# A Poisson bridge between fractional Brownian motion and stable Lévy motion

Raimundas Gaigalas

Department of Mathematics, Uppsala University
Box 480, S-751 06 Uppsala, Sweden

19th September 2005

**Abstract**

We study a non-Gaussian and non-stable process arising as the limit of sums of rescaled renewal processes under the condition of intermediate growth. The process has been characterized earlier by the cumulant generating function of its finite-dimensional distributions. Here we derive a more tractable representation for it as a stochastic integral of a deterministic function with respect to a compensated Poisson random measure. Employing the representation we show that the process is locally and globally asymptotically self-similar with fractional Brownian motion and stable Lévy motion as its tangent limits.

*Keywords*: long-range dependence, asymptotic self-similarity, Poisson random measure, infinitely divisible process.

## 1 Introduction

We investigate further the stochastic process arising as the limit of sums of rescaled renewal processes under the intermediate growth condition (Gaigalas and Kaj, 2003). The same process also appears as the aggregation limit in the so called "infinite source Poisson" model under the equivalent growth condition (Kaj and Taqqu, 2004; Kaj, 2005). This process together with fractional Brownian motion and stable Lévy motion provide the three possible aggregation limits for these and other related models of computer network traffic (Willinger et al., 2003; Gaigalas and Kaj, 2003, and references therein). Being a non-Gaussian and non-stable process with stationary, but strongly dependent, increments, it has originally been characterized by the cumulant generating function of the finite-dimensional distributions. Here we derive a stochastic-integral representation for the process and study its local and global structure.

The cumulant generating function for the increments of the process

$\{Y_\alpha(t), t \geq 0\}$ is given in Gaigalas and Kaj (2003) as

$$
\begin{aligned}
\Gamma(\bar\theta, \bar t) &:= \log E \exp\Big\{ \sum_{i=1}^{n} \theta_i (Y_\alpha(t_i) - Y_\alpha(t_{i-1}))\Big\} \\
&= \frac{1}{\alpha-1} \sum_{i=1}^{n} \theta_i^2 \int_0^{t_i - t_{i-1}} dx \int_0^x dy \, \exp\{\theta_i y\}\, y^{1-\alpha} \\
&\quad + \frac{1}{\alpha-1} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \theta_i \theta_j \exp\{ \sum_{k=i+1}^{j-1} \theta_k (t_k - t_{k-1})\} \\
&\quad \times \int_0^{t_i - t_{i-1}} dx \int_0^{t_j - t_{j-1}} dy \, \exp\{\theta_i x + \theta_j y\} (t_{j-1} - t_i + x + y)^{1-\alpha}
\end{aligned} \tag{1}
$$

where $\bar\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$, $0 = t_0 \leq t_1 \leq \ldots \leq t_n$ and $1 < \alpha < 2$ is the regular variation exponent of the tails of the interarrival distribution in the prelimit model. Note that we adopt here a different parametrization of the process: the parameter $\beta$ in the original paper and $\alpha$ above are related by $\alpha = \beta + 1$.

It has also been shown earlier that the process $Y_\alpha(t)$

- is not self-similar;
- has finite moments of all orders:

$$
\begin{aligned}
EY_\alpha(t) &= C_1(t) = 0, \\
EY_\alpha(t)^k &= C_k(t) + \sum_{j=2}^{k-2} \binom{k-1}{j-1} C_j(t)\, EY_\alpha(t)^{k-j}, \quad k \geq 2,
\end{aligned} \tag{2}
$$

  where the cumulants

$$
C_k(t) = \frac{(k-1)k}{(\alpha-1)(k-\alpha)(k+1-\alpha)}\, t^{k+1-\alpha},
$$

  and for $k = 2, 3$ the sum in the second term is interpreted as zero;
- asymptotically,

$$
EY_\alpha(t)^k \sim C_k(t), \quad \text{as } t \to \infty, \tag{3}
$$

  for any integer $k \geq 2$, where $f(x) \sim cg(x)$ means $\lim f(x)/g(x) = c$.
- has the same covariance as a multiple of fractional Brownian motion of index $H = (3-\alpha)/2$:

$$
EY_\alpha(t) Y_\alpha(s) = \sigma_\alpha^2 \, (t^{3-\alpha} + s^{3-\alpha} - |t-s|^{3-\alpha}),
$$

  where $\sigma_\alpha^2 = 2((\alpha-1)(2-\alpha)(3-\alpha))^{-1}$;
- is continuous;

- is Hölder continuous of order $\gamma$, for any $0 < \gamma < H$.

  Note that in Gaigalas and Kaj (2003, Property 4) the upper bound for the Hölder continuity of the process is stated erroneously as 1. The Kolmogorov-Chentsov criterion used in the proof yields in fact only $H$. Indeed, this is a consequence of the inequality

  $$0 \leq EY_\alpha(t)^{2k} \leq B_{\alpha,2k}(T \vee 1)^{(k-1)(\alpha-1)}t^{(3-\alpha)k},$$

  valid for any $0 \leq t \leq T$, $k \geq 1$ and some constants $B_{\alpha,k}$, which can be derived from formula (2). Thus, the process $Y_\alpha$ has the same order of Hölder continuity as fractional Brownian motion.

- locally can be approximated by fractional Brownian motion of index $H = (3-\alpha)/2$:

  $$\lambda^{-H}Y_\alpha(\lambda u) \xrightarrow{fdd} \sigma_\alpha B_H(u), \quad \text{as } \lambda \downarrow 0.$$

# 2 An integral representation for the process $Y_\alpha(t)$

The main result of this work is that the process $Y_\alpha(t)$ admits a representation as a stochastic integral of a deterministic function with respect to a compensated Poisson random measure. We start by recalling some facts about Poisson random measures.

## 2.1 Poisson random measures and integrals

A recent comprehensive account on Poisson random measures can be found in Kallenberg (2002, Chapter 12). Here we follow the general theory of integration with respect to independently scattered random measures as constructed in Rajput and Rosiński (1989) or Kwapień and Woyczyński (1992).

