_k(t) = EN_t^k$, where, to start with, no assumptions are made about the distribution of the first renewal. Note that the LS-transforms of $V_k(t)$ and $U(t)$,

$$v_k(s) = \int_0^\infty e^{-st}\, dV_k(t), \quad u(s) = \int_0^\infty e^{-st}\, dU(t),$$

exist and are finite. We proceed with some supplementary lemmas.

Lemma 3. *The LS-transforms $v_k(s)$ of $V_k(t)$, $k \geq 1$, satisfy the recurrence relation*

$$v_k(s) = (1 + u(s)) \sum_{i=1}^{k-1} \binom{k}{i} (-1)^{k-i+1} v_i(s) + (-1)^{k+1} v_1(s), \quad k \geq 2. \tag{31}$$

Proof. For $k \geq 0$ consider the generalized renewal measure

$$Z_k(t) = \sum_{n=1}^\infty n^k F_1 * F^{*n-1}(t).$$

Observe that $Z_0(t) = V_1(t)$ and for $k \geq 1$,

$$\begin{aligned} V_k(t) &= \sum_{n=1}^\infty n^k P(N(t) = n) = \sum_{n=1}^\infty n^k (F_1 * F^{*n-1}(t) - F_1 * F^{*n}(t)) \\ &= Z_k(t) - Z_k * F(t). \end{aligned}$$

In such a way, $V_k(t)$ is the coefficient of the renewal equation

$$Z_k(t) = V_k(t) + Z_k * F(t), \tag{32}$$

which involves the function $Z_k(t)$. By the classical renewal theorem (Asmussen (1987)), the unique solution of (32) is given by

$$Z_k(t) = V_k(t) + V_k * U(t). \tag{33}$$

On the other hand, the relation $n^k - (n-1)^k = -\sum_{i=0}^{k-1} \binom{k}{i} n^i (-1)^{k-i}$ inserted into $V_k(t)$ yields

$$\begin{aligned} V_k(t) &= \sum_{n=1}^\infty (n^k - (n-1)^k) F_1 * F^{*n-1}(t) \\ &= \sum_{n=1}^\infty (-\sum_{i=0}^{k-1} \binom{k}{i} n^i (-1)^{k-i}) F_1 * F^{*n-1}(t) \\ &= \sum_{i=0}^{k-1} \binom{k}{i} (-1)^{k-i+1} Z_i(t). \end{aligned} \tag{34}$$

Hence by (33)

$$V_k(t) = \sum_{i=1}^{k-1} \binom{k}{i} (-1)^{k-i+1} (V_i(t) + V_i * U(t)) + (-1)^{k+1} V_1(t), \tag{35}$$

and the recurrence property (31) is just the LS transform counterpart of (35). ☐

Lemma 4. *For any integer $k \geq 1$, the LS-transform $v_k(s)$ can be expressed in terms of the LS-transform $u(s)$ as follows:*

$$v_k(s) = v_1(s) \sum_{j=1}^{k} c_{kj} u(s)^{j-1}, \tag{36}$$

where $c_{kj} = j! \left\{ {k \atop j} \right\} = \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} i^k$, $j = 1, \ldots, k$ and $\left\{ {k \atop j} \right\}$ is the Stirling number of the second kind.

Proof. The existence of constants c_{kj}, $j = 1, \ldots, k$ such that (36) holds follows from the recurrence property (31) for $v_k(s)$ by a standard induction argument. Indeed, assuming that (36) is true for $v_i(s)$, $i \leq k-1$, we have

$$\begin{aligned}
v_k(s) &= (1 + u(s)) \sum_{i=1}^{k-1} \binom{k}{i} (-1)^{k-i+1} v_1(s) \sum_{j=1}^{i} c_{ij} u(s)^{j-1} + (-1)^{k+1} v_1(s) \\
&= v_1(s) (\sum_{i=1}^{k-1} \binom{k}{i} (-1)^{k-i+1} c_{i1} + (-1)^{k+1}) \\
&\quad + v_1(s) \sum_{j=2}^{k} \Big(\sum_{i=j}^{k-1} \binom{k}{i} (-1)^{k-i+1} c_{ij} + \sum_{i=j-1}^{k-1} \binom{k}{i} (-1)^{k-i+1} c_{i\,j-1} \Big) u(s)^{j-1} \\
&\quad + v_1(s) k c_{k-1\,k-1} u(s)^{k-1},
\end{aligned}$$

so that (36) is true for $v_k(s)$ as well, with the constants

$$c_{kj} = \begin{cases} \sum_{i=1}^{k-1} \binom{k}{i} (-1)^{k-i+1} c_{i1} + (-1)^{k+1} & j = 1, \\ \sum_{i=j}^{k-1} \binom{k}{i} (-1)^{k-i+1} c_{ij} & \\ \quad + \sum_{i=j-1}^{k-1} \binom{k}{i} (-1)^{k-i+1} c_{i\,j-1} & 2 \leq j \leq k-1, \\ k c_{k-1\,k-1} & j = k. \end{cases}$$

Now $c_{11} = 1$ yields $c_{k1} = 1$ and $c_{kk} = k!$ for all $k \geq 1$. Also, for $k \geq 2$,

$$c_{kj} = j(c_{k-1\,j} + c_{k-1\,j-1}), \quad j = 2, \ldots, k-1.$$

Thus if we define the triangular sequence $a_{kj} = \frac{c_{kj}}{j!}$, for $j = 1, \ldots, k$, $k \geq 1$, it will satisfy $a_{kk} = 1$, $k \geq 1$, and

$$a_{kj} = j a_{k-1\,j} + a_{k-1\,j-1}, \quad j = 2, \ldots, k-1.$$

This recurrence relation defines the sequence $\left\{ {k \atop j} \right\}$ of Stirling numbers of the second kind (see e. g. Rosen et al. (2000)), which gives us the claim of the lemma. $\square$

