A BOUNDARY POINT LEMMA FOR BLACK-SCHOLES TYPE OPERATORS

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Abstract. We prove a sharp version of the Hopf boundary point lemma for Black-Scholes type equations. We also investigate the existence and the regularity of the spatial derivative of the solutions at the spatial boundary.

1. Introduction

In this paper we study a parabolic initial-boundary value problem motivated by applications involving modeling of non-negative quantities described by diffusion processes that are absorbed at 0. To describe the problem in a financial setting, consider a market consisting of a risk-free asset

$$dB = r(t)B\,dt$$

where r is a deterministic function, and n risky assets with non-negative prices X_i, $i = 1, ..., n$, modeled by diffusion processes. According to standard arbitrage theory, to price options maturing at some future time T written on such risky assets, we need not specify the expected rate of return of the assets, compare for example Theorem 7.8 and Proposition 7.9 in [2] and Proposition 2.2.3 in [10]. Instead one specifies the price dynamics under a so-called risk-neutral measure, under which the expected rate of return of all assets equals the interest rate $r(t)$. For the sake of notational convenience, however, replacing $X_i(t)$ by $X_i(t)\exp\{\int_t^T r(s)\,ds\}$, we obtain processes with drift 0. Financially this corresponds to quoting the assets in terms of bonds maturing at time T. By abuse of notation, we denote also these new processes by X_i. Thus we model the price dynamics of the ith asset by

$$dX_i = \sum_{j=1}^{n} \alpha_{ij}(X_i, t)\,dW_j \qquad X_i(t) = x_i.$$

Here W_j, $j = 1, ..., n$, are independent standard Brownian motions, and $\alpha = (\alpha_{ij})_{i,j=1}^n$ is for each x and t an $n\times n$-matrix. We assume that the components of $X = (X_1, ..., X_n)$ are absorbed at 0, i.e. if some component at some point reaches 0, then it remains 0 forever. Therefore we let $\alpha_{ij}(0, t) = 0$ for all i

2000 *Mathematics Subject Classification.* Primary 35K65, 35B50; Secondary 91B28.

Key words and phrases. Hopf boundary point lemma, parabolic equations, degenerate equations, Black-Scholes equation.

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and j. We also assume that the rank of α is equal to the number of non-zero spatial coordinates (this assumption is connected to the completeness of the model; more precisely, under this assumption every option has a unique arbitrage-free price, compare Corollary 12.2.6 in [13] or Theorem 1.6.6 in [10]). Further assumptions on α are specified below. Given a continuous contract function $g : [0,\infty)^n \to \mathbb{R}$ of at most polynomial growth, the value $U(x,t)$ at time t of an option paying $g(X(T))$ at time T is

$$U(x,t) := E_{x,t}\, g\big(X(T)\big),$$

where the indices indicate that $X(t) = x$. Alternatively, the function $u(x,t) := U(x, T-t)$ solves the Black-Scholes equation

$$\begin{cases} \mathcal{L}u = 0 \text{ for } (x,t) \in (0,\infty)^n \times (0,\infty) \\ u(x,0) = g(x), \end{cases} \tag{1}$$

where

$$\mathcal{L}u = \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} - u_t,$$

$a_{ij} = a_{ij}(x,t)$ are the elements of the matrix

$$a(x,t) := \alpha(x, T-t)\alpha^*(x, T-t)/2$$

and

$$u_t := \frac{\partial u}{\partial t}, \qquad u_{x_i x_j} := \frac{\partial^2 u}{\partial x_j \partial x_j}.$$

In order to have uniqueness of solutions to the problem (1) in the class of functions with at most polynomial growth, one needs (in general) to impose boundary conditions at the faces $\{x_i = 0\}$, $i = 1, ..., n$. We assume that these boundary conditions are defined inductively by solving the partial differential equation in lower dimensional faces (the assumptions on α guarantee that the operator is parabolic in these faces). One thus starts with solving an ordinary differential equation ($u_t = 0$) along the t-axis, then one solves n parabolic equations in the faces spanned by t and one of the variables $x_1, ..., x_n$, and so on. According to Theorems 4.1 and 5.5 in [9], this procedure results in a unique classical solution of polynomial growth to equation (1), and this solution satisfies $u(x,t) = U(x, T-t)$.