A random set function $N(\cdot)$ on a measure space $(S, \mathcal{S}, n)$ is a *Poisson random measure* if (i) for any finite $A \in \mathcal{S}$, $N(A)$ is a $\mathrm{Po}(n(A))$-distributed random variable defined on the same probability space; (ii) for any disjoint finite $A_1, \ldots, A_n \in \mathcal{S}$, the random variables $N(A_1), \ldots, N(A_n)$ are independent; (iii) $N$ is a $\sigma$-additive set function, i.e. $N(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty N(A_i)$ a.s. for any disjoint finite sets $A_1, A_2, \ldots \in \mathcal{S}$. The measure $n(ds) = EN(ds)$ is called *the intensity measure* of $N$. A *compensated* Poisson random measure $\widetilde{N}(ds)$ with the intensity measure $n(ds)$ is defined as $\widetilde{N}(ds) = N(ds) - n(ds)$, where $N(ds)$ is a Poisson random measure.

The stochastic integral with respect to a compensated Poisson random measure $\widetilde{N}(ds)$ is constructed in a standard manner, starting from simple functions $f(s) = \sum_{i=1}^n c_i 1_{A_i}(s)$, where $A_i \in S$, for which

$$\int_S f(s)\widetilde{N}(ds) = \sum_{i=1}^n c_i \widetilde{N}(A_i).$$

A more general function $f : S \to \mathbb{R}$ is integrable with respect to the random measure $\widetilde{N}(ds)$ if there exists a sequence $\{f_k\}$ of simple functions such that $f_k \to f$ $n$-a.e. and the sequence $\{\int_S f_k(s)\,\widetilde{N}(ds)\}$ converges in probability.

By Kallenberg (2002, Lemma 12.13) or Rajput and Rosiński (1989, Section III), the integral $\int_S f(s)\,\widetilde{N}(ds)$ exists if and only if

$$\int_S (|f(s)|^2 \wedge |f(s)|)\, n(ds) < \infty. \tag{4}$$

The integral has zero-expectation whenever it exists.

Further, by Kallenberg (2002, Lemma 12.2), for any $f : [0,\infty) \times S \to \mathbb{R}$ such that $f(t,\cdot)$ satisfies condition (4) for all $t \geq 0$, the stochastic process

$$X(t) = \int_S f(t,s)\,\widetilde{N}(ds), \quad t \geq 0, \tag{5}$$

has the characteristic function

$$E\exp\{i\sum_{k=1}^n \theta_k X(t_k)\} = \exp\Big\{\int_S \Psi\Big(i\sum_{k=1}^n \theta_k f(t_k,s)\Big)\, n(ds)\Big\}, \tag{6}$$

where $(\theta_1,\dots,\theta_n)$ and $(t_1,\dots,t_n)$ are real numbers, and

$$\Psi(x) = e^x - 1 - x. \tag{7}$$

In particular, by differentiation, the covariance of the process $X$ is

$$E[X(t_1)X(t_2)] = \int_S f(t_1,s)f(t_2,s)\, n(ds). \tag{8}$$

### 2.2 An integral representation for the process $Y_\alpha(t)$

For $\bar\theta = (\theta_1,\dots,\theta_n)$, $\bar t = (t_1,\dots,t_n)$ denote

$$M(\bar\theta,\bar t) = \log E\exp\Big\{\sum_{i=1}^n \theta_i Y_\alpha(t_i)\Big\}.$$

**Theorem 1.** *The cumulant generating function $M(\bar\theta,\bar t)$ of the process $Y_\alpha(t)$ can be written as*

$$M(\bar\theta,\bar t) = \int_0^\infty dx \int_{\mathbb{R}} du\, \Psi\Big(\sum_{k=1}^n \theta_k h(t_k,x,u)\Big)\, \alpha x^{-\alpha-1}, \tag{9}$$

*where*

$$h(t,x,u) = ((t+u)\wedge 0 + x)_+ - (u \wedge 0 + x)_+,$$

*and $\Psi(x)$ is defined in (7). Hence, in the sense of finite-dimensional distributions the process $Y_\alpha(t)$ has representation*

$$Y_\alpha(t) = \int_0^\infty \int_{\mathbb{R}} (((t+u)\wedge 0 + x)_+ - (u\wedge 0 + x)_+)\,\widetilde{N}(dx,du), \tag{10}$$

*where $\widetilde{N}(dx,du) = N(dx,du) - n(dx,du)$ is a compensated Poisson random measure on $[0,\infty)\times\mathbb{R}$ with intensity measure*

$$n(dx,du) = \alpha x^{-\alpha-1}\, dxdu.$$

**The integration kernel.** For $x, t \geq 0,\ u \in \mathbb{R}$ the kernel $h(t,x,u)$ in (10) can be expressed in the following equivalent forms:

$$
\begin{aligned}
h(t,x,u) &= ((t+u)\wedge 0 + x)_+ - (u \wedge 0 + x)_+ \\
&= \int_0^x 1_{[-t,\,0]}(y+u)\,dy && (11)\\
&= \int_{-t}^0 1_{[0,x]}(y-u)\,dy && (12)\\
&= (x+u+t)_+ - (u+t)_+ - (x+u)_+ + u_+ && (13)\\
&= (x \wedge t \wedge (-u) \wedge (x+u+t))_+ && (14)\\
&= \begin{cases} -u & \text{if } -(x\wedge t) < u < 0,\ x > 0, \\ x & \text{if } -t < u < -x,\ 0 < x < t, \\ t & \text{if } -x < u < -t,\ x > t, \\ x+u+t & \text{if } -x-t < u < -(x \vee t),\ x > 0, \\ 0 & \text{otherwise.} \end{cases} && (15)
\end{aligned}
$$

**An alternative kernel.** Since the random measure $\widetilde{N}(dx,du)$ is shift-invariant with respect to the second variable, the change of variables $u' = -x-u$ in formula (9) yields an alternative representation for the process $Y_\alpha(t)$:

$$Y_\alpha(t) = \int_0^\infty \int_{\mathbb{R}} ((t-u)_+ \wedge x - (-u)_+ \wedge x)\,\widetilde{N}(dx,du).$$

As noted recently in Kaj and Taqqu (2004), this representation is a more appropriate one from applications view point, as it has a clear physical interpretation in the context of the infinite source Poisson model.

