Boundary values and finite difference methods for the single factor term structure equation

Erik Ekström[1], Per Lötstedt[2], and Johan Tysk[1] *

[1] *Department of Mathematics*
Uppsala University, P O Box 480, SE-75106 Uppsala, Sweden
[2] *Division of Scientific Computing, Department of Information Technology*
Uppsala University, P O Box 337, SE-75105 Uppsala, Sweden
emails: ekstrom@math.uu.se, perl@it.uu.se, johan.tysk@math.uu.se

Abstract

We study the classical single factor term structure equation for models that predict non-negative interest rates. For these models we develop a fast and accurate finite difference method using the appropriate boundary conditions at zero.

AMS subject classification: Primary 91B28; Secondary 35A05, 35K65, 60J60, 65M06

Keywords: term structure equation, degenerate parabolic equations, stochastic representation, finite difference method

1 Introduction

When determining option prices using the Black-Scholes equation with finite difference methods, boundary conditions need to be imposed both for vanishing asset values and for large asset values. In the case of one underlying

*Financial support has partially been obtained by Ekström from the Swedish National Graduate School of Mathematics and Computing (FMB) and by Tysk from the Swedish Research Council (VR). Telephone and fax numbers to the corresponding author Lötstedt are +46-18-4712972 and +46-18-523049.

asset, the appropriate value at zero for European options is simply the discounted value of the pay-off function at that point. This is the case since the boundary is absorbing corresponding to the underlying asset going bankrupt. The question of appropriate boundary values for several underlying assets is investigated in [9]. One should perhaps note that for several models, for instance geometric Brownian motion, the stock process reaches the boundary with probability zero and the boundary conditions are thus redundant to specify from a mathematical point of view. However, using finite difference methods (FDs) boundary conditions are needed, being mathematically redundant or not. Let us note that the conditions discussed above are valid for models predicting positive asset values as well as those that allow bankruptcy with positive probability.

The present note deals with FDs for the classical term structure equation in single factor models. Using this equation, bond prices and bond option prices can be determined. We consider models that predict non-negative interest rates. In most interest rate models the boundary is not absorbing since the short rate typically would not stay zero if the value zero is reached. Moreover, the diffusion coefficient tends to zero and the drift is non-negative at the boundary for models predicting non-negative rates. Consequently, it is not clear what boundary conditions should be specified for the term structure equation. Modelling the short rate $X(t)$ directly under the pricing measure as

$$dX(t) = \beta(X(t), t)\, dt + \sigma(X(t), t)\, dW,$$

the bond option price u corresponding to a pay-off function g is given, using risk neutral valuation, by

$$u(x,t) = E_{x,t}\left[e^{-\int_t^T X(s)\,ds} g(X(T))\right].$$

As indicated above, we assume that $\sigma(0,\cdot) = 0$ and $\beta(0,\cdot) \geq 0$. For the precise conditions on σ and β, see [5]. One important example is the Cox-Ingersoll-Ross model, for which $\beta(x,t) = a(b - x)$ and $\sigma(x,t) = c\sqrt{x}$, where a, b and c are positive constants. We note that if the pay-off $g \equiv 1$, then bond prices are obtained. The function u satisfies the term structure equation

$$u_t(x,t) + \frac{1}{2}\sigma^2(x,t)u_{xx}(x,t) + \beta(x,t)u_x(x,t) = xu(x,t) \tag{1}$$

with terminal condition $u(x,T) = g(x)$. The term structure equation holds at all interior points $(x,t) \in (0,\infty) \times [0,T)$. In [11], the issue of when boundary conditions at $x = 0$ are needed is discussed in detail. No boundary condition is needed if the so-called Fichera function, which in a one-dimensional time-homogeneous setting is $\beta(x) - \frac{1}{2}\frac{\partial \sigma^2}{\partial x}(x)$, satisfies

$$\lim_{x \searrow 0} \left(\beta(x) - \frac{1}{2}\frac{\partial \sigma^2}{\partial x}(x) \right) \geq 0.$$

In the example of the Cox-Ingersoll-Ross model, this condition reduces to $ab - \frac{1}{2}c^2 \geq 0$. This is of course consistent with the usual Feller condition that says that 0 is not attainable for the process X. However, to use FDs it is necessary to know the behaviour of the solution close to the boundary even though a boundary condition might be redundant from a mathematical perspective.