In this paper we are concerned with the behavior of the first order derivatives u_{x_i} at the boundary. We should note that the derivatives u_{x_i} are of special importance in financial applications. They are the so-called "deltas" of the option, and they represent the number of stocks X_i a hedger should have in a hedging portfolio, compare for example Theorem 8.5 in [2].

The following theorem, the Hopf boundary point lemma (adapted to our current setting), is well-known in the theory of parabolic equations, compare for example [6] (Theorem 2.14, p. 49) or [11] (Lemma 2.6, p. 10).

Theorem 1.1. (Hopf boundary point lemma.) *Assume that the differential operator is uniformly parabolic, i.e. that there exists $\gamma > 0$ such that $\xi a(x,t)\xi^* \geq \gamma|\xi|^2$ for all $\xi \in \mathbb{R}^n$, x and t. Let a point $P' = (0, x_2', ..., x_n', t_0)$ with $x_2', ..., x_n', t_0 > 0$ be given. Assume that in a neighborhood of P' we have*

$u(P) > u(P')$ for all points P with $x_1 > 0$ and $t \le t_0$. Then $u_{x_1}(P') > 0$ in the sense that

$$\liminf_{\epsilon \to 0} \frac{1}{\epsilon}\Big(u(P' + \epsilon e_1) - u(P')\Big) > 0,$$

where e_1 is the unit vector in the x_1-direction.

The following well-known example shows that the above theorem is not valid without the assumption of uniform parabolicity.

Example (The Black-Scholes price of a call option.) Let $n = 1$, and let $\alpha(x,t) = \sigma x$ and $g(x) = (x - K)^+$ where σ and K are positive constants (here and in the sequel we drop the subscripts of α_{11}, a_{11} and x_1 if $n = 1$). Then

$$u(x,t) = x\Phi\Big(\frac{\ln(x/K) + \sigma^2 t/2}{\sqrt{\sigma^2 t}}\Big) - K\Phi\Big(\frac{\ln(x/K) - \sigma^2 t/2}{\sqrt{\sigma^2 t}}\Big)$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} \exp\{-u^2/2\}\, du.$$

It is now straightforward to check that the delta of the call option is given by

$$u_x(x,t) = \Phi\Big(\frac{\ln(x/K) + \sigma^2 t/2}{\sqrt{\sigma^2 t}}\Big),$$

so $u_x(0,t) = 0$.

The outline of the present article is as follows. In Section 2 we provide a Hopf lemma for equations that are not uniformly parabolic, compare Theorems 2.2 and 2.4. To prove these results we need to assume that $a_{11} \ge C x_1^{\beta}$ for some $0 < \beta < 2$. In view of the above example (in which $a_{11} = \sigma^2 x_1^2/2$), our result can be regarded as a sharp version of the Hopf lemma. Along the lines of the above example we also show that if $n = 1$ and $a \le Cx^2$, then the result of the Hopf lemma always fails, compare Theorem 2.3.

In Section 3 we perform further investigations of the spatial derivative at the boundary. We use preservation of convexity in dimension one to show that $u_x(0,t)$ always exists finitely for a wide class of initial conditions, and we provide an example in which the function $t \mapsto u_x(0,t)$ is discontinuous.

2. Hopf boundary point lemma

In this section we prove a Hopf boundary point lemma for non-uniformly parabolic operators.

Hypothesis 2.1. *The $n \times n$-matrix $\alpha = \big(\alpha_{ij}(x,t)\big)_{i,j=1}^n$ is defined on $[0,\infty)^n \times (-\infty, T]$, and*

(i) *is continuous in the time variable, and α is also Lipschitz continuous in the spatial variables on every compact subset of $(0,\infty)^n \times (-\infty, T]$; we also assume the standard growth condition $|\alpha(x,t)| \le C(1 + |x|)$ for some constant C;*

(ii) *for each pair (i,j), α_{ij} is a function merely of x_i and t (this is automatic if $n = 1$), and for $x_i = 0$ we have $\alpha_{ij} = 0$;*

(iii) *the rank of α is equal to the number of non-zero spatial coordinates.*

If $n = 1$ the local Lipschitz condition may be replaced with a local Hölder(1/2) condition.