**A more symmetric measure.** Making the variable substitution $x' = -x-u,\ y' = -u$ in formula (9), we get an integral with respect to a more symmetric measure:

$$Y_\alpha(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} ((t \wedge y - x)_+ - (0 \wedge y - x)_+)\,\widetilde{L}(dx,dy),$$

where $\widetilde{L}(dx,dy) = L(dx,dy) - \ell(dx,dy)$ is a compensated Poisson random measure on $\mathbb{R}^2$ with intensity measure

$$\ell(dx,dy) = \alpha(y-x)_+^{-\alpha-1}\,dxdy.$$

### 2.3 Proof of Theorem 1

We start by the "final" expression (9), show that it is well-defined and then derive from it formula (1). The key-role is played by the function

$$R(\bar\theta, \bar t, x) = \int_{-\infty}^\infty \Psi\Big(\sum_{k=1}^n \theta_k h(t_k,x,u)\Big)\,du. \qquad (16)$$

**Lemma 1.** *For any $\bar{\theta} \in \mathbb{R}^n$, $0 = t_0 \leq t_1 \leq \ldots \leq t_n$, $x \geq 0$, the function $R(\bar{\theta}, \bar{t}, x)$ in (16)*

| | |
|---|---|
| *(i)* | *is well-defined;* |
| *(ii)* | $R(\bar{\theta}, \bar{t}, x) = \mathcal{O}(x^2)$, *as* $x \downarrow 0$; |
| *(iii)* | $R(\bar{\theta}, \bar{t}, x) = \mathcal{O}(x)$, *as* $x \to +\infty$; |
| *(iv)* | *is differentiable two times with respect to the variable* $x$*;* |
| *(v)* | $\frac{\partial}{\partial x} R(\bar{\theta}, \bar{t}, x) = \mathcal{O}(x)$, *as* $x \downarrow 0$; |
| *(vi)* | $\frac{\partial}{\partial x} R(\bar{\theta}, \bar{t}, x) = \mathcal{O}(0)$, *as* $x \to +\infty$. |

*Proof.* (i)-(iii) Observe that

$$R(\bar{\theta}, \bar{t}, x) = \int_0^{x+t_n} \Psi\Big( \sum_{k=1}^n \theta_k h(t_k, x, -u) \Big)\, du$$

and use the facts $|h(t, x, -u)| \leq (x \wedge t)_+$, $|\Psi(y)| \leq \Psi(|y|)$ to obtain

$$|R(\bar{\theta}, \bar{t}, x)| \leq (x + t_n) \Psi\Big( \sum_{k=1}^n |\theta_k| (x \wedge t_k) \Big). \tag{17}$$

This implies that $R(\bar{\theta}, \bar{t}, x)$ is well-defined and has the properties (ii) and (iii).

(iv) The differentiability of $R(\bar{\theta}, \bar{t}, x)$ follows from continuity of the functions $\Psi(x)$, $\Psi'(x)$, $h(t, x, u)$ and the property

$$\frac{\partial}{\partial x} h(t, x, u) = 1_{[-t,0]}(x + u),$$

derived from formula (11). With the change of variables $u' = -u - x$ and using that for $u \geq 0$, $h(t, x, -u - x) = x \wedge (t - u)_+$, we have

$$\frac{\partial}{\partial x} R(\bar{\theta}, \bar{t}, x) = \sum_{j=1}^n \theta_j \int_0^{t_j} \Big( \exp\{ \sum_{k=1}^n \theta_k (x \wedge (t_k - u)_+) \} - 1 \Big)\, du, \tag{18}$$

$$\frac{\partial^2}{\partial x^2} R(\bar{\theta}, \bar{t}, x) = \sum_{i=1}^n \sum_{j=1}^n \theta_i \theta_j \int_0^{t_j} 1_{[0,t_i]}(x + u) \exp\{ \sum_{k=1}^n \theta_k (x \wedge (t_k - u)_+) \}\, du. \tag{19}$$

(v)-(vi) Follows from (18) and the estimate

$$\Big|\frac{\partial}{\partial x} R(\bar{\theta}, \bar{t}, x)\Big| \leq \sum_{j=1}^n |\theta_j| t_j \Big( \exp\{ \sum_{k=1}^n |\theta_k| (x \wedge t_k) \} - 1 \Big). \quad \square$$

**Lemma 2.** *Formula (1) is equivalent to*

$$\begin{aligned} M(\bar\theta,\bar t) &= \frac{1}{\alpha-1}\sum_{i=1}^n\sum_{j=1}^n \theta_i\theta_j \qquad (20)\\ &\quad\times\int_0^{t_i}dx\int_0^{t_j}dy\exp\{\sum_{k=1}^n\theta_i(t_k\wedge y-x)_+\}(y-x)_+^{1-\alpha}. \end{aligned}$$

*Proof.* Follows from the relation $M(\bar\theta,\bar t)=\Gamma(\theta_1+\ldots+\theta_n,\theta_2+\ldots+\theta_n,\ldots,\theta_n)$ after rewriting (1) as

$$\begin{aligned} \Gamma(\bar\theta,\bar t) &= \frac{1}{\alpha-1}\sum_{i=1}^n\sum_{j=1}^n\theta_i\theta_j\exp\{\sum_{k=i}^{j-1}(\theta_k-\theta_{k+1})t_k\}\\ &\quad\times\int_{t_{i-1}}^{t_i}dx\int_{t_{j-1}}^{t_j}dy\ \exp\{\theta_j y-\theta_i x\}(y-x)_+^{1-\alpha}. \quad\Box \end{aligned}$$

**Proof of Theorem 1.** Since by property (i) of Lemma 1 the function $R(\bar\theta,\bar t,x)$ is well-defined, we can rewrite expression (9) as

$$M(\bar\theta,\bar t)=\alpha\int_0^\infty R(\bar\theta,\bar t,x)\,x^{-\alpha-1}\,dx.$$

Inserting estimate (17) and using the asymptotic properties of the function $\Psi(y)$ implies that the function $M(\bar\theta,\bar t)$ is also well-defined.