Lemma 5. *If the renewal distribution $F(t)$ has a regularly varying tail with index α, $1 < \alpha < 2$, then for any integer $j \geq 1$*

$$u(s)^j - (\mu s)^{-j} \sim \frac{j\,\Gamma(2-\alpha)}{\alpha-1} \frac{1}{\mu^{j+1} s^{j+1-\alpha}} L(s^{-1}), \quad \text{as } s \downarrow 0. \tag{37}$$

Proof. By binomial expansion,

$$u(s)^j - (\mu s)^{-j} = \sum_{r=1}^{j} \binom{j}{r} \big(u(s) - \frac{1}{\mu s}\big)^r \big(\frac{1}{\mu s}\big)^{j-r}. \tag{38}$$

Karamata's Tauberian theorem applied to Teugels' estimate (26) gives

$$u(s) - (\mu s)^{-1} \sim \frac{\Gamma(2-\alpha)}{\alpha-1} \frac{1}{\mu^2 s^{2-\alpha}} L(s^{-1}), \quad \text{as } s \downarrow 0.$$

In particular, $u(1/s) - s/\mu \in R_{2-\alpha}$ and property (i) implies $\big(u(1/s) - s/\mu\big)^r (s/\mu)^{j-r} \in R_{r(2-\alpha)+j-r}$. Property (iii) of Lemma 2 now shows that the dominating term in the expansion (38) is $\binom{j}{1}\big(u(s) - \frac{1}{\mu s}\big)\big(\frac{1}{\mu s}\big)^{j-1}$, corresponding to the regular variation index

$$j + 1 - \alpha = \max_{1 \leq r \leq j} \{r(2-\alpha) + j - r\}.$$

Thus, as $s \downarrow 0$

$$\begin{aligned} u(s)^j - (\mu s)^{-j} &\sim \binom{j}{1} \big(u(s) - \frac{1}{\mu s}\big)\big(\frac{1}{\mu s}\big)^{j-1} \\ &\sim j \frac{\Gamma(2-\alpha)}{\alpha-1} \frac{1}{\mu^2 s^{2-\alpha}} L(s^{-1}) \big(\frac{1}{\mu s}\big)^{j-1} \end{aligned}$$

and the claim of the lemma follows. $\square$

To prove parts (27) and (29) of Proposition 2, we will use Lemma 4 to express the function $v_k(s)$ in terms of the function $u(s)$ and then employ Lemma 5, which describes the behavior of the terms of the expansion at zero. Indeed, for a pure renewal process we have $\tilde{v}_1(s) = u(s)$ and by Lemma 4,

$$\tilde{v}_k(s) = \sum_{j=1}^{k} c_{kj} u(s)^j,$$

hence

$$\tilde{v}_k(s) - c_{kk}(\mu s)^{-k} = c_{kk}\big(u(s)^k - (\mu s)^{-k}\big) + \sum_{j=1}^{k-1} c_{kj} u(s)^j. \tag{39}$$

Lemma 5 yields $u(1/s)^k - \big(s/\mu\big)^k \in R_{k+1-\alpha}$. Moreover, $u(1/s)^j \in R_j$. Hence, again by property (iii), the dominating term in (39) as $s \downarrow 0$ is $k!\big(u(s)^k - (\mu s)^{-k}\big)$ with the regular variation index

$$k+1-\alpha = \max\{1,\dots,k-1,k+1-\alpha\},$$

so that

$$\tilde v_k(s) - k!(\mu s)^{-k} \sim k!\big(u(s)^k - (\mu s)^{-k}\big) \sim k! \frac{k\,\Gamma(2-\alpha)}{\alpha-1}\frac{1}{\mu^{k+1}s^{k+1-\alpha}}L(s^{-1}). \tag{40}$$

To finish the proof of (27), it remains to apply Karamata's Tauberian theorem to (40) (note that the LS transform of t^k is equal to $k!s^{-k}$).

The proof of (29) is completely analogous, we have only to take into account that $v_1(s) = 1/(\mu s)$. Again, Lemma 4 implies

$$v_k(s) = \frac{1}{\mu s}\sum_{j=1}^k c_{kj} u(s)^{j-1},$$

and consequently

$$v_k(s) - c_{kk}(\mu s)^{-k} = c_{kk}\frac{1}{\mu s}\big(u(s)^{k-1} - (\mu s)^{-(k-1)}\big) + \frac{1}{\mu s}\sum_{j=1}^{k-1} c_{k\,j} u(s)^{j-1}.$$

By an analogous argument as in the pure renewal case,

$$\begin{aligned} v_k(s) - k!(\mu s)^{-k} &\sim k!\frac{1}{\mu s}\big(u(s)^{k-1} - (\mu s)^{-(k-1)}\big) \\ &\sim k!\frac{1}{\mu s}\frac{(k-1)\Gamma(2-\alpha)}{\alpha-1}\frac{1}{\mu^k s^{k-\alpha}}L(s^{-1}), \end{aligned} \tag{41}$$

which is equivalent to (29) by Karamata's Tauberian theorem.

To prove (28), we first consider the shifted moment $E\big(\widetilde N_t - t/\mu\big)^k$. Exploiting the relation $\sum_{j=0}^k \binom{k}{j}(-1)^j = 0$, we get

$$\begin{aligned} E\big(\widetilde N_t - t/\mu\big)^k &= \sum_{j=0}^k \binom{k}{j} E\widetilde N_t^j (-t/\mu)^{k-j} \\ &= \sum_{j=0}^k \binom{k}{j} \big(E\widetilde N_t^j - (t/\mu)^j\big)(-t/\mu)^{k-j}. \end{aligned} \tag{42}$$

By (27), just proved, for any $j = 0,\dots,k$, we have

$$\big(E\widetilde N_t^j - (t/\mu)^j\big)(-t/\mu)^{k-j} \sim \frac{(-1)^{k-j} j\cdot j!\,\Gamma(2-\alpha)}{(\alpha-1)\,\Gamma(j+2-\alpha)}\frac{t^{k+1-\alpha}}{\mu^{k+1}}L(t).$$