Only recently the question of appropriate boundary behaviour for (1) has been treated mathematically. The main result of [5] states that the bond option price u is the unique classical solution to the term structure equation satisfying the boundary condition

$$u_t(0,t) + \beta(0,t)u_x(0,t) = 0. \tag{2}$$

Observe that this boundary condition is obtained by formally plugging in $x = 0$ into the equation. Alternatively, to obtain an intuitive explanation of (2), assume that u is sufficiently regular and use Ito's formula to compute

$$d(u(X(t),t)) = (u_t + \beta u_x + \frac{1}{2}\sigma^2 u_{xx})(X(t),t)\, dt + (\sigma u_x)(X(t),t)\, dW.$$

Standard arbitrage theory says that the local rate of return should equal the short rate X. At the boundary we therefore obtain (2) since σ vanishes there.

Remark It is perhaps misleading to always refer to (2) as a boundary *condition*. When the boundary is not attainable for the stochastic process X, (2) should perhaps rather be refered to as the boundary *behaviour* of u. For simplicity, however, we will refer to (2) as a boundary condition.

The recent book by D.J. Duffy [4] indeed has a section entitled "The thorny issue of boundary conditions" treating the term structure equation. Also other references in this area, such as [7], deal with this question. In these references, boundary conditions are only specified for certain models and for parameter values when the boundary is reached with positive probability, and the general case is avoided. In Example 1.1 in [8], the authors encounter several solutions to the pricing PDE when not considering the boundary behaviour, and they discuss these solutions as different possible prices. Our point of view is that only the solution that satisfies appropriate boundary conditions represents the price as given by the risk-neutral expected value.

In the present note we develop a fast and accurate FD using (2). The advantage of FDs compared to Monte Carlo methods for (1) is the accuracy and the efficiency for low dimensional problems, see e.g. [10]. Our method requires no tailoring for the specific model in question but is instead valid for *all* models that predict non-negative rates. We quote from [4]: "Much of the literature is very Spartan in the author's opinion when it comes to defining boundary conditions, and their assembly into the discrete system of equations". This note is one step towards filling this gap.

The paper is organized as follows. The term structure equation (1) with the boundary condition (2) is discretized by a FD of second order accuracy in Section 2. The FD is applied to the Cox-Ingersoll-Rubinstein (CIR) model [1, 2] and a model with a diffusion proportional to $x^{3/4}$ in Section 3. Finally, some conclusions are drawn.

2 Numerical method

The term structure equation (1) is solved by a FD on the grid $x_n = nh,\ n = 0, \ldots, N$. The upper boundary of the computational domain is $x_{\max}$ and the step size h is $x_{\max}/N$. The constant time step is $\Delta t = T/M$ between the discrete time points $t^m = m\Delta t,\ m = 0, \ldots, M$. The numerical solution at (x_n, t^m) is denoted by u_n^m and the spatial derivatives there are approximated by

$$u_x \approx \frac{1}{2}h^{-1}(u_{n+1}^m - u_{n-1}^m),\ u_{xx} \approx h^{-2}(u_{n+1}^m - 2u_n^m + u_{n-1}^m). \quad (3)$$

At the lower boundary, $x_0 = 0$,

$$u_x \approx -h^{-1}(\frac{3}{2}u_0^m - 2u_1^m + \frac{1}{2}u_2^m) \quad (4)$$

in (2) and at $x_N = x_{\max}$,

$$\begin{aligned} u_x &\approx h^{-1}(\tfrac{3}{2}u_N^m - 2u_{N-1}^m + \tfrac{1}{2}u_{N-2}^m), \\ u_{xx} &\approx h^{-2}(2u_N^m - 5u_{N-1}^m + 4u_{N-2}^m - u_{N-3}^m). \end{aligned} \quad (5)$$

In this way, only values of the solution between x_0 and x_N appear in the approximations. The second formula in (5) is a linear extrapolation of the difference approximations of u_{xx} at x_{N-1} and x_{N-2}. All approximations are second order accurate.