Furthermore, due to the differentibility of the function $R(\bar\theta,\bar t,x)$ we can integrate the above expression by parts. The asymptotic properties (ii), (iii), (v), (vi) from Lemma 1 yields

$$R(\bar\theta,\bar t,x)\,x^{-\alpha}\Big|_0^\infty=0,\quad \frac{\partial}{\partial x}R(\bar\theta,\bar t,x)\,x^{1-\alpha}\Big|_0^\infty=0,$$

whence

$$M(\bar\theta,\bar t)=\frac{1}{\alpha-1}\int_0^\infty\frac{\partial^2}{\partial x^2}R(\bar\theta,\bar t,x)\,x^{1-\alpha}\,dx.$$

It remains to insert formula (19) and make the change of variables $x'=x+u$ to get expression (20). This, in turn is equivalent to (1) by Lemma 2.$\Box$

# 3 Related processes and other properties

## 3.1 Fractional Brownian motion and stable Lévy motion

Three processes are known to appear as aggregation limits for sums of processes of counting type under different scaling conditions (Willinger et al., 2003; Gaigalas and Kaj, 2003, and references therein): fractional Brownian motion, stable Lévy motion and the process $Y_\alpha(t)$. Integral representations can serve as a unified framework to investigate all three processes and to understand the relation between them.

A representation for fractional Brownian motion as a double stochastic integral is given in Kurtz (1996, Section 4). Due to the properties of multiple Gaussian integrals, to derive such a representation, it is enough to factorize the covariance as an inner product in a selected $L_2$-space (see e.g. Nualart, 1995). Since the process $Y_\alpha(t)$ has the same covariance as fractional Brownian motion of index $H = (3-\alpha)/2$, formula (8) yields

$$EB_H(t)B_H(s) = \int_0^\infty dx \int_{\mathbb{R}} du\, h(t_1, x, u)h(t_2, x, u)\, \alpha x^{-\alpha-1},$$

resulting in a stochastic-integral representation

$$B_H(t) = \int_0^\infty \int_{\mathbb{R}} h(t, x, u)W(dx, du), \tag{21}$$

where $W(dx, du)$ is a Gaussian random measure on $[0,\infty) \times \mathbb{R}$ with control measure $x^{2H-4}\, dxdu$. Thus, we conclude that fractional Brownian motion and the process $Y_\alpha(t)$ have the same dependence structure and differ only in distribution of random "noise" used in their construction.

The third limit process, $\alpha$-stable Lévy motion has independent increments and hence can not be assigned the same integration kernel. Nevertheless, it can be considered as a degenerate case of the process below.

### 3.2 Stable Telecom process

The stable *Telecom process* has first been defined in Levy and Taqqu (2000) as one of the possible scaling limits of sums of heavy-tailed renewal reward processes. As proved in Pipiras and Taqqu (2000) (see also Pipiras and Taqqu (2004)), it can be written as the stochastic integral

$$Z_{\alpha,\beta}(t) = \int_0^\infty \int_{\mathbb{R}} h(t, x, u)\, M(dx, du), \tag{22}$$

where $\alpha$ and $\beta$ are the regular variation indices of the tails of the distributions of renewals and rewards respectively, subject to the condition $0 < \alpha < \beta < 2$, and $M(dx, du)$ is a symmetric $\beta$-stable random measure on $[0,\infty) \times \mathbb{R}$ with control measure $x^{-\alpha-1}dxdu$. From the point of view of limit results, natural extensions of the Telecom process for $\beta = \alpha$ is $\alpha$-stable Lévy motion and for $\beta = 2$ fractional Brownian motion. The process $Y_\alpha(t)$ can be regarded as such extension for $\beta = 0$, for the reason explained below.

Being a stable process, the Telecom process $Z_{\alpha,\beta}(t)$ can be expressed as an integral with respect to a compensated Poisson random measure. We shall compare such representation with the representation for the process $Y_\alpha(t)$. Indeed, if the distribution of the random measure $M(dx, du)$ in (22) is totally skewed to the right, then by Samorodnitsky and Taqqu (1994, Theorem 3.12.2),

$$Z_{\alpha,\beta}(t) = \int_0^\infty \int_{\mathbb{R}} \int_0^\infty h(t,x,u)w\, \widetilde{Q}(dx,du,dw),$$

where $\widetilde{Q}(dx,du,dw) = Q(dx,du,dw) - q(dx,du,dw)$ is a compensated Poisson random measure on $[0,\infty) \times \mathbb{R} \times [0,\infty)$ with intensity measure

$$q(dx,du,dw) = x^{-\alpha-1}dxdu\, w^{-\beta-1}dw.$$

On the other hand, (10) can be rewritten as

$$Y_\alpha(t) = \int_0^\infty \int_{\mathbb{R}} \int_0^\infty h(t,x,u)w\, \widetilde{R}(dx,du,dw),$$

where $\widetilde{R}(dx,du,dw) = R(dx,du,dw) - r(dx,du,dw)$ is a compensated Poisson random measure with intensity measure

$$r(dw,dx,du) = x^{-\alpha-1}dxdu\, \delta_1(dw).$$

### 3.3 Fractal sums of pulses

Following Cioczek-Georges and Mandelbrot (1996), for $\epsilon > 0$ and $t \geq 0$ consider the process

$$\mathcal{M}_\epsilon(t) = \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \epsilon\Big[p\Big(\frac{t-u}{x}\Big) - p\Big(\frac{-u}{x}\Big)\Big]xw\, \widetilde{N}_\epsilon(dx,du,dw), \tag{23}$$

where $\widetilde{N}_\epsilon(dx,du,dw)$ is a compensated Poisson random measure on $[0,\infty) \times \mathbb{R} \times \mathbb{R}$ with intensity measure

$$n_\epsilon(dx,du,dw) = \epsilon^{-2}x^{-\delta-1}\, dxduF(dw),$$

for some parameter $1 < \delta < 2$ and a probability measure $F(dw)$ with a finite second moment. The integrand $p(u)$ is a deterministic function satisfying the condition

$$\int_0^\infty dx \int_{\mathbb{R}} du \,\Big[p\Big(\frac{t-u}{x}\Big) - p\Big(\frac{-u}{x}\Big)\Big]^2 x^{1-\delta} < \infty. \tag{24}$$

It is proved in Cioczek-Georges and Mandelbrot (1996) that if the function $p(u)$ is taken to be a "pulse", i.e. it has finite support, then as $\epsilon \downarrow 0$,

$$\mathcal{M}_\epsilon(t) \stackrel{fdd}{\to} B_H(t), \tag{25}$$

where $B_H(t)$ is fractional Brownian motion of index $H = (3-\delta)/2$.