Thus property (iv) applied to (42) yields

$$\begin{aligned} E\big(\widetilde{N}_t - t/\mu\big)^k &\sim \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^{k-j} j \cdot j!\,\Gamma(2-\alpha)}{(\alpha-1)\,\Gamma(j+2-\alpha)} \frac{t^{k+1-\alpha}}{\mu^{k+1}} L(t) \\ &= \frac{k(-1)^k}{(k-\alpha)(k+1-\alpha)} \frac{t^{k+1-\alpha}}{\mu^{k+1}} L(t). \end{aligned} \tag{43}$$

But due to property (iii), the expansion

$$E\big(\widetilde{N}_t - E\widetilde{N}_t\big)^k = \sum_{j=0}^{k} \binom{k}{j} E\big(\widetilde{N}_t - t/\mu\big)^{k-j} \big(-(E\widetilde{N}_t - t/\mu)\big)^j$$

now gives

$$E\big(\widetilde{N}_t - E\widetilde{N}_t\big)^k \sim E\big(\widetilde{N}_t - t/\mu\big)^k,$$

hence (28). The proof of (30) is analogous to that of (43). □

5 Proof of the theorem

Consider

$$Z^{(m)}(t) = -\mu Y^{(m)}(t) = -\frac{\mu}{a_m}\big(W(m, a_m t) - \frac{m a_m t}{\mu}\big), \quad t \geq 0. \tag{44}$$

It will be proved that the process $\{Z^{(m)}(t)\}$ converges weakly in the scaling regime (ICR) to a process $\{Y_\beta(t)\}$ with finite-dimensional distributions characterized by (6). For convenience, we write $a_m = a$ in this section.

5.1 Convergence of one-dimensional distributions

The proof is based on the method of moments. We will prove that all cumulants of the marginal distributions of $Z^{(m)}(t)$ defined in (44) converge to those of the limit process $Y_\beta(t)$, from which convergence of the one-dimensional distributions follow.

Indeed, due to independence, for the cumulant generating function of $Z^{(m)}(t)$ we have

$$\log E \exp\{\theta Z^{(m)}(t)\} = m \log E \exp\{-\theta\mu(N_{at} - at/\mu)/a\}.$$

The kth order cumulants of $Z^{(m)}(t)$,

$$C_k^{(m)}(t) = m \frac{d^k}{d\theta^k} \Big(\log E e^{-\theta\mu\,(N_{at} - at/\mu)/a} \Big)\Big|_{\theta=0}, \quad k \geq 1,$$

are hence determined by the cumulants of the process $-\mu(N_{at} - at/\mu)/a$, which is a rescaled and centered renewal counting process up to a constant.

It is well-known that such cumulants can be expressed as polynomials with respect to the moments, i.e. there exist constants α_{kj}, $j = 0, \dots, k$, such that

$$C_k^{(m)}(t) = m \sum_{j=0}^{k} \alpha_{kj} (-1)^j \mu^j E(N_{at} - at/\mu)^j / a^j. \tag{45}$$

Moreover, it can be proved that $\alpha_{kk} = 1$ for all k. Since $a \to \infty$ as $m \to \infty$, by Proposition 2 we have

$$E(N_{at} - at/\mu)^k \sim \frac{(-1)^k (k-1)k}{\beta(k-1-\beta)(k-\beta)} \frac{(at)^{k-\beta}}{\mu^{k+1}} L(a), \quad \text{as } m \to \infty.$$

Hence the terms of expansion (45) are regularly varying and property (iii) from Lemma 2 yields

$$\begin{aligned} C_k^{(m)}(t) &\sim \alpha_{kk} m\, (-1)^k \mu^k E(N_{at} - at/\mu)^k / a^k \\ &\sim m a^{-\beta} L(a) \frac{(k-1)k}{\beta(k-1-\beta)(k-\beta)} \frac{t^{k-\beta}}{\mu}. \end{aligned}$$

But $ma^{-\beta}L(a) \to \mu$ by assumption (ICR), thus

$$C_k^{(m)}(t) \to \frac{(k-1)k}{\beta(k-1-\beta)(k-\beta)} t^{k-\beta},$$

where the limit expressions are the cumulants of $Y_\beta(t)$ by formula (11). Hence $Z^{(m)}(t)$ converges in distribution to a random variable $Y_\beta(t)$, where $Y_\beta(t)$ satisfies (7).

5.2 Convergence of n-dimensional distributions

Here we use the asymptotic relation

$$\begin{aligned} \log E \exp\Big\{ \sum_{i=1}^{n} \theta_i Z^{(m)}(t_i) \Big\} &= m \log E \exp\Big\{ -\sum_{i=1}^{n} \theta_i \mu (N_{at_i} - at_i/\mu)/a \Big\} \\ &\sim mE\big[e^{-\sum_{i=1}^n \theta_i \mu (N_{at_i} - at_i/\mu)/a} - 1\big] + \mathcal{O}(\frac{m}{a}), \qquad \text{as } m \to \infty. \end{aligned} \tag{46}$$

For $n \geq 1$ and $1 \leq k \leq n$, put

$$\bar\theta_{k,n} = (\theta_k, \dots, \theta_n), \quad \bar t_{k,n} = (t_k, \dots, t_n),$$

where $0 = t_0 \leq t_1 \leq \dots \leq t_n$ and let

$$\Phi_n(\bar\theta_{1,n}; \bar t_{1,n}) = E[e^{\sum_{i=1}^n \theta_i (N_{t_i} - t_i/\mu)}]$$

be the multi-variate moment generating function for the centered renewal process $\{N_t - t/\mu\}_{t\geq 0}$. Similarly, let $\widetilde\Phi_n(\bar\theta_{1,n}; \bar t_{1,n})$ denote the corresponding function for the pure renewal process $\{\widetilde N_t - t/\mu\}_{t \geq 0}$. By Proposition 1,

$$\Phi_n(\bar\theta_{1,n}; \bar t_{1,n}) = \Phi_{n-1}(\bar\theta_{2,n}; \bar t_{2,n}) e^{-\theta_1 t_1/\mu} + \frac{1 - e^{-\theta_1}}{1 - e^{-\sum_{i=1}^n \theta_i}} \tag{47}$$

$$\times \int_0^{t_1} e^{-\theta_1(t_1-u)/\mu} \widetilde{\Phi}_{n-1}(\bar\theta_{2,n};\, \bar t_{2,n}-u) e^{-u\sum_{i=1}^n \theta_i/\mu}\, dE[e^{N_u \sum_{i=1}^n \theta_i}].$$

Here and in the sequel the subtraction $\bar t_{k,n} - u = (t_k - u, \dots, t_n - u)$ is interpreted component-wise.