Note that we allow discontinuities of the coefficients at the spatial boundary. Also note that condition (ii) does of course not rule out the possibility of dependence between the different assets. It merely means that the instantaneous covariance a_{ij} between two assets X_i and X_j only depends on time and the present values of X_i and X_j. The condition (iii) is connected to the issue of completeness of the model, compare the discussion in the introduction.

Theorem 2.2. *Assume Hypothesis 2.1 and that u satisfies the Black-Scholes equation (1). Let a point $P' = (0, x_2', ..., x_n', t_0)$ with $t_0 > 0$ and $x_i' > 0$ for $i = 2, ..., n$ be given. Assume that in some neighborhood of P' we have*

$$u(x_1, x_2, ..., x_n, t) > u(0, x_2, ..., x_n, t)$$

and $a_{11}(x_1, t) \geq C x_1^\beta$ for some constants $C > 0$ and $\beta \in [0, 2)$ for all points with $x_1 > 0$ and $t \leq t_0$. Then $u_{x_1}(P') > 0$ (in the same sense as described in Theorem 1.1).

Proof. Introduce the function

$$v(x,t) := x_1 + x_1^{1+\epsilon} - |t - t_0|^N - \sum_{i=2}^n |x_i - x_i'|^{2N}$$

for some constants $\epsilon > 0$ and $N \geq 1$, both to be chosen later. Let

$$D := \{P : v(P) \geq 0, 0 \leq x_1 \leq \eta, t \leq t_0\}$$

for some small constant $\eta > 0$. Then there exists a positive constant C_1 (depending on η) such that

$$-C_1 x_1^{1/N} \leq t - t_0 \leq 0$$

and

$$-C_1 x_1^{1/(2N)} \leq -|x_i - x_i'| \leq 0,$$

$i = 2, ...n$, for points in D. Consequently,

$$\begin{aligned} \mathcal{L}v &= \sum_{i,j=1}^n a_{ij} v_{x_i x_j} - v_t = \sum_{i=1}^n a_{ii} v_{x_i x_i} - v_t \\ &= a_{11}(1+\epsilon)\epsilon x_1^{\epsilon - 1} - N|t - t_0|^{N-1} - (4N^2 - 2N) \sum_{i=2}^n a_{ii} |x_i - x_i'|^{2N-2} \\ &\geq C(1+\epsilon)\epsilon x_1^{\beta + \epsilon - 1} - N C_2 x_1^{(N-1)/N} - (4N^2 - 2N) C_2 x_1^{(N-1)/N} \end{aligned}$$

in D for some constant C_2 satisfying

$$C_2 > \max \left\{ C_1^{N-1}, C_1^{N-2}(n-1) \max_{2 \leq i \leq n} \{a_{ii}(P')\} \right\}.$$

Thus, choosing ϵ small and N large so that $\beta + \epsilon - 1 < (N-1)/N < 1$, it is clear that $\mathcal{L}v \geq 0$ in D (at least if η is small enough; note that C_1, and thus also C_2, can be held fixed when decreasing η). The parabolic boundary $\partial_p D$ of D can be written as $\partial_p D = S_1 \cup S_2$ where

$$S_1 = \{P : v(P) = 0, t \leq t_0, x_1 \leq \eta\}$$

and

$$S_2 = \{P : v(P) \geq 0, t \leq t_0, x_1 = \eta\}.$$

Since, by assumption,

$$u(x_1, x_2, ..., x_n, t) > u(0, x_2, ..., x_n, t) \text{ for } (x_1, x_2, ..., x_n, t) \in D \setminus \{P'\}$$

there exists $\delta > 0$ such that $u(x_1, x_2, ..., x_n, t) - u(0, x_2, ..., x_n, t) \geq \delta$ for $(x_1, x_2, ..., x_n, t) \in S_2$. Since v is bounded by 1 in D (at least if η is small enough) it follows that

(2) $u(x_1, x_2, ..., x_n, t) - u(0, x_2, ..., x_n, t) - \delta v(x_1, x_2, ..., x_n, t) \geq 0$ on $\partial_p D$.

