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Convexity preserving jump-diffusion models for option pricing

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Abstract

We investigate which jump-diffusion models are convexity preserving. The study of convexity preserving models is motivated by monotonicity results for such models in the volatility and in the jump parameters. We give a necessary condition for convexity to be preserved in several-dimensional jump-diffusion models. This necessary condition is then used to show that, within a large class of possible models, the only convexity preserving models are the ones with linear coefficients.

Keywords: Convexity; Jump-diffusions; Integro-differential equations; Options; Option price orderings

1. Introduction

A model for a set of stock prices is said to be convexity preserving if the price of any convex European claim is convex as a function of the underlying stock prices at all times prior to maturity. As is well known, this property is intimately connected to certain monotonicity properties of the option price with respect to volatility and other parameters of the model. Generally speaking, if the option price is convex at all fixed times, then it is also increasing in the volatility. This robustness property motivates the study of convexity preserving models in finance.

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Although these issues have been studied quite extensively during the last decade, compare [3,6,8,9,11] for the case of one-dimensional diffusion models, [2,4,10] for several-dimensional diffusion models and [5] for one-dimensional jump-diffusion models, the general picture for more advanced models is not yet fully understood. In [5], a sufficient condition for the preservation of convexity in one-dimensional models with jumps is provided. That condition, however, is not a necessary condition for preservation of convexity. The main contribution of the present paper is to give a necessary condition for convexity to be preserved in jump-diffusion models in arbitrary dimensions. We also use this necessary condition to show that, within a large class of possible models, the only higher-dimensional convexity preserving models are the ones with linear coefficients.

To analyze the convexity of an option price we employ the characterization of the price as the unique viscosity solution to a parabolic integro-differential equation

$$u_t = \mathcal{A}u + \mathcal{B}u \tag{1}$$

with an appropriate terminal condition. In this equation, $\mathcal{A}$ is an elliptic differential operator associated with the continuous fluctuations of the stock price processes, whereas $\mathcal{B}$ is an integro-differential operator associated with the possible jumps of the stock price processes, compare Section 3 below. Preservation of convexity of the solution to Eq. (1) is dealt with using the notion of *locally convexity preserving* (LCP) operators. This concept was introduced and analyzed in [10], and also used in [4,5]. Following these references, we show that the condition that $\mathcal{M} = \mathcal{A} + \mathcal{B}$ is LCP at all points is necessary for convexity to be preserved. We also show that $\mathcal{M}$ is LCP if and only if both $\mathcal{A}$ and $\mathcal{B}$ are LCP, i.e.

$$\left\{\begin{matrix}\text{The model is}\\ \text{convexity}\\ \text{preserving}\end{matrix}\right\} \implies \left\{\begin{matrix}\mathcal{M}\text{ is LCP}\\ \text{at all points}\end{matrix}\right\} \iff \left\{\begin{matrix}\text{Both }\mathcal{A}\text{ and }\mathcal{B}\\ \text{are LCP}\\ \text{at all points}\end{matrix}\right\},$$

compare Theorem 4.2. Thus the characterization of LCP models breaks down into two easier problems: (i) to describe which diffusion models are LCP, and (ii) to describe which jump structures are LCP. Problem (i) has been dealt with in [10] and [4], and problem (ii) is dealt with in Theorem 5.1 below.

The present paper is organized as follows. In Section 2 we introduce the model and we motivate the study of convexity preserving models by means of a monotonicity result. In Section 3 we prove a technical regularity result which is used in the sequel. In Section 4 we introduce the LCP-condition, and we show that a model is convexity preserving only if both the differential operator $\mathcal{A}$ and the integro-differential operator $\mathcal{B}$ are LCP at all points. In Section 5 we investigate which jump structures are LCP. This investigation is continued in Section 6 for models with only a finite number of possible jump sizes, where we show that, within a large class of possible models, all convexity preserving models have linear diffusion coefficients and jump structures.

2. The model and a monotonicity result

We consider a market consisting of n different stocks, the prices of which are modeled by an n-dimensional stochastic process $X(t)$. To specify X, let W be an n-dimensional Brownian motion, and let υ be a Poisson random measure on $[0, T] \times [0, 1]$ with intensity measure

$$q(dt, dz) = \lambda(t)\, dt\, dz,$$

where λ is a deterministic function. Let X be a jump-diffusion satisfying the stochastic differential equation

$$dX = \beta\bigl(X(t-), t\bigr)\, dW + \int_0^1 \phi\bigl(X(t-), t, z\bigr)\tilde{v}(dt, dz).$$

Here $\beta = (\beta_{ij})_{ij=1}^n$ is an $(n \times n)$-matrix, $\phi = (\phi_1, \ldots, \phi_n)$ is an n-dimensional vector and $\tilde{v}$ is the compensated jump measure defined by

$$\tilde{v}(dt, dz) = (v - q)(dt, dz).$$

Remark. In this model, jumps occur according to a Poisson process

$$Y(t) := \int_0^t \int_0^1 v(dt, dz)$$

with intensity $\lambda(t)$. Associated with each jump is a label $z \in [0, 1]$. The interpretation is that a jump of Y at time t with label z results in a jump of size $\phi_i(X(t-), t, z)$ in the ith coordinate of X. Between jumps, X follows a continuous diffusion governed by the diffusion coefficient $\beta(X(t), t)$ and the drift $-\lambda \int_0^1 \phi(X(t), t, z)\, dz$.