For $m \geq 1$ and the given scaling sequence $a = a_m$, consider the scaled functions

$$\begin{aligned} \Lambda_n^{(m)}(\bar\theta_{1,n};\, \bar t_{1,n}) &= m(\Phi_n(-\mu\bar\theta_{1,n}/a;\, a\bar t_{1,n}) - 1) \\ &= mE[e^{-\sum_{i=1}^n \theta_i \mu (N_{at_i} - at_i/\mu)/a} - 1], \end{aligned}$$

and, analogously,

$$\widetilde\Lambda_n^{(m)}(\bar\theta_{1,n};\, \bar t_{1,n}) = m(\widetilde\Phi_n(-\mu\bar\theta_{1,n}/a;\, a\bar t_{1,n}) - 1).$$

According to (46) it is the limit functions of $\Lambda_n^{(m)}(\bar\theta_{1,n};\, \bar t_{1,n})$ as $m \to \infty$ that determine the CGF of Y_β.

Lemma 6. *The limit functions*

$$\Lambda_n(\bar\theta_{1,n};\, \bar t_{1,n}) = \lim_{m\to\infty} \Lambda_n^{(m)}(\bar\theta_{1,n};\, \bar t_{1,n}), \quad n \geq 1,$$

exist and satisfy the system of recursive integral equations

$$\begin{aligned} \Lambda_n(\bar\theta_{1,n};\, \bar t_{1,n}) &= \Lambda_{n-1}(\bar\theta_{2,n};\, \bar t_{2,n} - t_1) + \Lambda_1(\textstyle\sum_{i=1}^n \theta_i;\, t_1) \\ &\quad -(1 - \theta_1/\textstyle\sum_{i=1}^n \theta_i) \displaystyle\int_0^{t_1} e^{\theta_1(t_1-u)} d\Lambda_1(\textstyle\sum_{i=1}^n \theta_i;\, u) \\ &\quad -(1 + \theta_1/\textstyle\sum_{i=2}^n \theta_i) \displaystyle\int_0^{t_1} e^{\theta_1(t_1-u)}\, d\Lambda_{n-1}(\bar\theta_{2,n};\, \bar t_{2,n} - u), \end{aligned}$$

where in the case $n = 1$ we put $\Lambda_{n-1} = 0$.

Proof. The proof is by induction on n. The relation (7), established in the previous subsection, provides the existence of a limit function $\Lambda_1(\theta;\, t)$ for the case $n = 1$. The integral equation for $n = 1$ is trivial. Fix $n \geq 2$ and assume that $\Lambda_{n'}(\bar\theta_{1,n'};\, \bar t_{1,n'})$ exist for $n' \leq n-1$. It follows in particular that the limit functions $\Lambda_{n-k+1}(\bar\theta_{k,n};\, \bar t_{k,n})$ exist for $2 \leq k \leq n$.

To study the asymptotic behavior of $\Lambda_n^{(m)}(\bar\theta_{1,n};\, \bar t_{1,n})$ as $m \to \infty$ we apply the defining scaling relation to equation (47). This gives

$$\Lambda_n^{(m)}(\bar\theta_{1,n};\, \bar t_{1,n}) = \Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\, \bar t_{2,n})\, e^{\theta_1 t_1} + I_1^{(m)} + I_2^{(m)} + I_3^{(m)}, \tag{48}$$

where

$$\begin{aligned} I_1^{(m)} &= -\frac{a}{\mu}(e^{\mu\theta_1/a} - 1) \int_0^{t_1} e^{\theta_1(t_1-u)}\, \widetilde\Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\, \bar t_{2,n} - u)\, du, \\ I_2^{(m)} &= -\frac{a}{\mu}(e^{\mu\theta_1/a} - 1) \int_0^{t_1} e^{\theta_1(t_1-u)}(1 + \frac{1}{m}\widetilde\Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\, \bar t_{2,n} - u))\, H^{(m)}(du), \end{aligned}$$

$$I_3^{(m)} = m\big(1 - \frac{a(e^{\mu\theta_1/a}-1)}{\mu\theta_1}\big)(e^{\theta_1 t_1}-1),$$

and the integration in $I_2^{(m)}$ is with respect to the signed measure

$$H^{(m)}(du) = m\Big(\frac{\mu\, e^{u\sum_{i=1}^n \theta_i}}{a(1-e^{\mu\sum_{i=1}^n \theta_i/a})}\, dE[e^{-\mu N_{au}\sum_{i=1}^n \theta_i/a}] - du\Big).$$

Since

$$I_3^{(m)} = \mathcal{O}(\frac{m}{a}), \quad \text{as } m \to \infty,$$

we need only to investigate the terms $I_1^{(m)}$ and $I_2^{(m)}$.