Let $\mathbf{u}^m$ denote the solution vector at t^m with the components u_n^m. The time derivative is approximated in the same manner as the space derivative in (4). Then the complete integration scheme backward in time for (1) is

$$(\frac{3}{2}I - \Delta t A)\mathbf{u}^{m-1} = 2\mathbf{u}^m - \frac{1}{2}\mathbf{u}^{m+1},\ m = M-1, M-2, \ldots, 1, \qquad (6)$$

where the constant matrix A represents the space discretizations in (3), (4), and (5). The implicit time integration method is the backward differentiation formula of order two (BDF2). The first step is taken with a first order method, the Euler backward method or BDF1 of order one,

$$(I - \Delta t A)\mathbf{u}^{M-1} = \mathbf{u}^M,\ u_n^M = g(x_n). \qquad (7)$$

Both methods are stable if all eigenvalues $\lambda(A)$ of A satisfy $\Re\lambda(A) \leq 0$ [6]. The error in the solution after the first time step is of $\mathcal{O}(\Delta t^2)$ and the truncation error is of order two in both time and space at all points (m, n) with $m < M$. The matrices in (6) and (7) are almost tridiagonal and the systems of equations are both solved easily in a number of operations proportional to N. If σ and β are time-independent then A is constant and a LU-factorization is first computed for the system matrices in (6) and (7) [3, Ch. 5.4]. This factorization is then used in every time step with a cost of about $5N$ operations to obtain $\mathbf{u}^{m-1}$.

3 Numerical results

We solve (1) by the scheme in Section 2 for two different models: the CIR model [2, 1] and a model with $\sigma \sim x^{3/4}$. An exact solution is known for the CIR equation and the convergence properties of our method can then be investigated. The second model is chosen to demonstrate the flexibility of the FD in a case without analytical solution.

Let

$$\beta(x) = a(b - x), a = 0.55, b = 0.035,\ \sigma(x) = 0.39\sqrt{x} \qquad (8)$$

in (1) with similar parameters as in the CIR model in [7] and let $g(x) = 1$. The end points in space and time are $x_{\text{max}} = 0.1$ and $T = 1$. The analytical solution v at grid and time points $v_n^m = v(x_n, t^m)$ is found in [1, p. 58].

Remark The analytical solution given in [1] is only specified for parameter values such that the Fichera function is strictly positive at $x = 0$, and the parameter specification (8) does not fulfill this condition. However, the same formula is correct also in the case when the Fichera function is negative at $x = 0$. To see this it suffices to check that the correct boundary condition (2) is satisfied.

The difference $\mathbf{d}^m$ between $\mathbf{u}^m$ and $\mathbf{v}^m$ at all time points is measured in the norm defined by

$$\|\mathbf{d}\|^2 = \sum_{m=0}^{M} \sum_{n=0}^{N} h \Delta t |u_n^m - v_n^m|^2.$$

In Figure 1, the second order convergence rate is confirmed with our choice of T and $x_{\max}$. For long integration times, the solution error has its maximum at $x = x_{\max}, t = 0$.

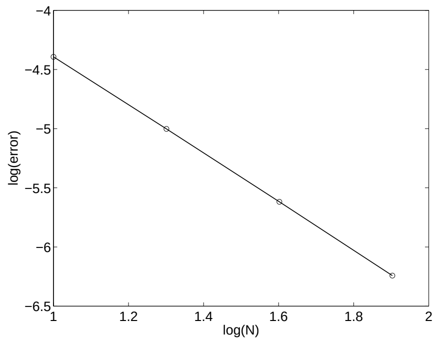

Figure 1: The difference $\log_{10} \|\mathbf{d}\|$ between the exact solution and the solution computed with the FD in Section 2 versus $\log_{10} N$ for the same number time steps M and space steps N.

N	10	20	40	80
$\min \lvert\lambda(A)\rvert$	0.0294	0.0294	0.0294	0.0294
$\max \lvert\lambda(A)\rvert$	$0.0186 \cdot 10^4$	$0.0926 \cdot 10^4$	$0.4152 \cdot 10^4$	$1.7711 \cdot 10^4$

Table 1: The minimum and maximum modulus of the eigenvalues of A.