We denote by $\mathbb{R}_+^n$ the space $(0, \infty)^n$, and we say that a function $g : \mathbb{R}_+^n \to \mathbb{R}$ is of at most polynomial growth if there exist constants m and C such that

$$\bigl|g(x)\bigr| \leqslant C\bigl(1 + |x|^m\bigr)$$

for all $x \in \mathbb{R}_+^n$. Given a continuous pay-off function $g : \mathbb{R}_+^n \to \mathbb{R}$ of at most polynomial growth, the price at time $t \in [0, T_0]$ of an option paying $g(X(T_0))$ at time $T_0 \in [0, T]$, is $u(X(t), t)$. Here the function $u : \mathbb{R}_+^n \times [0, T_0] \to \mathbb{R}$ is given by

$$u(x, t) := E_{x,t}\, g\bigl(X(T_0)\bigr),$$

where the indices indicate that $X(t) = x$. Note that the conditions (A1)–(A7) specified below and the polynomial growth of g implies that all moments of $X(T)$ are finite, compare Section 7 in [7]. Consequently, the option value u is finite.

Remark. We do not address the issue of how to choose an appropriate pricing measure, but we rather assume that the model is specified directly under the measure used for pricing options. Also note that there is no discounting factor in the definition of the option price. Thus we implicitly assume, without loss of generality for our purposes, that all prices are quoted in terms of some bond price.

We will throughout this paper work under the regularity and growth assumptions (A1)–(A7). When specifying these, D is a positive constant and the Hölder exponent α is a constant between 0 and 1.

(A1) For all $i, j = 1, \ldots, n$, $\beta_{ij} : \mathbb{R}^+ \times [0, T] \to \mathbb{R}$ is in $C_\alpha^{2,0}(\mathbb{R}_+^n \times [0, T])$.
(A2) $\lambda \in C_\alpha([0, T])$.
(A3) $\sum_{j=1}^n |\beta_{ij}(x, t)|^2 + |\phi_i(x, t, z)|^2 \leqslant D x_i^2$ for all $i = 1, \ldots, n$.

(A4) $|\beta(x,t)-\beta(y,t)|+|\phi(x,t,z)-\phi(y,t,z)| \leqslant D|x-y|$.
(A5) The matrix $\beta(x,t)$ is non-singular for all $(x,t)\in\mathbb{R}_+^n\times[0,T]$.
(A6) $\phi:\mathbb{R}_+^n\times[0,T]\times[0,1]\to\mathbb{R}^n$ is measurable, and $\phi_i(\cdot,\cdot,z)\in C_\alpha^{2,0}(\mathbb{R}_+^n\times[0,T])$ with the Hölder norms being uniformly bounded in the z-variable. Moreover, for all x and t we have $\phi(x,t,z)\neq 0$ for almost all z.
(A7) There exists $\gamma>-1$ such that $\phi_i(x,t,z)>\gamma x_i$ for $i=1,\dots,n$.

Definition 2.1. A model (β,λ,ϕ) is convexity preserving on $[0,T]$ if for all t and T_0 with $0\leqslant t\leqslant T_0\leqslant T$, the price $u(x,t)$ of the option with pay-off $g(X(T_0))$ at T_0 is convex in x for any convex pay-off function g of at most polynomial growth.

The main reason for studying preservation of convexity is, as mentioned in the introduction, that convexity implies certain monotonicity properties of the option price with respect to the parameters of the model. The following result can be proven in a similar way as Theorem 5.1 in [5], in which the one-dimensional case is treated.

Theorem 2.2. *Let two models be given with parameters (β,λ,ϕ) and $(\tilde\beta,\tilde\lambda,\tilde\phi)$, respectively. Assume that*

(i) $\tilde\lambda(t)\leqslant\lambda(t)$ *for all* $t\in[0,T]$;
(ii) *for each fixed* $(x,t)\in\mathbb{R}_+^n\times[0,T]$ *we have* $\tilde\phi(x,z,t)=k(z)\phi(x,t,z)$ *for some* $k(z)\in[0,1]$;
(iii) *for all* $(x,t)\in\mathbb{R}_+^n\times[0,T]$ *we have* $\tilde\beta\tilde\beta^*\leqslant\beta\beta^*$ *as quadratic forms (here* β^* *denotes the transpose of* β*).*

Also assume that at least one of the two models is convexity preserving. Then, for any convex contract function g of at most polynomial growth we have

$$\tilde u(x,t)\leqslant u(x,t)$$

for all $(x,t)\in\mathbb{R}_+^n\times[0,T_0]$, where $\tilde u$ and u are the two option prices corresponding to the two different models.

Remark. Note that the most important special case of (ii) is when for all x, t and z there exists $i\in\{1,\dots,n\}$ such that $\phi_j(x,t,z)=\tilde\phi_j(x,t,z)=0$ for all $j\neq i$ (i.e. the case when at most one component of X and one component of $\tilde X$ jump at each given time), and $\phi_i(x,t,z)/\tilde\phi_i(x,z,t)\geqslant 1$ if $\tilde\phi_i(x,z,t)\neq 0$. Also note that condition (iii) is the same as the one used for diffusion models in higher dimensions, compare [4].