To evaluate $I_1^{(m)}$ we apply Lemma 1, which implies

$$E[e^{\sum_{i=1}^n \theta_i \widetilde{N}_{t_i-u}}] = -\frac{\mu}{e^{\sum_{i=1}^n \theta_i}-1}\,\frac{d}{du}E[e^{\sum_{i=1}^n \theta_i N_{t_i-u}}].$$

Hence

$$\begin{aligned}
&\frac{d}{du}E[e^{\sum_{i=1}^n \theta_i(N_{t_i-u}-\frac{1}{\mu}(t_i-u))}]\\
&\quad = -\frac{1}{\mu}(e^{\sum_{i=1}^n \theta_i}-1)\, E[e^{\sum_{i=1}^n \theta_i(\widetilde{N}_{t_i-u}-\frac{1}{\mu}(t_i-u))}]\\
&\qquad +\frac{1}{\mu}\textstyle\sum_{i=1}^n \theta_i\, E[e^{\sum_{i=1}^n \theta_i(N_{t_i-u}-\frac{1}{\mu}(t_i-u))}],
\end{aligned}$$

which can be written

$$\begin{aligned}
\widetilde{\Phi}_n(\bar\theta_{1,n};\, \bar t_{1,n}-u) &= \frac{\sum_{i=1}^n \theta_i}{e^{\sum_{i=1}^n \theta_i}-1}\Phi_n(\bar\theta_{1,n};\, \bar t_{1,n}-u)\\
&\qquad -\frac{\mu}{e^{\sum_{i=1}^n \theta_i}-1}\,\frac{d}{du}\Phi_n(\bar\theta_{1,n};\, \bar t_{1,n}-u).
\end{aligned}$$

Under the given rescaling scheme the same relation takes the form

$$\begin{aligned}
\widetilde{\Lambda}_n^{(m)}(\bar\theta_{1,n};\, \bar t_{1,n}-u) &= \frac{\mu\sum_{i=1}^n \theta_i}{a(1-e^{-\mu\sum_{i=1}^n \theta_i/a})}\Lambda_n^{(m)}(\bar\theta_{1,n};\, \bar t_{1,n}-u)\\
&-\frac{\mu}{a(e^{-\mu\sum_{i=1}^n \theta_i/a}-1)}\,\frac{d}{du}\Lambda_n^{(m)}(\bar\theta_{1,n};\, \bar t_{1,n}-u)\\
&+m\Big(\frac{\mu\sum_{i=1}^n \theta_i}{a(1-e^{-\mu\sum_{i=1}^n \theta_i/a})}-1\Big).
\end{aligned}$$

If we apply the preceeding identity with the choice of index $n-1$, then the induction hypothesis allows us to replace the coefficients with their asymptotic limits, adding a remainder term, thus

$$\begin{aligned}
\widetilde{\Lambda}_{n-1}^{(m)}(\bar\theta_{2,n};\, \bar t_{2,n}-u) &= \Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\, \bar t_{2,n}-u)\\
&\qquad +\frac{1}{\sum_{i=2}^n \theta_i}\,\frac{d}{du}\Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\, \bar t_{2,n}-u)+\mathcal{O}(\frac{m}{a}).
\end{aligned}$$

It follows that

$$
\begin{aligned}
I_1^{(m)} &= -\theta_1 \int_0^{t_1} e^{\theta_1(t_1-u)}\, \Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\, \bar t_{2,n}-u)\, du \\
&\qquad - \frac{\theta_1}{\sum_{i=2}^n \theta_i} \int_0^{t_1} e^{\theta_1(t_1-u)}\, d\Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\, \bar t_{2,n}-u) + \mathcal{O}(\frac{m}{a}).
\end{aligned}
$$

Moreover, the integration by parts

$$
\begin{aligned}
\theta_1 \int_0^{t_1} e^{\theta_1(t_1-u)}\, \Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\, \bar t_{2,n}-u)\, du &= e^{\theta_1 t_1}\, \Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\, \bar t_{2,n}) \\
- \Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\, \bar t_{2,n}-t_1) &+ \int_0^{t_1} e^{\theta_1(t_1-u)}\, d\Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\, \bar t_{2,n}-u)
\end{aligned}
$$

gives us

$$
\begin{aligned}
I_1^{(m)} &= \Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\, \bar t_{2,n}-t_1) - e^{\theta_1 t_1}\, \Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\, \bar t_{2,n}) \\
&\quad - \Big(1 + \frac{\theta_1}{\sum_{i=2}^n \theta_i}\Big) \int_0^{t_1} e^{\theta_1(t_1-u)}\, d\Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\, \bar t_{2,n}-u) + \mathcal{O}(\frac{m}{a}).
\end{aligned}
$$

Turning to $I_2^{(m)}$, this integral is evaluated with respect to the measure

$$
\begin{aligned}
H^{(m)}(du) &= m\Big(\frac{\mu\, e^{u\sum_{i=1}^n \theta_i}}{a(1-e^{\mu\sum_{i=1}^n \theta_i/a})}\, dE[e^{-\mu N_{au}\sum_{i=1}^n \theta_i/a}] - du\Big) \\
&= -\frac{1}{\sum_{i=1}^n \theta_i}\, d(mE[e^{-\mu\frac{N_{au}-au/\mu}{a}\sum_{i=1}^n \theta_i} - 1]) \\
&\qquad + mE[e^{-\mu\frac{N_{au}-au/\mu}{a}\sum_{i=1}^n \theta_i} - 1]\, du + \mathcal{O}(\frac{m}{a}) \\
&= -\frac{1}{\sum_{i=1}^n \theta_i}\, d\Lambda_1^{(m)}(\textstyle\sum_{i=1}^n \theta_i;\, u) + \Lambda_1^{(m)}(\textstyle\sum_{i=1}^n \theta_i;\, u)\, du + \mathcal{O}(\frac{m}{a}).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\theta_1 \int_0^{t_1} e^{\theta_1(t_1-u)}\, \Lambda_1^{(m)}(\textstyle\sum_{i=1}^n \theta_i;\, u)\, du \\
&\qquad = -\Lambda_1^{(m)}(\textstyle\sum_{i=1}^n \theta_i;\, t_1) + \displaystyle\int_0^{t_1} e^{\theta_1(t_1-u)}\, d\Lambda_1^{(m)}(\textstyle\sum_{i=1}^n \theta_i;\, u),
\end{aligned}
$$

by a partial integration, it follows that

$$
\begin{aligned}
I_2^{(m)} &= \Lambda_1^{(m)}(\textstyle\sum_{i=1}^n \theta_i;\, t_1) \\
&\quad -(1-\theta_1/\textstyle\sum_{i=1}^n \theta_i) \displaystyle\int_0^{t_1} e^{\theta_1(t_1-u)}\, d\Lambda_1^{(m)}(\textstyle\sum_{i=1}^n \theta_i;\, u) + \mathcal{O}(\frac{m}{a}).
\end{aligned}
$$