The eigenvalues of A are all real except for 2 and all have a negative real part for $N = 10, 20, 40, 80$ implying a stable integration in (6) in those cases. The minimum and maximum modulus of $|\lambda(A)|$ are found in Table 1. The minimal value is associated with an eigenvector which is almost constant in n. This is the mode with the slowest decay. With an explicit method the time step is restricted by

$$\Delta t \sim 1/\max |\lambda(A)|$$

for a stable integration. From the table, it follows that with $N = 80$ we have an upper bound on an explicit time step $\Delta t \sim 0.56 \cdot 10^{-4}$. With the implicit method in Section 2 $\Delta t = 1/M = 0.0125$, about 200 times longer. The work per time step for the implicit method is less than two times the work for the simplest explicit method and the error in the solution is dominated by the spatial error in both cases.

The term structure equation is solved with $g(x) = 1$ and the same drift term as in (8) but with $\sigma(x) = 0.39x^{3/4}$. Only a minor change in the code is necessary. The solution is displayed in Figure 2.

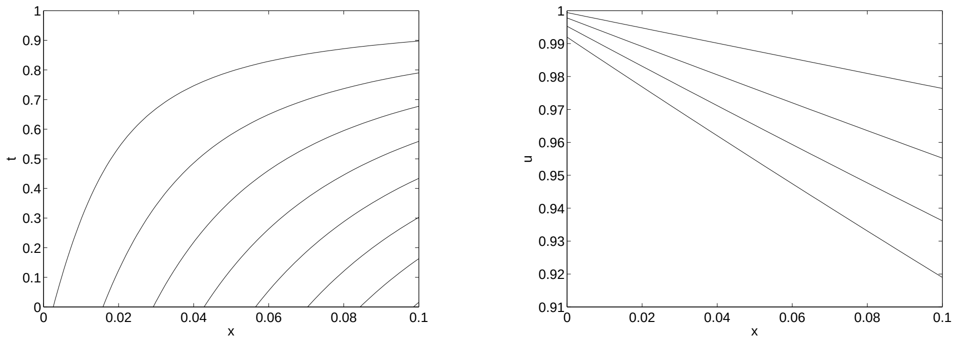

Figure 2: The solution of (1) with a diffusion $\sim x^{3/4}$ and $M = 21$ and $N = 21$: Isolines from $u = 0.92$ to 1.0 with step 0.01 (left) and u at $t = 0.75, 0.5, 0.25, 0$, from top to bottom (right).

4 Conclusions

We implement a general boundary condition at $x = 0$ for the term structure equation and propose a finite difference method based on this boundary condition. In this way, we partly resolve "the thorny issue of boundary conditions" which is the title of Section 25.6 treating the term structure equation in [4]. The numerical method is implicit in time and second order accurate. The flexibility of a finite difference method makes it easy to change the drift and diffusion terms in the model.

References

[1] Brigo, D. and Mercurio, F. Interest Rate Models. Theory and Practice, Springer Verlag, Berlin, 2001.

[2] Cox, J.C., Ingersoll Jr, J.E., and Ross, S.A. An intertemporal general equilibrium model of asset prices. Econometrica. 53 (1985), 363–384.

[3] Dahlquist, G. and Björck, Å. Numerical Methods. Prentice-Hall, Englewood Cliffs, NJ, 1974.

[4] Duffy, D.J. Finite Difference Methods in Financial Engineering. Wiley, Chichester, 2006.

[5] Ekström, E. and Tysk, J. Existence and uniqueness theory for the term structure equation. Preprint 2008.

[6] Hairer, E, Nørsett, S. P., and Wanner, G. Solving Ordinary Differential Equations, Nonstiff Problems, 2nd ed., Springer-Verlag, Berlin, 1993.

[7] d'Halluin, Y., Forsyth, P.A., Vetzal, K.R., and Labahn, G. A numerical PDE approach for pricing callable bonds, Appl. Math. Finance 8 (2001), 49–77.

[8] Heston, S., Loewenstein, M. and Willard, G. Options and bubbles. Rev. Financ. Stud. 20 (2007), 359–390.

[9] Janson, S. and Tysk, J. Feynman-Kac formulas for Black-Scholes-type operators. Bull. London Math. Soc. 38 (2006), 269–282.

[10] Lötstedt, P., Persson, J., von Sydow, L., and Tysk, J. Space-time adaptive finite difference method for European multi-asset options. Comp. Math. Appl. 53 (2007), 1159–1180.

[11] Oleinik, O. A.; Radkevic, E. V. Second order equations with nonnegative characteristic form. Plenum Press, New York-London, 1973.