Since all one-dimensional diffusion models and all geometric Brownian motions (not necessarily one-dimensional) are known to be convexity preserving, see [3,6,8] or [9], the following consequence of Theorem 2.2 is immediate. It is the higher-dimensional analogue to a result in [1].

Corollary 2.3. *Assume that a model (β,λ,ϕ) and a convex contract function g of at most polynomial growth are given. If $n\geqslant 2$, also assume that β is the (possibly time-dependent) diffusion matrix of a geometric Brownian motion, i.e. $\beta_{ij}(x,t)=\gamma_{ij}(t)x_i$ for some deterministic functions γ_{ij}. Then a lower bound for the corresponding option price is given by the option price in the model $(\beta,0,\phi)$ with no jumps.*

3. Regularity of the value function

Under weak conditions, see for example [12], the pricing function u is the unique viscosity solution of a parabolic integro-differential equation

$$u_t + \mathcal{M}u = 0 \tag{2}$$

with terminal condition

$$u(x, T_0) = g(x).$$

In this equation, the operator $\mathcal{M} = \mathcal{A} + \mathcal{B}$, where the second-order differential operator $\mathcal{A}$ and the integro-differential operator $\mathcal{B}$ are given by

$$\mathcal{A}u(x,t) := \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i x_j}(x,t)$$

and

$$\mathcal{B}u(x,t) := \lambda(t)\int_0^1 \big(u\big(x+\phi(x,t,z),t\big) - u(x,t) - \phi(x,t,z)\cdot\nabla u(x,t)\big)\,dz \tag{3}$$

respectively, and a_{ij} are the coefficients of the matrix $\beta\beta^*/2$. Under assumptions (A1)–(A7), u is not merely a viscosity solution to (2), but it is also a classical solution. Indeed, we show below that the value function u is regular. The proof has certain similarities to the proof of Theorem 3.2 in [5] in which the one-dimensional case is treated. However, for the convenience of the reader, and since the proofs differ at some points, we include the higher-dimensional version in its full detail.

Theorem 3.1. *Assume that $g \in C^4_\alpha(\mathbb{R}^n_+)$ and that g is globally Lipschitz continuous. Then $u \in C^{4,1}_\alpha(\mathbb{R}^n_+ \times [0, T_0])$.*

Proof. Let $\psi : \mathbb{R}_+ \to \mathbb{R}$ be a smooth function with $\psi' > 0$ such that

$$\psi(s) = \begin{cases} s & \text{if } s > 2, \\ -1/s & \text{if } s < 1. \end{cases}$$

It follows from Itô's lemma that the n-dimensional stochastic process $Y(t)$, where

$$Y_i(t) = \psi\big(X_i(t)\big),$$

satisfies

$$dY_i = \tilde{b}_i\big(Y(t-),t\big)\,dt + \sum_{j=1}^{n}\tilde{\beta}_{ij}\big(Y(t-),t\big)\,dW_j + \int_0^1 \tilde{\phi}_i\big(Y(t-),t,z\big)\,\tilde{v}(dt,dz)$$

on $\mathbb{R}^n \times [0, T]$. Here

$$\begin{aligned}\tilde{b}_i(y,t) &= \frac{1}{2}\psi''\big(\psi^{-1}(y_i)\big)\sum_{j=1}^{n}\beta_{ij}^2\big(\psi^{-1}(y),t\big) \\ &\quad + \lambda(t)\int_0^1 \big(\tilde{\phi}_i(y,t,z) - \psi'\big(\psi^{-1}(y_i)\big)\phi_i\big(\psi^{-1}(y),t,z\big)\big)\,dz,\end{aligned}$$

$$\tilde{\beta}_{ij}(y,t) = \psi'\big(\psi^{-1}(y_i)\big)\beta_{ij}\big(\psi^{-1}(y),t\big),$$

and

$$\tilde{\phi}_i(y,t,z) = \psi\big(\psi^{-1}(y_i) + \phi_i\big(\psi^{-1}(y),t,z\big)\big) - y_i,$$

where $\psi^{-1}(y) := (\psi^{-1}(y_1),\dots,\psi^{-1}(y_n))$. Now it is straightforward to check that $\tilde{b}$, $\tilde{\beta}$ and $\tilde{\phi}$ together with the initial condition $\tilde{g}(y) := g(\psi^{-1}(y))$ satisfy conditions (2.2)–(2.5) in [12]. According to Theorem 3.1 in [12], the function $v(y,t) := u(\psi^{-1}(y),t)$ is a viscosity solution of the integro-differential equation

$$\begin{cases} v_t + \sum_{i,j=1}^n \tilde{a}_{ij} v_{y_i y_j} + \sum_{i=1} \left(\tilde{b}_i - \lambda \int_0^1 \tilde{\phi}_i \, dz\right) v_{y_i} + h = 0, \\ v(y,T_0) = \tilde{g}(y), \end{cases} \tag{4}$$

where

$$h(y,t) = \lambda(t) \int_0^1 \big(v(y+\tilde{\phi},t) - v(y,t)\big)\, dz$$

and $\tilde{a}_{ij}$ are the coefficients of the matrix $\tilde{\beta}\tilde{\beta}^*/2$.