Going back to representation (10), we notice that

$$h(t,x,u) = \Big[g\Big(\frac{t-u}{x}\Big) - g\Big(\frac{-u}{x}\Big)\Big]x,$$

where $g(u) = (u \wedge 0 + 1)_+$. The function $g(u)$ satisfies the integrability condition (24) and hence the process

$$\mathcal{Y}_\epsilon(t) = \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \epsilon h(t,x,u) w \, \widetilde{N}_\epsilon(dx,du,dw) \stackrel{fdd}{=} \epsilon^{-\frac{3-\delta}{\delta-1}} Y_\delta(\epsilon^{\frac{2}{\delta-1}} t)$$

is well-defined and in the class (23). Furthermore, it follows from the results of Section 4 below that this process also shares property (25). Since the function $g(u)$ has an infinite support and a shape that reminds of a "step with a shifted upper part" rather than a "pulse", the process $\mathcal{Y}_\epsilon(t)$ can be regarded as an extension of the class of micropulses constructed in Cioczek-Georges and Mandelbrot (1996).

### 3.4 Infinite divisibility

Integral representation (10) implies that the process is infinitely divisible, i.e. all its finite-dimensional distributions are infinitely divisible. Expanding the function $\sum_{k=1}^n \theta_k h(t_k, x, u)$ in all different domains, it is possible to rewrite the cumulant generating function $M(\bar{\theta}, \bar{t})$ in the Lévy-Khinchine form. In the general case of $n$-dimensional distributions the resulting expression is quite cumbersome. For the marginal distributions it reads

$$\begin{aligned} \log E e^{\theta Y_\alpha(t)} &= \frac{t^{1-\alpha}}{\alpha - 1}(e^{\theta t} - 1 - \theta t) \\ &\quad + \int_0^t (e^{\theta x} - 1 - \theta x)(\alpha t x^{-\alpha-1} + (2-\alpha) x^{-\alpha})\, dx, \quad (26) \end{aligned}$$

corresponding to the Lévy measure

$$L_t(dx) = \frac{t^{1-\alpha}}{\alpha - 1}\delta_t(dx) + 1_{(0,t)}(x)(\alpha t x^{-\alpha-1} + (2-\alpha)x^{-\alpha})\, dx.$$

## 4 Local and global structure of the process $Y_\alpha(t)$

In this section we use the integral representation (10) to study local and global structure of the process $Y_\alpha(t)$. We show that it can be regarded as a "bridge" between fractional Brownian motion and stable Lévy motion and exhibits an intrinsic duality in its features.

### 4.1 Locally and globally asymptotically self-similar processes

A stochastic process $X$ is *locally asymptotically self-similar (lass)* at the point $t$ with index $H$ if there exists a process $T(u)$ such that

$$\frac{X(t+\lambda u) - X(t)}{\lambda^H} \xrightarrow{fdd} T(u), \quad \text{as } \lambda \downarrow 0,$$

where $\xrightarrow{fdd}$ means convergence of the finite-dimensional distributions. The process $T(u)$ is called the *tangent process* at the point $t$.

A stochastic process $X$ is *asymptotically self-similar at infinity (iass)* with index $H$ if there exists a process $R(u)$ such that

$$\lambda^{-H} X(\lambda u) \xrightarrow{fdd} R(u), \quad \text{as } \lambda \to +\infty.$$

The process $R(u)$ is called the *asymptotic process.*

Locally asymptotically self-similar processes were first formalized in Benassi et al. (1997) and Peltier and Lévy Véhel (1995) as a generalization of self-similar processes. Recently, such processes with càdlàg sample paths have been studied by Falconer (2003) in a more general setting. The processes asymptotically self-similar at infinity were defined in Benassi et al. (2002).

From applications point of view, a very interesting class of processes is that of "bridges" between two self-similar processes, i.e. those that are both lass and iass. Along those lines is the *real harmonizable fractional Lévy motion* constructed by Benassi et al. (2002) and the *moving average fractional Lévy motion* introduced by the same authors in Benassi et al. (2004). The process $Y_\alpha(t)$ is another example from this class, different from the other two.

## 4.2 Local scaling properties of the process $Y_\alpha(t)$

**Proposition 1.** *The process $Y_\alpha(t)$ is locally asymptotically self-similar at any point $t \geq 0$ with exponent $H = (3-\alpha)/2$ and fractional Brownian motion as the tangent process, that is*

$$\frac{Y_\alpha(t+\lambda u) - Y_\alpha(t)}{\lambda^H} \xrightarrow{fdd} \sigma_\alpha B_H(u), \quad \textit{as } \lambda \downarrow 0,$$

*where* $\sigma_\alpha^2 = 2((\alpha-1)(2-\alpha)(3-\alpha))^{-1}$.

Due to the stationarity of increments of $Y_\alpha(t)$, the proposition is equivalent to Gaigalas and Kaj (2003, Corollary 1), saying that

$$\lambda^{-H} Y_\alpha(\lambda u) \xrightarrow{fdd} \sigma_\alpha B_H(u), \quad \text{as } \lambda \downarrow 0.$$

For the sake of completness, we include here an alternative proof of this fact based on the integral representation. Heuristically, it can be derived by formal calculations involving stochastic integrals.

Indeed, the relation

$$h(\lambda t, \lambda x, \lambda u) = \lambda h(t, x, u)$$

and the change of variables $x' = \lambda^{-1} x$, $u' = \lambda^{-1} u$ implies that for any $\lambda > 0$,

$$\lambda^{-H} Y_\alpha(\lambda t) \overset{fdd}{=} \lambda^{\frac{\alpha-1}{2}} \int_0^\infty \int_{\mathbb{R}} h(t,x,u)\, \widetilde{N}(\lambda^{1-\alpha} dx, du). \tag{27}$$

Due to the central limit theorem, for any finite $A \in \mathcal{B}([0,\infty) \times \mathbb{R})$, as $r \to +\infty$,

$$r^{-1/2}\widetilde{N}(rA) \xrightarrow{d} W(A),$$

where $W(A)$ is a Gaussian random variable and hence $W(\cdot)$ is a Gaussian random measure. Since $h(t,x,u)$ is sufficiently regular, taking $\lambda \downarrow 0$ in (27) yields

$$\lambda^{-H} Y_\alpha(\lambda t) \xrightarrow{fdd} \int_0^\infty \int_{\mathbb{R}} h(t,x,u)\, W(dx,du).$$

Below we make these calculations precise.