Summarizing,

$$
\Lambda_n^{(m)}(\bar\theta_{1,n};\, \bar t_{1,n}) = \Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\, \bar t_{2,n}-t_1) + \Lambda_1^{(m)}(\textstyle\sum_{i=1}^n \theta_i;\, t_1)
$$

$$-(1+\theta_1/\textstyle\sum_{i=2}^n\theta_i)\displaystyle\int_0^{t_1} e^{\theta_1(t_1-u)}\,d\Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\,\bar t_{2,n}-u)$$
$$-(1-\theta_1/\textstyle\sum_{i=1}^n\theta_i)\displaystyle\int_0^{t_1} e^{\theta_1(t_1-u)}d\Lambda_1^{(m)}(\textstyle\sum_{i=1}^n\theta_i;\,u)+\mathcal{O}(\frac{m}{a}).$$

Now take $m\to\infty$ and apply the induction hypothesis to conclude that the limit function $\Lambda_n(\bar\theta_{1,n};\,\bar t_{1,n})$ exists and satisfies the equation

$$\begin{aligned}\Lambda_n(\bar\theta_{1,n};\,\bar t_{1,n}) &= \Lambda_{n-1}(\bar\theta_{2,n};\,\bar t_{2,n}-t_1)+\Lambda_1(\textstyle\sum_{i=1}^n\theta_i;\,t_1)\\ &-(1+\theta_1/\textstyle\sum_{i=2}^n\theta_i)\displaystyle\int_0^{t_1} e^{\theta_1(t_1-u)}\,d\Lambda_{n-1}(\bar\theta_{2,n};\,\bar t_{2,n}-u)\\ &-(1-\theta_1/\textstyle\sum_{i=1}^n\theta_i)\displaystyle\int_0^{t_1} e^{\theta_1(t_1-u)}d\Lambda_1(\textstyle\sum_{i=1}^n\theta_i;\,u).\end{aligned}$$

This completes the proof of the lemma and the proof of convergence of the finite-dimensional distributions. □

5.3 Cumulant generating function for the increment process

The logarithmic moment generating function for the increments of the limit process Y_β is given by

$$\Gamma_n(\bar\theta_{1,n},\bar t_{1,n})=\log E\exp\Big\{\sum_{i=1}^n\theta_i(Y_\beta(t_i)-Y_\beta(t_{i-1}))\Big\}.$$

In particular,

$$\Gamma_n(\bar\theta_{1,n},\bar t_{1,n})=\Lambda_n((\theta_1-\theta_2,\ldots,\theta_{n-1}-\theta_n,\theta_n);\,\bar t_{1,n}).$$

Lemma 6 shows that these functions satisfy the recursive system

$$\begin{aligned}\Gamma_n(\bar\theta_{1,n};\,\bar t_{1,n}) &= \Gamma_{n-1}(\bar\theta_{2,n};\,\bar t_{2,n}-t_1)+\Lambda_1(\theta_1;\,t_1)\\ &+\frac{\theta_2}{\theta_1}\int_0^{t_1} e^{(\theta_1+\theta_2)u}d\Lambda_1(\theta_1;\,t_1-u)\\ &+\frac{\theta_1}{\theta_2}\int_0^{t_1} e^{(\theta_1+\theta_2)u}\,d\Gamma_{n-1}(\bar\theta_{2,n};\,\bar t_{2,n}-t_1+u)\\ &=: I_1+I_2+I_3+I_4.\end{aligned}\tag{49}$$

Note that in the integral terms of Lemma 6 we also made the change of variable $u\to t_1-u$.

To complete the proof of the characterization of the limit process Y_β it remains to verify that the functions given in (6) are the solutions of the integral equation stated above. To this end, assume that $\Gamma_{n-1}(\bar\theta_{2,n};\,\bar t_{2,n}-t_1)$ is given by the representation (6). Then by (49)

$$\Gamma_n(\bar\theta_{1,n};\,\bar t_{1,n})=\frac{1}{\beta}\sum_{i=1}^n\theta_i^2\int_0^{t_i-t_{i-1}}\int_0^v e^{\theta_i u}u^{-\beta}\,dudv$$

$$
\begin{aligned}
&+\frac{1}{\beta}\sum_{i=2}^{n-1}\sum_{j=i+1}^{n}\theta_i\theta_j e^{\sum_{k=i+1}^{j-1}\theta_k(t_k-t_{k-1})}\\
&\quad\times\int_0^{t_i-t_{i-1}}\int_0^{t_j-t_{j-1}} e^{\theta_j u}e^{\theta_i v}(t_{j-1}-t_i+u+v)^{-\beta}\,dudv\\
&+I_3+I_4,
\end{aligned}
$$

and it remains to establish that

$$
\begin{aligned}
I_3+I_4 &= \frac{1}{\beta}\sum_{j=2}^{n}\theta_1\theta_j e^{\sum_{k=2}^{j-1}\theta_k(t_k-t_{k-1})}\\
&\quad\times\int_0^{t_1}\int_0^{t_j-t_{j-1}} e^{\theta_j u}e^{\theta_1 v}(t_{j-1}-t_1+u+v)^{-\beta}\,dudv. \qquad (50)
\end{aligned}
$$