Proposition 3.3 in [12] yields the estimate

$$\big|v(y,t) - v(\tilde{y},\tilde{t})\big| \leqslant C\big(\big(1+|y|\big)|t-\tilde{t}|^{1/2} + |y-\tilde{y}|\big) \tag{5}$$

for some constant C. Together with the assumptions on ϕ, this implies that $h \in C_\alpha(\mathbb{R}^n \times [0,T_0]) \cap C_{\mathrm{pol}}(\mathbb{R}^n \times [0,T_0])$. Consequently, Theorems A.14 and A.18 in [10] ensure the existence of a unique classical solution $w \in C_{\mathrm{pol}}(\mathbb{R}^n \times [0,T_0]) \cap C_\alpha^{2,1}(\mathbb{R}^n \times [0,T_0])$ to Eq. (4). This classical solution w to (4) can also be represented (through the Feynman–Kac representation theorem) as

$$w(y,t) = E_{y,t}\left(\int_t^{T_0} h\big(Z(s),s\big)\, ds + \tilde{g}\big(Z(T_0)\big)\right)$$

where $Z = (Z_1,\dots,Z_n)$ is the continuous diffusion process given by

$$dZ_i = \left(\tilde{b}_i\big(Z(t),t\big) - \lambda(t)\int_0^1 \tilde{\phi}_i\big(Z(t),t,z\big)\, dz\right) dt + \sum_{j=1}^n \tilde{\beta}_{ij}\big(Z(t),t\big)\, dW_j$$

and $Z(t) = y$. Since h is Lipschitz continuous in y, it follows from Lemma 3.1 in [12] that w is Lipschitz continuous in y, uniformly in t. From the uniqueness result Theorem 4.1 in [12] we deduce that $v = w$. Consequently, $v \in C_{\mathrm{pol}}(\mathbb{R}^n \times [0,T_0]) \cap C_\alpha^{2,1}(\mathbb{R}^n \times [0,T_0])$, and therefore $h \in C_\alpha^{2,0}(\mathbb{R}^n \times [0,T_0])$. Applying Theorem A.18 in [10] to Eq. (4) once again we find that $v = w \in C_\alpha^{4,1}(\mathbb{R}^n \times [0,T_0])$. Transforming back to the original coordinates we get $u \in C_\alpha^{4,1}(\mathbb{R}_+^n \times [0,T_0])$. □

4. The LCP condition as a necessary condition for preservation of convexity

Following [10], see also [4] and [5], to investigate which models are convexity preserving we introduce the notion of *locally convexity preserving* (LCP) models.

Definition 4.1. An operator $\mathcal{D}$, where $\mathcal{D}$ equals either $\mathcal{M}$, $\mathcal{A}$ or $\mathcal{B}$ specified above, is LCP at a point $(x,t) \in \mathbb{R}^n_+ \times [0,T]$ if for any direction $v \in \mathbb{R}^n \setminus \{0\}$ we have that

$$\partial_v^2(\mathcal{D}f)(x,t) \geqslant 0$$

for all convex functions $f \in C^4_\alpha(\mathbb{R}^n_+) \cap C^2_{\mathrm{pol}}(\mathbb{R}^n_+)$ with $f_{vv}(x) = 0$.

We then have the following key result.

Theorem 4.2. *Consider the following statements*:

(i) *The model is convexity preserving.*
(ii) *The operator $\mathcal{M} = \mathcal{A} + \mathcal{B}$ is LCP at all points $(x,t) \in \mathbb{R}^n_+ \times [0,T]$.*
(iii) *The operators $\mathcal{A}$ and $\mathcal{B}$ are both LCP at all points $(x,t) \in \mathbb{R}^n_+ \times [0,T]$.*

We have that (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).

Remark. Under a few additional growth conditions on β, ϕ and their derivatives, it is possible to prove also (ii) $\Rightarrow$ (i), compare [5, Theorem 4.3] for the one-dimensional case. Thus, under such additional assumptions, all three statements in Theorem 4.2 are equivalent. We do not pursue this further since we only use LCP as a necessary condition in the analysis below.

Proof. To prove (i) $\Rightarrow$ (ii) we argue as in the proof of Lemma 3.3 in [10]. Choose $(x_0, T_0) \in \mathbb{R}^n_+ \times [0,T]$ and let $g \in C^4_\alpha(\mathbb{R}^n_+) \cap C^2_{\mathrm{pol}}(\mathbb{R}^n_+)$ with $g_{vv}(x_0) = 0$ for some direction v. Define u to be the solution to

$$u_t + \mathcal{M}u = 0$$

on $\mathbb{R}^n_+ \times [0, T_0)$ with terminal condition $u(x, T_0) = g(x)$. Let $\tilde{g} \in C^4_\alpha(\mathbb{R}^n_+)$ be convex, Lipschitz continuous and satisfy $\tilde{g} = g$ inside a box which contains x_0 and all possible values of $x_0 + \phi$. It follows that $\mathcal{B}g = \mathcal{B}\tilde{g}$ in a (spatial) neighborhood of x_0, so

$$\partial_v^2(\mathcal{M}g)(x_0, T_0) = \partial_v^2(\mathcal{M}\tilde{g})(x_0, T_0).$$