**An alternative proof of Proposition 1.** For a fixed $\lambda > 0$ the cumulant generating function for the rescaled process $\lambda^{-H} Y_\alpha(\lambda t)$ reads

$$M(\lambda^{-H}\bar{\theta}, \lambda\bar{t}) = \int_0^\infty dx \int_{\mathbb{R}} du\, \Psi\Big(\sum_{k=1}^n \lambda^{-H}\theta_k h(\lambda t_k, x, u)\Big)\, \alpha x^{-\alpha-1}.$$

Making the change of variables $x' = \lambda^{-1}x$, $u' = \lambda^{-1}u$, using that

$$h(\lambda t, \lambda x, \lambda u) = \lambda h(t,x,u)$$

and inserting $H = (3-\alpha)/2$, we get

$$M(\lambda^{-H}\bar{\theta}, \lambda\bar{t}) = \lambda^{1-\alpha} \int_0^\infty dx \int_{\mathbb{R}} du\, \Psi\Big(\sum_{k=1}^n \lambda^{\frac{\alpha-1}{2}}\theta_k h(t_k, x, u)\Big)\, \alpha x^{-\alpha-1}.$$

Since for any real $u$, $\lim_{a\to 0} a^{-2}\Psi(au) = u^2/2$, taking $\lambda \downarrow 0$, the dominated convergence theorem yields

$$M(\lambda^{-H}\bar{\theta}, \lambda\bar{t}) \to \frac{1}{2}\int_0^\infty dx \int_{\mathbb{R}} du\, \Big(\sum_{k=1}^n \theta_k h(t_k, x, u)\Big)^2 \alpha x^{-\alpha-1}.$$

This expression is the cumulant generating function of a Gaussian process with covariance function

$$C(t_1,t_2) = \int_0^\infty dx \int_{\mathbb{R}} du\, h(t_1,x,u)h(t_2,x,u)\, \alpha x^{-\alpha-1}.$$

But due to formula (8) this is also the covariance of the process $Y_\alpha(t)$, which is equal to the covariance of fractional Brownian motion. □

### 4.3 Global scaling properties of the process $Y_\alpha(t)$

**Proposition 2.** *The process $Y_\alpha(t)$ is asymptotically self-similar at infinity with exponent $\kappa = 1/\alpha$ and $\alpha$-stable Lévy motion totally skewed to the right as the asymptotic process, that is*

$$\lambda^{-\kappa} Y_\alpha(\lambda t) \xrightarrow{fdd} c_\alpha \Lambda_\alpha(t), \quad \textit{as } \lambda \to +\infty,$$

*where* $c_\alpha = \big(-\cos(\pi\alpha/2)\Gamma(2-\alpha)/(\alpha-1)\big)^{1/\alpha}$ *and* $\Lambda_\alpha(t) \sim S_\alpha(t^{1/\alpha}, 1, 0)$.

Before proceeding with the proof, we give here a sketch of it in terms of integral representations.

The key observation is that writing the kernel in the form (11) and making the substitution $y' = \lambda^{-\frac{1}{\alpha}} y$, for any $t, x \geq 0$, $u \in \mathbb{R}$ we obtain

$$\lim_{\lambda \to +\infty} \lambda^{-\frac{1}{\alpha}} h(\lambda t, \lambda^{\frac{1}{\alpha}} x, \lambda u) = \lim_{\lambda \to +\infty} \int_0^x 1_{[-t,\,0]}(\lambda^{\frac{1}{\alpha}-1} y + u)\, dy = x 1_{[-t,\,0]}(u). \tag{28}$$

On the other hand, for any fixed $\lambda > 0$ the change of variables $x' = \lambda^{-\frac{1}{\alpha}} x$, $u' = \lambda^{-1} u$ in representation (10) implies

$$\lambda^{-\kappa} Y_\alpha(\lambda t) \stackrel{fdd}{=} \int_0^\infty \int_{\mathbb{R}} \lambda^{-\frac{1}{\alpha}} h(\lambda t, \lambda^{\frac{1}{\alpha}} x, \lambda u)\, \widetilde{N}(dx, du).$$

Combining with (28), as $\lambda \to +\infty$, we get

$$\lambda^{-\kappa} Y_\alpha(\lambda t) \stackrel{fdd}{\to} \int_0^\infty \int_{-t}^0 x\, \widetilde{N}(dx, du) \stackrel{fdd}{=} c_\alpha \int_{-t}^0 \Lambda_\alpha(du) \stackrel{fdd}{=} c_\alpha \Lambda_\alpha(t).$$

**Proof of Proposition 2.** For $\bar\theta = (\theta_1, \ldots, \theta_n)$ and $\bar t = (t_1, \ldots, t_n)$ consider

$$\begin{aligned}
\Lambda(\bar\theta, \bar t) \quad &:= \quad \log E \exp\Big\{ i \sum_{k=1}^n \theta_k Y_\alpha(t_k) \Big\} \\
&= \quad \Big( \sum_{j=1}^n \int_{t_{j-1}}^{t_j} du \int_0^{u-t_{j-1}} dx + \sum_{j=1}^n \int_{t_{j-1}}^{t_j} du \int_{u-t_{j-1}}^\infty dx \\
&\qquad + \int_{t_n}^\infty du \int_{u-t_n}^\infty dx \Big) \Psi\Big( i \sum_{k=1}^n \theta_k h(t_k, x, -u) \Big)\, \alpha x^{-\alpha-1} \\
&=: \quad I_1(\bar\theta, \bar t) + I_2(\bar\theta, \bar t) + I_3(\bar\theta, \bar t).
\end{aligned}$$

Given $\lambda > 0$, we are interested in the function $\Lambda(\lambda^{-\kappa}\bar\theta, \lambda\bar t)$ corresponding to the rescaled process $\lambda^{-\kappa} Y_\alpha(\lambda t)$.