Moreover, applying the differential operator

$$\mathcal{L} = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial}{\partial t}$$

in D to the function $u(0, x_2, ..., x_n, t)$ we get

$$(3) \quad \mathcal{L}\Big(u(0, x_2, ..., x_n, t)\Big) = \sum_{i,j=2}^{n} a_{ij}(x_i, x_j, t) \frac{\partial^2}{\partial x_i \partial x_j} u(0, x_2, ..., x_n, t) - \frac{\partial}{\partial t} u(0, x_2, ..., x_n, t) = 0$$

since $u(0, x_2, ..., x_n, t)$ satisfies the $(n-1)$-dimensional Black-Scholes equation in the face $x_1 = 0$. Thus we have

(4) $\mathcal{L}\Big(u(x_1, x_2, ..., x_n, t) - u(0, x_2, ..., x_n, t) - \delta v(x_1, x_2, ..., x_n, t)\Big) \leq 0$ in D.

Applying the weak maximum principle to the inequalities (2) and (4) we find that

$$u(x_1, x_2, ..., x_n, t) - u(0, x_2, ..., x_n, t) - \delta v(x_1, x_2, ..., x_n, t) \geq 0 \text{ in } D.$$

Using $v(P') = 0$ we get

$$u_{x_1}(P') \geq \delta v_{x_1}(P') = \delta > 0$$

which finishes the proof. □

Remark Note that the assumption that α_{ij} is a function merely of x_i and t (condition (ii) in Hypothesis 2.1) is essential. Indeed, if α instead would depend on the whole vector x and t, then the equality (3) would not be true in general.

Remark Theorem 2.2 also holds if one includes lower order terms in the differential operator $\mathcal{L}$. In addition to the assumptions on the coefficients specified in Theorem 2.4 below, one also needs to assume that c and b_i, $i = 2, ..., n$ are independent of x_1, where c and b_i are as defined in that theorem.

The theorem also generalizes to any non-tangential direction with non-positive time-component. In that case, N has to be chosen strictly larger than 1.

To illustrate the result of Theorem 2.2 we give two examples.

Example (The Margrabe exchange option.) Let $n = 2$, assume that

$$dX_i = \sigma_i X_i^{\beta_i/2}\, dW_i,$$

where $\sigma_i > 0$ and $0 \leq \beta_i < 2$ for $i = 1, 2$, and let $g(x_1, x_2) = (x_2 - x_1)^+$. From Theorem 2.2 it follows that $u_{x_2}(x_1, 0, t) > 0$ for $x_1 > 0$ and $t > 0$. To investigate the derivative at the boundary $x_1 = 0$, note that adding the affine function x_1 to the contract function g gives $(x_2 - x_1)^+ + x_1 = \max\{x_1, x_2\}$. By Theorem 2.2 the value $\tilde{u}$ of this contract has a spatial derivative $\tilde{u}_{x_1}(0, x_2, t) > 0$. It follows that $u_{x_1} = \frac{\partial}{\partial x_1}(\tilde{u} - x_1) > -1$ for points $(0, x_2, t)$.

On the other hand, if $\beta_1 = \beta_2 = 2$, i.e. if X_1 and X_2 are geometric Brownian motions, then standard formulas for the value of the exchange option (compare [12]) can be used to show that $u_{x_2}(x_1, 0, t) = 0$ and $u_{x_1}(0, x_2, t) = -1$.

Example (The call option in a CEV-model.) Let $n = 1$, $g(x) = (x - K)^+$ and

$$dX = \sigma X^{\beta/2}\, dW$$

for some constants $K > 0$, $\sigma > 0$ and $\beta \in [0, 2]$. If $\beta = 2$, then X is a geometric Brownian motion and we know that $u_x(0, t) = 0$ (compare the example in the introduction), and if $\beta = 1$, then one can use the valuation formula in [3] (p. 161, equation (36)) to explicitly calculate that $u_x(0, t) = \exp\{-\frac{2K}{\sigma^2 t}\} > 0$ for $t > 0$. Theorem 2.2 tells us that also in the remaining cases, i.e. for all $\beta < 2$, we have $u_x(0, t) > 0$ for $t > 0$.