Now, let $\tilde{u}$ be the solution to

$$\tilde{u}_t + \mathcal{M}\tilde{u} = 0$$

on $\mathbb{R}^n_+ \times [0, T_0)$ with terminal condition $u(x, T_0) = \tilde{g}(x)$. Then, since the model is convexity preserving, $\tilde{u}$ is convex in x at all times t prior to T_0, and in particular $\tilde{u}_{vv}(x_0, t) \geqslant 0$. Consequently, $\partial_t \tilde{u}_{vv}(x_0, T_0) \leqslant 0$, so it follows from Theorem 3.1 that

$$\begin{aligned} 0 &\leqslant -\partial_t \tilde{u}_{vv}(x_0, T_0) = -\partial_v^2 \tilde{u}_t(x_0, T_0) = \partial_v^2 \mathcal{M}\tilde{u}(x_0, T_0) \\ &= \partial_v^2(\mathcal{M}\tilde{g})(x_0, T_0) = \partial_v^2(\mathcal{M}g)(x_0, T_0), \end{aligned}$$

which finishes the proof of (i) $\Rightarrow$ (ii).

The implication (iii) $\Rightarrow$ (ii) is immediate from the definition of LCP. It remains to show that (ii) $\Rightarrow$ (iii), i.e. that if $\mathcal{M}$ is LCP, then both $\mathcal{A}$ an $\mathcal{B}$ are LCP. This follows from Lemmas 4.3 and 4.4 below. □

Lemma 4.3. *Let $t_0 \in [0,T]$ be given, and let $f \in C^4_\alpha(\mathbb{R}^n_+) \cap C^2_{\mathrm{pol}}(\mathbb{R}^n_+)$ be a convex function with $f_{vv}(x_0) = 0$ at some point x_0 and for some direction v. Then, for any $\epsilon > 0$ there exists* *a convex function* $h \in C^4_\alpha(\mathbb{R}^n_+) \cap C^2_{\mathrm{pol}}(\mathbb{R}^n_+)$ *with* $h_{vv}(x_0) = 0$ *such that* $\partial^2_v(\mathcal{A}h)(x_0, t_0) = 0$ *and* $|\partial^2_v(\mathcal{B}(f - h))(x_0, t_0)| \leqslant \epsilon$. *Consequently, if* $\mathcal{M}$ *is LCP, then also* $\mathcal{B}$ *is LCP.*

Proof. Let $(x_0, t_0) \in \mathbb{R}^n_+ \times [0, T]$ be given, and assume that $f \in C^4_\alpha(\mathbb{R}^n_+) \cap C^2_{\mathrm{pol}}(\mathbb{R}^n_+)$ is a convex function with $f_{vv}(x_0) = 0$ for some direction v. Without loss of generality we assume that $f(x_0) = 0$ and $\nabla f(x_0) = 0$ (this can be done since $\mathcal{A}\tilde{f} = \mathcal{B}\tilde{f} = 0$ for all affine functions $\tilde{f}$). Let D be a constant such that for all unit directions w we have $f_{ww}(x) \leqslant D$ for all x in a neighborhood of x_0. Let $\varphi : \mathbb{R}^n_+ \to [0, 1]$ be a smooth function such that

$$\varphi(y) = \begin{cases} 0 & \text{if } |y| \leqslant 1, \\ 1 & \text{if } |y| \geqslant 2, \end{cases}$$

and let C_1 be a constant such that $|\varphi_w(y)| \leqslant C_1$ and $|\varphi_{ww}(y)| \leqslant C_1$ for all $y \in \mathbb{R}^n_+$ and all unit directions w. Further, let $\psi : [0, \infty) \to [0, \infty)$ be a smooth and non-decreasing function satisfying

$$\psi(s) = \begin{cases} 0 & \text{for } s \in [0, 1/2], \\ Ms & \text{for } s \in [1, 2], \\ 3M & \text{for } s \geqslant 3, \end{cases}$$

where M is a constant satisfying $M > 8C_1 D$. Now, for $\delta > 0$, define the function $h = h^\delta$ by

$$h(x) = f\big(\varphi\big(\delta^{-1}(x - x_0)\big)(x - x_0) + x_0\big) + \delta^2 \int_0^{|x-x_0|/\delta} \psi(s)\, ds.$$

Then h is the wanted function for some δ small enough. Indeed, first note that h is 0 if $|x - x_0| \leqslant \delta/2$. Consequently, $\partial^2_v(\mathcal{A}h)(x_0, t_0) = 0$. Moreover, h is convex if δ is small enough. To see that $\partial^2_v(\mathcal{B}(f - h))$ can be made small, note that

$$\begin{aligned} \partial^2_v(\mathcal{B}f)(x_0, t_0) &= \lambda \int_0^1 \big(\partial^2_v\big(f(x_0 + \phi)\big) - f_{vv}(x_0) - \partial^2_v\big(\phi \cdot \nabla f(x_0)\big)\big)\, dz \\ &= \lambda \int_0^1 \big(\partial^2_v\big(f(x_0 + \phi)\big) - \phi_{vv} \cdot \nabla f(x_0)\big)\, dz \end{aligned}$$

where we have used $f_{vv}(x_0) = 0$ and $f_{vw}(x_0) = 0$ for any direction w (the latter statement follows from $f_{vv} = 0$ and the convexity of f). Similarly,

$$\partial^2_v(\mathcal{B}h)(x_0, t_0) = \lambda \int_0^1 \big(\partial^2_v\big(h(x_0 + \phi)\big) - \phi_{vv} \cdot \nabla h(x_0)\big)\, dz.$$