Making the change of variables $x' = \lambda^{-1} x$, $u' = \lambda^{-1} u$ and employing the facts that for any $y \in \mathbb{R}$, $|\Psi(iy)| \leq 2|y|$ and $|h(t, x, u)| \leq (x \wedge t)_+$ yields

$$\begin{aligned}
&|I_2(\lambda^{-\kappa}\bar\theta, \lambda\bar t) + I_3(\lambda^{-\kappa}\bar\theta, \lambda\bar t)| \\
&\leq \; 2\lambda^{2-\alpha-\kappa} \sum_{j=1}^n \sum_{k=1}^n |\theta_k| \int_{t_{j-1}}^{t_j} du \int_{u-t_{j-1}}^\infty dx\, (x \wedge t_k)\, \alpha x^{-\alpha-1} \\
&\qquad + 2\lambda^{2-\alpha-\kappa} \sum_{k=1}^n |\theta_k| \int_{t_n}^\infty du \int_{u-t_n}^\infty dx\, (x \wedge t_k)\, \alpha x^{-\alpha-1}.
\end{aligned}$$

Since both integrals on the right-hand side are finite and $2 - \alpha - \kappa = -(\alpha - 1)^2/\alpha < 0$, taking $\lambda \to +\infty$, we obtain

$$I_2(\lambda^{-\kappa}\bar\theta, \lambda\bar t) + I_3(\lambda^{-\kappa}\bar\theta, \lambda\bar t) \to 0.$$

Turning to the term $I_1(\lambda^{-\kappa}\bar\theta, \lambda\bar t)$, observe that in the integration domain $\{t_{j-1} \le u \le t_j, 0 \le x \le u - t_{j-1}\}$, $1 \le j \le n$,

$$\sum_{k=1}^{n} \theta_k h(t_k, x, -u) = x \sum_{k=j}^{n} \theta_k = x \sum_{k=1}^{n} \theta_k 1_{[0,\,t_k]}(u).$$

Hence, variable substitution $x' = \lambda^{-\kappa}x$, $u' = \lambda^{-1}u$ gives

$$I_1(\lambda^{-\kappa}\bar\theta, \lambda\bar t) = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} du \int_0^{\lambda^{1-\kappa}(u-t_{j-1})} dx\, \Psi\Big(ix \sum_{k=1}^{n} \theta_k 1_{[0,\,t_k]}(u)\Big)\, \alpha x^{-\alpha-1}.$$

Since $1-\kappa > 0$, and the real and imaginary parts of the integrand are monotone functions with respect to $x$, by the monotone convergence theorem, as $\lambda \to +\infty$,

$$I_1(\lambda^{-\kappa}\bar\theta, \lambda\bar t) \to \int_0^{t_n} du \int_0^{\infty} dx\, \Psi\Big(ix \sum_{k=1}^{n} \theta_k 1_{[0,\,t_k]}(u)\Big)\, \alpha x^{-\alpha-1}.$$

By Samorodnitsky and Taqqu (1994, Exercise 3.24), this is the logarithm of the characteristic function of $c_\alpha \Lambda_\alpha(t)$. $\square$

### 4.4 Absolute moments of small orders

Recall formula (3), showing the asymptotic behaviour of the moments of the process of order $k \ge 2$. Due to the asymptotic self-similarity at infinity, such behaviour is different for the absolute moments of orders $0 < p < \alpha$.

**Corollary 1.** *Let $\Lambda_\alpha(t)$ be stable Lévy motion totally skewed to the right, i.e. $\Lambda_\alpha(t) \sim S_\alpha(t^{1/\alpha}, 1, 0)$ and $c_\alpha$ be defined as in Proposition 2. For $0 < r < \alpha$, the absolute moments of the process $Y_\alpha(t)$ satisfy the relation*

$$E|Y_\alpha(t)|^r \sim c_\alpha^r E|\Lambda_\alpha(t)|^r = \rho_{\alpha,r} t^{\frac{r}{\alpha}}, \quad \text{as } t \to \infty,$$

*where $\rho_{\alpha,r} = c_\alpha^r E|\Lambda_\alpha(1)|^r$.*

To derive the corollary, we need some bounds for the moments implied by the following estimate.

**Lemma 3.** *For $\theta \in \mathbb{R}$, $t \ge 0$ the characteristic function $\Phi(\theta, t)$ of $Y_\alpha(t)$ satisfies*

$$|\Phi(\theta, t)|^2 \ge \exp\{-2d_\alpha\, t|\theta|^\alpha\},$$

*where $d_\alpha = c_\alpha^\alpha + 2^{1-\alpha}(\alpha-1)^{-1}(3\alpha - 1 - \alpha^2)$ with $c_\alpha$ defined in Proposition 2.*

*Proof.* Employing (26), we have

$$|\Phi(\theta, t)|^2 = \exp\{-2(J_1(\theta, t) + J_2(\theta, t) + J_3(\theta, t))\},$$

where

$$\begin{aligned} J_1(\theta,t) &= \alpha t \int_0^t (1-\cos(\theta x))\, x^{-\alpha-1}\, dx, \\ J_2(\theta,t) &= \frac{t^{1-\alpha}}{\alpha-1}(1-\cos(\theta t)), \\ J_3(\theta,t) &= (2-\alpha)\int_0^t (1-\cos(\theta x))\, x^{-\alpha}\, dx. \end{aligned}$$

The fact that for any $u \in \mathbb{R}$ and $0 < r < 2$,

$$\int_0^\infty (1-\cos(xu)))\, x^{-r-1}\, dx = q_r |u|^r,$$

where

$$q_r = r^{-1} c_r^r = \int_0^\infty (1-\cos(x))\, x^{-r-1}\, dx, \tag{29}$$

yields the estimate

$$|J_1(\theta,t)| \le \alpha t \int_0^\infty (1-\cos(\theta x))\, x^{-\alpha-1}\, dx = c_\alpha^\alpha\, t|\theta|^\alpha.$$

The inequality $|1-\cos x| \le 2^{1-\alpha}|x|^\alpha$, valid for $x \in \mathbb{R}$ and $1 \le \alpha \le 2$ gives the bounds for the remaining terms:

$$\begin{aligned} |J_2(\theta,t)| &\le 2^{1-\alpha}(\alpha-1)^{-1} t|\theta|^\alpha, \\ |J_3(\theta,t)| &\le 2^{1-\alpha}(2-\alpha) t|\theta|^\alpha. \quad \square \end{aligned}$$

**Proof of Corollary 1.** By Proposition 2, the random variables $\{t^{-\kappa} Y_\alpha(t)\}$ converge in distribution to the random variable $c_\alpha \Lambda_\alpha(1)$, as $t \to +\infty$. The statement of Corollary 1 is just another way of writing that the absolute moments of $t^{-\kappa} Y_\alpha(t)$ also converge to the corresponding moments of $c_\alpha \Lambda_\alpha(1)$. The second equality follows from the expression for the moments of an $\alpha$-stable random variable given in Samorodnitsky and Taqqu (1994, Property 1.2.17).