In dimension $n = 1$ we have the following result that for instance covers the example given in the introduction. It shows that the condition that $\alpha_{11} \geq C x_1^\beta$ for some $\beta \in (0, 2)$ in Theorem 2.2 cannot be substantially weakened.

Theorem 2.3. *Let $n = 1$, and assume that $a(x, t) \leq C x^2$ for all x and t for some constant C. Also assume that g' exists at $x = 0$. Then $u_x(x, t)$ exists at $x = 0$ and $u_x(0, t) = g'(0)$ for all t.*

Proof. By subtracting a constant and a suitable multiple of x we may without loss of generality assume that $g(0) = 0$ and $g'(0) = 0$ (note that all affine functions w satisfy $\mathcal{L}w = 0$). Let $\epsilon > 0$ be given. Since g is of at most polynomial growth, we can find $C_1 > 0$ and $N > 1$ such that

$$g(x) \leq \epsilon x + C_1 x^N$$

for all x. Note that C_1 depends on ϵ, whereas N can be held fixed if varying ϵ. Define the function v by

$$v(x, t) := \epsilon x + C_1 x^N + C_2 t x^N$$

for some constant $C_2 > 0$ to be chosen. Then

$$\begin{aligned} v_t - a(x, t) v_{xx} &\geq C_2 x^N - C x^2 C_1 N(N-1) x^{N-2} - C x^2 C_2 t N(N-1) x^{N-2} \\ &= (C_2 - C(C_1 + C_2 t) N(N-1)) x^N, \end{aligned}$$

so for times $0 \leq t \leq \frac{1}{2C(N^2 - N)} =: t_0$ we have

$$v_t - a(x, t) v_{xx} \geq (C_2/2 - C C_1 N(N-1)) x^N \geq 0$$

if C_2 is chosen large enough. Thus v is a supersolution to the Black-Scholes equation satisfying $v(x,0) \geq g(x)$. It follows from the maximum principle that $u \leq v$ for all times $0 \leq t \leq t_0$. Thus $u_x(0,t) \leq v_x(0,t) = \epsilon$ for such t. The same argument applied to $-u$ gives $u_x(0,t) \geq -v_x(0,t) = -\epsilon$. Letting $\epsilon \to 0$ we arrive at $u_x(0,t) = 0$ for times $t \leq t_0$. Viewing $u(x,t_0)$ as the initial condition the above argument shows that $u_x(0,t) = 0$ also for $t_0 \leq t \leq 2t_0$ (note that the same constant N can be used again) and thus by iteration also for all $t \geq 0$, which finishes the proof. □

There is no immediate generalization of Theorem 2.3 to higher dimensions. For instance, the price $u(x,t)$ in a geometric Brownian motion model of a call option on the sum of two assets (i.e. $g(x_1,x_2) = (x_1 + x_2 - K)^+$) satisfies $u_{x_1}(0,x_2,t) \neq g_{x_1}(0,x_2)$. On the other hand, the example with the exchange option shows that the condition on a_{11} in Theorem 2.2 is sharp also in several dimensions.

Note that in the version of the Hopf boundary point lemma provided above (Theorem 2.2) it is not assumed that P' is a minimum of the function u but rather of the function $u(x_1,x_2,...,x_n,t) - u(0,x_2,...,x_n,t)$. This is appropriate in the case when the boundary conditions are defined inductively by solving the equation in lower dimensional faces. Our proof also works in the situation where $u(x_1,x_2,...,x_n,t)$ has a minimum in P'. This situation is not the typical one when the boundary conditions are defined as in the introduction, but it can of course occur in other types of degenerate initial-boundary value problems.

Theorem 2.4. *Assume that a is continuous and parabolic in $(0,\infty)^n \times (0,\infty)$, i.e. $\xi a(x,t)\xi^* > 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and for all x,t. Assume also that u satisfies*

$$\sum_{i,j=1}^n a_{ij}u_{x_ix_j} + \sum_{i=1}^n b_i u_{x_i} + cu - u_t \leq 0 \tag{5}$$

in $(0,\infty)^n \times (0,\infty)$, where $b_i = b_i(x,t)$ and $c = c(x,t)$ are continuous functions. Let a point $P' = (0,x_2',...,x_n',t_0)$ with $x_2',...,x_n',t_0 > 0$ be given, and assume that in a neighborhood of P' we have