But by the induction hypothesis, changing the order of integration, and with the change of variable $u'=t_1+v-u$,

$$
\begin{aligned}
I_3 &= -\frac{\theta_1\theta_2}{\beta}\int_0^{t_1} e^{-(\theta_1-\theta_2)(u-t_1)}\int_0^{u} e^{\theta_1 v}v^{-\beta}\,dv\,du\\
&= -\frac{\theta_1\theta_2}{\beta}\int_0^{t_1}\int_v^{t_1} e^{-(\theta_1-\theta_2)(u-v-t_1)}\,du\,e^{\theta_2 v}v^{-\beta}\,dv\\
&= -\frac{\theta_1\theta_2}{\beta}\int_0^{t_1}\int_v^{t_1} e^{(\theta_1-\theta_2)u}\,du\,e^{\theta_2 v}v^{-\beta}\,dv\\
&= -\frac{\theta_1\theta_2}{\beta}\int_0^{t_1} e^{(\theta_1-\theta_2)u}\int_0^{u} e^{\theta_2 v}v^{-\beta}\,dv\,du.
\end{aligned}
$$

Further, since

$$
\begin{aligned}
\frac{d}{dw}\Gamma_{n-1}(\bar\theta_{2,n};\bar t_{2,n}-t_1+w) &= \frac{\theta_2^2}{\beta}\int_0^{t_2-t_1+w} e^{\theta_2 v}v^{-\beta}\,dv\\
&+\frac{1}{\beta}\sum_{j=3}^{n}\theta_2\theta_j e^{\sum_{k=3}^{j-1}\theta_k(t_k-t_{k-1})}\\
&\quad\times\int_0^{t_j-t_{j-1}} e^{\theta_j v}e^{\theta_2(t_2-t_1+w)}(t_{j-1}-t_1+w+v)^{-\beta}\,dv,
\end{aligned}
$$

we get

$$
\begin{aligned}
I_4 &= \frac{\theta_1\theta_2}{\beta}\int_0^{t_1} e^{(\theta_1-\theta_2)u}\int_0^{t_2-t_1+u} e^{\theta_2 v}v^{-\beta}\,dvdu\\
&\quad+\frac{1}{\beta}\sum_{j=3}^{n}\theta_1\theta_j e^{\sum_{k=3}^{j-1}\theta_k(t_k-t_{k-1})}\\
\times&\int_0^{t_1} e^{(\theta_1-\theta_2)u}e^{\theta_2(t_2-t_1+u)}\int_0^{t_j-t_{j-1}} e^{\theta_j v}(t_{j-1}-t_1+u+v)^{-\beta}\,dvdu
\end{aligned}
$$

and (50) follows. □

5.4 Tightness

To finish the proof of the convergence result in Theorem 1, it remains to establish tightness as the sequence of laws of $Y^{(m)}$ converges to the law of $-\mu^{-1}Y_\beta$. As usual the trajectories of $Y^{(m)}$ are considered to be elements in the Skorokhod space $D(0,T)$ of cádlág functions on a real interval $[0,T]$, equipped with the Skorokhod topology. To prove tightness in $D(0,T)$ for any fixed T, fix $0<\beta<1$ and consider time points $0<t_1<t<t_2<T$. By stationarity of $Y^{(m)}$,

$$\begin{aligned}&E(|Y^{(m)}(t)-Y^{(m)}(t_1)|\,|Y^{(m)}(t_2)-Y^{(m)}(t)|)\\&\qquad\le \mathrm{Var}(Y^{(m)}(t-t_1))^{1/2}\,\mathrm{Var}(Y^{(m)}(t_2-t))^{1/2}\end{aligned}$$

By (45),

$$\mathrm{Var}(Y^{(m)}(t))=C_2^{(m)}(t)=\frac{m}{a^2}E(N_{at}-at/\mu)^2.$$

As proved in Proposition 2, the function $E(N_t-t/\mu)^2$ is regularly varying and

$$E(N_t-t/\mu)^2\sim\frac{2}{\beta(1-\beta)(2-\beta)\mu^3}\,t^{2-\beta}L(t),\quad \text{as } t\to\infty. \tag{51}$$

The Potter bounds for a regularly varying function (Bingham et al. (1987, Theorem 1.5.6)) yield that for any $\epsilon>0$ there exists m_0 such that

$$\frac{E(N_{at}-at/\mu)^2}{E(N_a-a/\mu)^2}<(1+\epsilon)\max\{t^{2-\beta+\epsilon},t^{2-\beta-\epsilon}\},\quad \text{as } m\ge m_0.$$

Hence, for $m\ge m_0$,

$$\begin{aligned}&E(|Y^{(m)}(t)-Y^{(m)}(t_1)|\,|Y^{(m)}(t_2)-Y^{(m)}(t)|)\\&\quad\le\frac{m}{a^2}\big(E[N_{a(t-t_1)}-a(t-t_1)/\mu]^2\big)^{1/2}\big(E[N_{a(t_2-t)}-a(t_2-t)/\mu]^2\big)^{1/2}\\&\quad\le\frac{m}{a^2}(1+\epsilon)E(N_a-a/\mu)^2\,C(t_1,t,t_2),\end{aligned}$$

where

$$\begin{aligned}C(t_1,t,t_2)&=\max\{[(t-t_1)(t_2-t)]^{1-(\beta-\epsilon)/2},[(t-t_1)(t_2-t)]^{1-(\beta+\epsilon)/2}\}\\&\le\max\{(t_2-t_1)^{2-\beta+\epsilon},(t_2-t_1)^{2-\beta-\epsilon}\}.\end{aligned}$$

Since (51) and the condition (ICR) implies that $\frac{m}{a^2}E(N_a-a/\mu)^2\to\sigma_\beta^2\mu^{-2}$, we have that for any $\delta>0$ there exists m_1 such that

$$\begin{aligned}&E(|Y^{(m)}(t)-Y^{(m)}(t_1)|\,|Y^{(m)}(t_2)-Y^{(m)}(t)|)\\&\qquad\le(1+\epsilon)(\sigma_\beta^2\mu^{-2}+\delta)\max\{(t_2-t_1)^{2-\beta+\epsilon},(t_2-t_1)^{2-\beta-\epsilon}\}.\end{aligned}$$

for $m\ge m_1$. Take $\epsilon<1-\beta$. Then the desired tightness property follows from Billingsley (1968, Theorem 15.6).