We shall prove that for any $1 \le p < \alpha$,

$$\sup_{t \ge 0} E|t^{-\kappa} Y_\alpha(t)|^p \le E|Z|^p, \tag{30}$$

where $Z$ is a $S_\alpha(d_\alpha, 0, 0)$-distributed random variable, with $d_\alpha$ given in Lemma 3. As known, this implies convergence of moments of order $0 < r < p$ (e.g. Chung, 1974, Theorem 4.5.2).

By an elegant Lemma 2 in von Bahr and Esseen (1965), for a random variable $X$ and $0 < r < 2$,

$$E|X|^r = q_r^{-1} \int_0^\infty (1 - \mathcal{R}(\phi_X(\theta)))\, \theta^{-r-1}\, d\theta, \tag{31}$$

where $q_r$ is defined by (29). Further, by Lemma 4 of the same authors, if $EX = 0$, then for $1 \le r < 2$,

$$E|X|^r \le E|\tilde{X}|^r,$$

where $\tilde{X} = X - X'$ has the symmetrized distribution, i.e. $\tilde{X}$ has the characteristic function $|\phi_X(\theta)|^2$. Hence, by symmetrization, for any $t \ge 0$,

$$E|t^{-\kappa} Y_\alpha(t)|^p \le E|t^{-\kappa} \tilde{Y}_\alpha(t)|^p = q_p^{-1} \int_0^\infty (1 - |\Phi(t^{-\kappa}\theta, t)|^2)\, \theta^{-p-1}\, d\theta,$$

where $\Phi(\theta, t) = E \exp\{i\theta Y_\alpha(t)\}$. Now due to Lemma 3, for any $t \ge 0$, $|\Phi(t^{-\kappa}\theta, t)|^2 \ge \exp\{-2d_\alpha|\theta|^\alpha\}$, which gives

$$E|t^{-\kappa} Y_\alpha(t)|^p \le q_p^{-1} \int_0^\infty (1 - \exp\{-2d_\alpha|\theta|^\alpha\})\, \theta^{-p-1}\, d\theta = E|Z|^p.$$

Here the last equality is obtained by formula (31) applied to the stable random variable $Z$. This completes the proof. □


# Acknowledgment

Since this paper is a part of my PhD thesis defended at Uppsala University, I would like to thank my supervisor Ingemar Kaj who “always has time for his students”. I am also grateful to Murad S.Taqqu and Vladas Pipiras for many fruitful discussions while writing the paper. A special credit should be given to Lasse Leskelä for pointing out a mistake in the construction of the space of the integrands.


# References


Benassi, A., Cohen, S., and Istas, J. (2002). Identification and properties of real harmonizable fractional Lévy motions. *Bernoulli*, 8(1):97–115.

Benassi, A., Cohen, S., and Istas, J. (2004). On roughness indices for fractional fields. *Bernoulli*, 10(2):357–373.

Benassi, A., Jaffard, S., and Roux, D. (1997). Elliptic Gaussian random processes. *Rev. Mat. Iberoamericana*, 13(1):19–90.

Chung, K. L. (1974). *A course in probability theory.* Academic Press, New York, second edition.

Cioczek-Georges, R. and Mandelbrot, B. B. (1996). Alternative micropulses and fractional Brownian motion. *Stochastic Process. Appl.*, 64(2):143–152.

Falconer, K. J. (2003). The local structure of random processes. *J. London Math. Soc. (2)*, 67(3):657–672.

Gaigalas, R. and Kaj, I. (2003). Convergence of scaled renewal processes and a packet arrival model. *Bernoulli*, 9(4):671–703.

Kaj, I. (2005). Limiting fractal random processes in heavy-tailed systems. In *Fractals in Engineering, New Trends in Theory and Applications*, pages 199–218. Springer-Verlag, London.

Kaj, I. and Taqqu, M. S. (2004). Convergence to fractional Brownian motion and to the Telecom process: the integral representation approach. Technical report, no. 2004:16. Uppsala University, Department of Mathematics.

Kallenberg, O. (2002). *Foundations of modern probability.* Springer-Verlag, New York, second edition.

Kurtz, T. G. (1996). Limit theorems for workload input models. In *Stochastic networks: theory and applications*, pages 119–139. Clarendon Press, Oxford.

Kwapień, S. and Woyczyński, W. A. (1992). *Random series and stochastic integrals: single and multiple.* Birkhäuser Boston, Boston, MA.

Levy, J. B. and Taqqu, M. S. (2000). Renewal reward processes with heavy-tailed inter-renewal times and heavy-tailed rewards. *Bernoulli*, 6(1):23–44.

Nualart, D. (1995). *The Malliavin calculus and related topics.* Springer-Verlag, New York.

Peltier, R. and Lévy Véhel, J. (1995). Multifractional Brownian motion: definition and preliminary results. Technical report, no. 2645, INRIA. Available at `http://www-syntim.inria.fr/fractales/`.

Pipiras, V. and Taqqu, M. S. (2000). The limit of a renewal reward process with heavy-tailed rewards is not a linear fractional stable motion. *Bernoulli*, 6(4):607–614.

Pipiras, V. and Taqqu, M. S. (2004). Dilated fractional stable motions. *J. Theoret. Probab.*, 17(1):51–84.

Rajput, B. S. and Rosiński, J. (1989). Spectral representations of infinitely divisible processes. *Probab. Theory Related Fields*, 82(3):451–487.

Samorodnitsky, G. and Taqqu, M. S. (1994). *Stable non-Gaussian random processes.* Chapman and Hall, New York.

von Bahr, B. and Esseen, C. G. (1965). Inequalities for the $r$th absolute moment of a sum of random variables, $1 \leq r \leq 2$. *Ann. Math. Statist*, 36:299–303.

Willinger, W., Paxson, V., Riedi, R. H., and Taqqu, M. S. (2003). Long-range dependence and data network traffic. In *Theory and applications of long-range dependence*, pages 373–407. Birkhäuser Boston, Boston, MA.