$$a_{11} \geq Cx_1^\beta,$$
$$a_{ii} \leq C$$

for $i = 2,...,n$,

$$b_1 \geq -Cx_1^{\beta-1+\delta}, \tag{6}$$
$$|b_i| \leq Cx_1^{\beta-2+\delta}$$

for $i = 2,...,n$ and

$$c \geq -Cx_1^{\beta-2+\delta} \tag{7}$$

for some constants $C > 0$, $\delta > 0$ and $\beta \in [0,2)$. Also, in the same neighborhood, assume that $u(P) > u(P')$ for all points P with $x_1 > 0$ and $t \leq t_0$. Then $u_{x_1}(P') > 0$.

Proof. The proof follows along the lines of the proof of Theorem 2.2 and is therefore omitted. □

As is shown in the next example, the lower bounds on b and c in Theorem 2.4 are sharp in the sense that the result fails for $\delta = 0$.

Example Let $n = 1$. The function $u(x,t) := x^2/2$ is the solution of

$$\begin{cases} u_t = x^\beta u_{xx} - x^{\beta-1}u_x \\ u(x,0) = x^2/2. \end{cases}$$

Note that (6) is not satisfied and that $u_x(0,t) = 0$. Moreover, the same function u also is the solution of

$$\begin{cases} u_t = x^\beta u_{xx} - 2x^{\beta-2}u \\ u(x,0) = x^2/2. \end{cases}$$

For this system (7) is not satisfied.

Remark If $\beta = 0$ in Theorem 2.4, i.e. if the differential operator is uniformly parabolic, then one only needs the inequality (5) to hold in a parabolic frustrum, compare Lemma 2.6, p. 10 in [11]. Examining the proof of the Hopf lemma for non-uniformly operators, it is clear that this also is true for $\beta \in [0,1)$, since N in that case can be chosen to be 1.

3. The spatial derivative in dimension one

In this section we use preservation of convexity for $n = 1$ to deduce the existence of the derivative $u_x(0,t)$ and also some regularity properties of this function as a function of time. If the operator is uniformly parabolic, then $u_x(0,t)$ exists and is continuous for $t > 0$ (even if $g'(0)$ does not exist). To see this, note first that we can assume, without loss of generality, that $g(0) = 0$. Then, extending g to a continuous odd function on $\mathbb{R}$ and α to an even function in x, standard interior regularity results for parabolic PDE:s yield that $u_x(0,t)$ is continuous. Note that, indeed, the solution to the extended problem agrees with the solution to the original problem for non-negative x, since the extended solution is odd in x and thus vanishes at $x = 0$.

We start this section with an example that shows that this is not true in general for degenerate operators.

Example (Power options in a geometric Brownian motion model.) Let $a(x,t) = x^2$ and $g(x) = x^\gamma$ for some constant $\gamma > 0$. Then it is easy to check that

$$u(x,t) = x^\gamma \exp\{(\gamma^2 - \gamma)t\}.$$

Thus $u_x(0,t)$ does not exist if $0 < \gamma < 1$.

Theorem 3.1. *Assume that the contract function g is convex and that $g'(0+) > -\infty$. Then the derivative $u_x(0,t)$ exists for all t. Moreover, the function $t \mapsto u_x(0,t)$ is increasing and upper semi-continuous.*

Proof. Recall, see [8], that convexity is preserved and that the option price increases in time to maturity, i.e. $x \mapsto u(x,t)$ is convex for each fixed $t \geq 0$ and $t \mapsto u(x,t)$ is increasing for each fixed x, see also [1], [5] and [7]. Since $u(0,t) = g(0)$ it follows that $u_x(0,t)$ exists and that $u_x(0,t)$ is increasing. Moreover, using the continuity in t and the spatial convexity of u, it follows that $u_x(0,t)$ is upper semi-continuous. □

Remark Note that $u_x(0,t)$ also exists if the contract function can be written as a difference of two convex functions (both with finite derivative in the origin). Also note that the example above with $g(x) = x^\gamma$ where $0 < \gamma < 1$ (or rather $g(x) = -x^\gamma$) shows that the assumption in Theorem 3.1 about $g'(0) > -\infty$ is essential.