Thus, since $\nabla f(x_0) = \nabla h(x_0)$,

$$\begin{aligned} \partial^2_v\big(\mathcal{B}(f - h)\big)(x_0, t_0) &= \lambda \int_0^1 \partial^2_v\big(f(x_0 + \phi) - h(x_0 + \phi)\big)\, dz \\ &= \lambda \int_0^1 \partial^2_v\big(f(x_0 + \phi) - h(x_0 + \phi)\big) 1_{\{z:\ |\phi| < 3\delta\}}\, dz \end{aligned}$$

$$
\begin{aligned}
&+ \lambda \int_0^1 \partial_v^2 \left(\delta^2 \int_0^{|\phi|/\delta} \psi(s)\,ds \right) 1_{\{z:\, |\phi| \geqslant 3\delta\}}\,dz \\
&= I_1 + I_2.
\end{aligned}
$$

Here I_1 converges to 0 as δ goes to 0 since $\phi(x,t,z) \neq 0$ for almost all z by (A6) and $\partial_v^2(h(x+\phi))1_{\{z:\, |\phi|<3\delta\}}$ is bounded uniformly in δ at $x = x_0$. Similarly,

$$
\begin{aligned}
I_2 &= \lambda \int_0^1 \big(\delta|\phi|_{vv}\psi\big(|\phi|/\delta\big) + |\phi|_v^2\psi'\big(|\phi|/\delta\big)\big) 1_{\{z:\, |\phi| \geqslant 3\delta\}}\,dz \\
&= \lambda \int_0^1 \delta|\phi|_{vv} 3M 1_{\{z:\, |\phi| \geqslant 3\delta\}}\,dz,
\end{aligned}
$$

so it follows from (A6) that also I_2 converges to 0. Thus, by choosing δ small enough, we find that $\partial_v^2(\mathcal{B}h)(x_0,t_0)$ is arbitrarily close to $\partial_v^2(\mathcal{B}f)(x_0,t_0)$. $\square$

Lemma 4.4. *Let $t_0 \in [0,T]$ be given, and let $f \in C_\alpha^4(\mathbb{R}_+^n) \cap C_{\mathrm{pol}}^2(\mathbb{R}_+^n)$ be a convex function with $f_{vv}(x_0) = 0$ at some point x_0 and for some direction v. Then, for any $\epsilon > 0$ there exists a convex function $h \in C_\alpha^4(\mathbb{R}_+^n) \cap C_{\mathrm{pol}}^2(\mathbb{R}_+^n)$ with $h_{vv}(x_0) = 0$ such that $\partial_v^2(\mathcal{A}h)(x_0,t_0) = \partial_v^2(\mathcal{A}f)(x_0,t_0)$ and $|\partial_v^2(\mathcal{B}h)(x_0,t_0)| \leqslant \epsilon$. Consequently, if $\mathcal{M}$ is LCP, then also $\mathcal{A}$ is LCP.*

Proof. Without loss of generality, assume that $f(x_0) = 0$ and $\nabla f(x_0) = 0$. As is seen in the proof of Theorem 5.1 below,

$$
\begin{aligned}
\partial_v^2(\mathcal{B}h)(x_0,t_0) = \int_0^1 \big(& h_{vv}(x_0+\phi) + 2\phi_v \cdot \nabla h_v(x_0+\phi) + \phi_v Hh(x_0+\phi)\phi_v^* \\
&+ \phi_{vv} \cdot \big(\nabla h(x_0+\phi) - \nabla h(x_0)\big)\big)\,dz
\end{aligned}
\tag{6}
$$

provided h is convex and $h_{vv}(x_0) = 0$ (here Hh denotes the Hessian of h). Thus it suffices to find $\delta > 0$ and h satisfying $h = f$ on $\{|x - x_0| \leqslant \delta\}$ and such that ∇h and Hh are small on $\{|x - x_0| \geqslant \delta\}$.

To do this, let C_1 be a constant such that $f_{ww} \leqslant C_1$ in a neighborhood of x_0 and for all unit directions w. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a smooth non-negative function satisfying $0 \leqslant \varphi \leqslant 1$ such that

$$
\varphi(x) = \begin{cases} 1 & \text{if } |x - x_0| \leqslant 1, \\ 0 & \text{if } |x - x_0| \geqslant 2. \end{cases}
$$

Further, let C_2 be a constant such that $|\varphi_w| \leqslant C_2$ and $|\varphi_{ww}| \leqslant C_2$ for all unit directions w. Let $M > 6C_1C_2$ and let $\psi : [0,\infty) \to [0,\infty)$ be a smooth and non-decreasing function satisfying

$$
\psi(s) = \begin{cases} 0 & \text{for } s \in [0, 1/2), \\ Ms & \text{for } s \in (1,2), \\ 3M & \text{for } s \in (3,\infty). \end{cases}
$$

Now let

$$
h^\delta(x) := f(x)\varphi\big((x - x_0)/\delta\big) + \delta^2 \int_0^{|x-x_0|/\delta} \psi(s)\,ds.
$$