5.5 Proof of the convergence of $Y^{(m)}$ under condition (FCR)

Selected parts of the proof of Theorem 1 can be modified to provide the limit result (3) for the process $Y^{(m)}(t)$ under condition (FCR) with the normalizing sequence

$$b_m = \left(m a_m^{2-\beta} L(a_m)\right)^{\frac{1}{2}}. \tag{52}$$

Since in this case the limit process is Gaussian, it is enough to show that the marginal distributions of $Y^{(m)}(t)$ converge to Gaussian distributions and that the covariance function converges to that of the multiple of FBM.

The convergence of the marginal distributions of $Y^{(m)}(t)$ under new scaling $b_m = b$ can be obtained by the method of moments along the same lines as for the marginals of $Z^{(m)}(t)$ in Section 5.1. Now the cumulants of $Y^{(m)}(t)$ read as follows:

$$D_k^{(m)}(t) = m\frac{d^k}{d\theta^k}\Big(\log E e^{\theta(N_{at}-at/\mu)/b}\Big)\Big|_{\theta=0} = m\sum_{j=0}^{k}\alpha_{kj}E(N_{at}-at/\mu)^j/b^j.$$

Continuing as earlier, due to Proposition 2 we have

$$\begin{aligned} D_k^{(m)}(t) &\sim \alpha_{kk} m E(N_{at}-at/\mu)^k/b^k \\ &\sim \frac{ma^{k-\beta}L(a)}{b^k}\frac{(-1)^k(k-1)k}{\beta(k-1-\beta)(k-\beta)}\frac{t^{k-\beta}}{\mu^{k+1}}. \end{aligned}$$

Observe that (52) and condition (FCR) yield

$$\frac{ma^{k-\beta}L(a)}{b^k} = \begin{cases} 1, & \text{if } k=2, \\ \left(ma^{-\beta}L(a)\right)^{1-k/2} \to 0, & \text{if } k>2. \end{cases}$$

Hence,

$$D_k^{(m)}(t) \to \begin{cases} \mu^{-3}\sigma_\beta^2 t^{2-\beta}, & \text{if } k=2, \\ 0 & \text{if } k>2. \end{cases} \tag{53}$$

and since $D_1^{(m)}(t)=0$, it follows that the cumulants of the random variable $Y^{(m)}(t)$ converge to those of a Gaussian random variable with the same distribution as $\mu^{-3/2}\sigma_\beta B_H(t)$.

It remains to prove that the covariance function of the process $Y^{(m)}(t)$ converges to that of $\mu^{-3/2}\sigma_\beta B_H(t)$. But the process $Y^{(m)}(t)$ has stationary increments, whence

$$E[Y^{(m)}(t)\,Y^{(m)}(s)] = \frac{1}{2}\left(\mathrm{Var}[Y^{(m)}(t)] + \mathrm{Var}[Y^{(m)}(s)] - \mathrm{Var}[Y^{(m)}(t-s)]\right),$$

and the convergence follows from (53).

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References

M. Abramowitz and I. A. Stegun, eds. (1992) *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* Reprint of the 1972 edition. New York: Dover Publications.

S. Asmussen (1987) *Applied probability and queues.* Chichester: John Wiley and Sons.

P. Billingsley (1968) *Convergence of probability measures.* New York: John Wiley and Sons.

N. H. Bingham, C. M. Goldie and J. L. Teugels (1987) *Regular variation.* Cambridge: Cambridge University Press.

J. J. Hunter (1974) Renewal theory in two dimensions: basic results. *Adv. Appl. Prob.*, **6**, 376-391.

I. Kaj and S. Sagitov (1998) Limit processes for age-dependent branching particle systems. *J. Theoretical Prob.*, **11**, 225-257.

I. Kaj (2002) *Stochastic modeling in broadband communications systems.* To appear in SIAM Monographs in Mathematical Modeling and Computation.

J. B. Levy and M. S. Taqqu (2000) Renewal reward processes with heavy-tailed inter-renewal times and heavy-tailed rewards. *Bernoulli*, **6**(1), 23-44.

B. B. Mandelbrot (1969) Long-run linearity, locally Gaussian processes, H-spectra and infinite variances. *International Economic Review*, **10**, 82-113.

Th. Mikosch, S. Resnick, H. Rootzén, A. Stegeman (2002) Is network traffic approximated by stable Lévy motion or fractional Brownian motion? *Ann. Appl. Probab.*, **12**(1), 23-68.

I. Norros (1995) On the use of fractional Brownian motion in the theory of connectionless networks. *IEEE J. Select. Areas Commun.*, **13**, 953-962.

V. Pipiras, M. S. Taqqu and J. B. Levy (2002) Slow, fast and arbitrary growth conditions for renewal reward processes when the renewals and the rewards are heavy-tailed. Boston University, preprint.

K. H. Rosen, J. G. Michaels, J. L. Gross, J. W. Grossman, D. R. Shier, eds. (2000) *Handbook of discrete and combinatorial mathematics.* Boca Raton: CRC Press.

G. Samorodnitsky and M. S. Taqqu (1994) *Stable non-Gaussian random processes. Stochastic models with infinite variance.* New York: Chapman and Hall.

M. S. Taqqu and J. Levy (1986) Using renewal processes to generate long-range dependence and high variability. In: E. Eberlein and M. S. Taqqu, eds. *Dependence in Probability and Statistics*. Boston: Birkhäuser, 73-89.

M. S. Taqqu, W. Willinger and R. Sherman (1997) Proof of a fundamental result in self-similar traffic modeling. *Computer Communications Review*, **27**(2), 5-23.

J. L. Teugels (1968) Renewal theorems when the first and the second moment is infinite. *Ann. Math. Statist.*, **39**, 1210-1219.

W. Willinger, M. S. Taqqu, R. Sherman, D. V. Wilson (1997) Self-similarity through high-variability: statistical analysis of Ethernet LAN traffic at the source level. *IEEE/ACM Transactions on Networking*, **5**(1), 71-86.