Remark In higher dimensions convexity is in general no longer preserved, compare [4]. Therefore the above proof of Theorem 3.1 is not applicable to problems with several underlying assets. It remains an open question to determine conditions under which the spatial derivatives exist at the boundary if $n \geq 2$.

The conclusions of Theorem 3.1 that can be drawn more or less immediately from the preservation of convexity cannot be improved in the sense that $u_x(0,t)$ need not be continuous in t. Indeed, we end this article with the construction of an example where α is continuous and locally Lipschitz in x, the contract function g is convex and in $C^\infty([0,\infty))$ and yet $u_x(0,t)$ fails to be lower semi-continuous.

Example ($u_x(0,t)$ need not be continuous as a function of t). The example is constructed by patching together two functions $v : [0,\infty) \times [0,t_0) \to [0,\infty)$ and $w : [0,\infty) \times [t_0,\infty) \to [0,\infty)$ for some given $t_0 > 0$.

To define v, let $h \in C^\infty([0,\infty))$ be convex, non-negative and satisfy $h(0) = 0$, $h'(0) = 0$, and $h'(y) = 1$ and $h''(y) = 0$ for $y \geq y_0 > 0$. Let

$$C := \lim_{y\to\infty}(yh'(y) - h(y)) = y_0 - h(y_0).$$

Define $v : [0,\infty) \times [0,t_0) \to [0,\infty)$ by

$$v(x,t) = x^2 e^t + (t_0 - t)h(\frac{x}{t_0 - t}).$$

It follows that

$$v_t = \tilde{a}(x,t)v_{xx}$$

in $(0,\infty) \times (0,t_0)$ where

$$\tilde{a}(x,t) = \frac{x^2e^t(t_0-t) + xh'(\frac{x}{t_0-t}) - (t_0-t)h(\frac{x}{t_0-t})}{2e^t(t_0-t) + h''(\frac{x}{t_0-t})}.$$

Note that $v_x(0,t) = 0$ for $0 \leq t < t_0$, and for $x > 0$ we have

$$\lim_{(y,t)\to(x,t_0)} v(x,t) = x^2e^{t_0} + x,$$

$$\lim_{(y,t)\to(x,t_0)} v_t(x,t) = x^2e^{t_0} + C,$$

$$\lim_{(y,t)\to(x,t_0)} v_x(x,t) = 2xe^{t_0} + 1$$

and

$$\lim_{(y,t)\to(x,t_0)} v_{xx}(x,t) = 2e^{t_0}.$$

Next define w as the unique solution to

$$\begin{cases} w_t = \frac{x^2+Ce^{-t_0}}{2}w_{xx} & \text{for } (x,t) \in (0,\infty)\times(t_0,\infty) \\ w = 0 & \text{if } x = 0 \\ w = x^2e^{t_0} + x & \text{if } t = t_0. \end{cases}$$

Then it is straightforward to check that

$$w_t(x,t_0) = x^2 e^{t_0} + C,$$
$$w_x(x,t_0) = 2xe^{t_0} + 1$$

and

$$w_{xx}(x,t_0) = 2e^{t_0}$$

for $x > 0$. It follows that

$$u(x,t) = \begin{cases} v(x,t) & \text{if } 0 \leq t < t_0 \\ w(x,t) & \text{if } t_0 \leq t < \infty \end{cases}$$

solves

$$u_t = a(x,t)u_{xx}$$

where

$$a(x,t) := \begin{cases} \tilde{a}(x,t) & \text{if } 0 \leq t < t_0 \\ \frac{x^2+C}{2} & \text{if } t_0 \leq t < \infty. \end{cases}$$

Note that $\alpha := \sqrt{2a}$ is continuous on $(0,\infty) \times (0,\infty)$, locally Lipschitz in the x-variable and that α satisfies the growth condition

$$\alpha(x,t) \leq D(1+x)$$

for some positive constant D. Recall that option prices with convex contract functions are increasing in the time to maturity, compare [8], i.e. the function $t \mapsto w(x,t)$ is increasing. Thus $w_x(0,t) \geq 1$ for $t \geq t_0$. Therefore the function $u_x(0,t)$ is not continuous.

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