Then $h = h^\delta$ is the wanted function for some δ small enough. Indeed, first note that $\mathcal{A}f(x_0, t_0) = \mathcal{A}h(x_0, t_0)$ since $h \equiv f$ for $|x - x_0| \leqslant \delta$. Also note that straightforward calculations yield that h is convex if δ is small enough. Moreover, $h(x) = k|x - x_0| + b$ for $|x - x_0| \geqslant 3\delta$, where $k = 3M\delta$ and b is some constant. If w is a unit vector with the same direction as ϕ, i.e. if $\phi = |\phi|w$, then it follows that

$$h_{x_i}(x_0 + \phi) - h_{x_i}(x_0) = \int_0^{|\phi|} h_{x_i w}(x_0 + sw)\, ds$$

can be made arbitrarily small (when varying δ) since $h_{x_i w}$ is bounded inside $|x - x_0| \leqslant 3\delta$ (uniformly in δ) and vanishes outside this region. Thus the last term in the right-hand side of (6) can be made arbitrarily small.

Moreover, examining the first three terms of (6) one finds that these together form a second derivative of h, evaluated at $x_0 + \phi$, in the direction $w := v + \phi_v$. Now

$$0 \leqslant h_{ww}(x_0 + \phi) \leqslant k|w|^2/|\phi|$$

for $|\phi| \geqslant 3\delta$. Since k is linear in δ it follows that also the three first terms in (6) can be made arbitrarily small when decreasing δ. □

5. A characterization of LCP models

In [4] it is shown that, within a large class of models, the only differential operators of the form $\mathcal{A}$ which are LCP in dimension $n \geqslant 2$ are the ones corresponding to geometric Brownian motions. In that sense, there are not very many convexity preserving diffusion models in higher dimensions. In this section we study the LCP-condition for the operator $\mathcal{B}$ corresponding to the jump part of X.

Let Hf denote the Hessian of a function f. The following theorem gives a precise description of which jump structures ϕ give rise to an integro-differential operator $\mathcal{B}$ which is LCP.

Theorem 5.1. *The operator $\mathcal{B}$ is LCP at a point (x, t) if and only if for all directions $v \in \mathbb{R}^n \setminus \{0\}$ we have*

$$\int_0^1 \big(f_{vv}(x + \phi) + 2\phi_v \cdot \nabla f_v(x + \phi) + \phi_v Hf(x + \phi)\phi_v^* \\ + \phi_{vv} \cdot \big(\nabla f(x + \phi) - \nabla f(x)\big)\big)\, dz \geqslant 0 \tag{7}$$

for all convex functions $f \in C_\alpha^4(\mathbb{R}_+^n) \cap C_{\mathrm{pol}}^2(\mathbb{R}_+^n)$ with $f_{vv}(x) = 0$.

Proof. Assume that $f \in C_\alpha^4(\mathbb{R}_+^n) \cap C_{\mathrm{pol}}^2(\mathbb{R}_+^n)$ is convex and that $f_{vv} = 0$ at x. Straightforward calculations yield that

$$\partial_v^2(\mathcal{B}f)(x) = \int_0^1 \big(f_{vv}(x + \phi) + 2\phi_v \cdot \nabla f_v(x + \phi) + \phi_v Hf(x + \phi)\phi_v^* \\ + \phi_{vv} \cdot \nabla f(x + \phi) - \phi_{vv} \cdot \nabla f(x) - 2\phi_v \cdot \nabla f_v - \phi \cdot \nabla f_{vv}\big)\, dz.$$

From the assumption $f_{vv} = 0$ at x it follows, due to the convexity of f, that $\nabla f_v = \nabla f_{vv} = 0$ at x. Thus

$$\partial_v^2(\mathcal{B}f)(x) = \int_0^1 \big(f_{vv}(x+\phi) + 2\phi_v \cdot \nabla f_v(x+\phi) + \phi_v Hf(x+\phi)\phi_v^* \\ + \phi_{vv} \cdot \big(\nabla f(x+\phi) - \nabla f(x)\big)\big)\, dz.$$

Consequently, $\mathcal{B}$ is LCP if and only if (7) holds for all directions $v \in \mathbb{R}^n \setminus \{0\}$ and all $f \in C_\alpha^4(\mathbb{R}_+^n) \cap C_{\mathrm{pol}}^2(\mathbb{R}_+^n)$ with $f_{vv}(x) = 0$. $\square$

Corollary 5.2. *Let $(x,t) \in \mathbb{R}_+^n \times [0,T]$. If*

$$\int_0^1 \phi_{vv} \cdot \big(\nabla f(x+\phi) - \nabla f(x)\big)\, dz \geqslant 0 \tag{8}$$

for all convex functions $f \in C_\alpha^4(\mathbb{R}_+^n) \cap C_{\mathrm{pol}}^2(\mathbb{R}_+^n)$ and all directions $v \in \mathbb{R}^n \setminus \{0\}$, then the operator $\mathcal{B}$ is LCP at (x,t).

Proof. Since f is convex, and since $f_{vv} + 2\phi_v \cdot \nabla f_v + \phi_v Hf\phi_v^*$ is the second derivative of f in the $(v + \phi_v)$-direction, it is clear that (8) is sufficient for the LCP-condition. $\square$

Remark. If for all $i = 1, \dots, n$, the function ϕ_i is convex in x at all points (x,t,z) where $\phi_i(x,t,z)$ is positive, and ϕ_i is concave in x at all points (x,t,z) where $\phi_i(x,t,z)$ is negative, then (8) is clearly satisfied. This sufficient condition for preservation of convexity was used in [5] in a one-dimensional setting. Also note that it is possible to show that condition (8) is strictly weaker than condition (7).

6. The case of only finitely many possible jump sizes

In this section we investigate models with only finitely many possible jump sizes at each time. More specifically, we assume that for each fixed x and t, the function $z \mapsto \phi(x,t,z)$ takes at most finitely many values.

Theorem 6.1. *Assume there are only finitely many jump sizes, and let $(x,t) \in \mathbb{R}_+^n \times [0,T]$. Then the following conditions are equivalent*:

(i) *$\mathcal{B}$ is LCP at (x,t).*
(ii) (7) *holds for all directions v and all convex $f \in C_\alpha^4(\mathbb{R}_+^n) \cap C_{\mathrm{pol}}^2(\mathbb{R}_+^n)$ with $f_{vv} = 0$.*
(iii) (8) *holds for all directions v and all convex $f \in C_\alpha^4(\mathbb{R}_+^n) \cap C_{\mathrm{pol}}^2(\mathbb{R}_+^n)$.*

Proof. In view of Theorem 5.1 and Corollary 5.2 we only need to show the implication (ii) $\Rightarrow$ (iii).

To do this, let $f \in C_\alpha^4(\mathbb{R}_+^n) \cap C_{\mathrm{pol}}^2(\mathbb{R}_+^n)$ be convex. Since there are only finitely many possible values of ϕ, we can deform f to be flat around all possible values of $x + \phi$ and also in a neighborhood of x without altering the first derivative at these points. Accordingly, the first three terms of the integrand in (7) vanish, whereas the last term remains unchanged, which finishes the proof. □

Theorem 6.2. *Assume there are only finitely many jump sizes and that the model is convexity preserving. Then, for each $i = 1, \dots, n$, the "jump volatility"*

$$\sqrt{\frac{\lambda^2(t) \int_0^1 \phi_i^2(x,t,z)\,dz}{x_i^2}}$$

of the ith asset is increasing as a function of x_i at each fixed time t.

Proof. Since the model is convexity preserving, it follows from Theorems 4.2 and 6.1 that inequality (8) holds at all points and for all convex functions $f \in C_\alpha^4(\mathbb{R}_+^n) \cap C_{\mathrm{pol}}^2(\mathbb{R}_+^n)$. Choosing $f = x_i^2$ in (8) gives that for all $i = 1, \dots, n$ and for all directions v we have

$$\int_0^1 (\phi_i)_{vv}\phi_i\,dz \geqslant 0 \tag{9}$$

at all points (x,t). Fix i and let

$$\psi(x,t) := \frac{\int_0^1 \phi_i^2(x,t,z)\,dz}{x_i^2}.$$

Differentiating ψ with respect to x_i gives

$$x_i^4 \psi_{x_i} = 2\int_0^1 \left(x_i^2 \phi_i (\phi_i)_{x_i} - x_i \phi_i^2\right) dz.$$

Using (9) with $v = e_{x_i}$ (here e_{x_i} denotes the ith unit coordinate vector), integration by parts, $\phi_i \to 0$ as $x_i \to 0$ and Jensen's inequality we find

$$\begin{aligned} 0 &\leqslant x_i^2 \int_0^{x_i}\int_0^1 (\phi_i)_{x_i x_i}\phi_i\,dz\,dx_i \\ &= \int_0^1 \left(x_i^2 \phi_i (\phi_i)_{x_i} - x_i^2 \int_0^{x_i} (\phi_i)_{x_i}^2\,dx_i\right) dz \\ &\leqslant \int_0^1 \left(x_i^2 \phi_i (\phi_i)_{x_i} - x_i \phi_i^2\right) dz. \end{aligned} \tag{10}$$

This shows that ψ_{x_i} is non-negative, which finishes the proof. □

To the best of our knowledge, not very many models in finance have increasing volatilities. Instead, models have typically large volatilities for small values of the underlyings. If we restrict our attention to these typical models, we show below that preservation of convexity is a rather special property.

Theorem 6.3. *Let $n \geqslant 2$. Assume that there are only finitely many possible jump sizes and that for all i and j, β_{ij} is a function merely of x_i and t. Also assume that for all $i = 1, \dots, n$ and for each fixed time t the "total volatility"*

$$\sqrt{\frac{\beta_{i1}^2(x_i,t) + \cdots + \beta_{in}^2(x_i,t) + \lambda^2(t)\int_0^1 \phi_i^2(x,t,z)\,dz}{x_i^2}}$$

of the ith asset is not an increasing function of x_i, unless it is constant. If the model is convexity preserving, then β_{ij} and ϕ_i are linear in x_i for all i and j, and ϕ_i does not depend on x_j for $j \neq i$. More explicitly, there exist functions $\gamma_{ij} : [0,T] \to \mathbb{R}$ and $\gamma_i : [0,T] \times [0,1]$ such that

$$\beta_{ij}(x_i,t) = x_i \gamma_{ij}(t) \tag{11}$$

and

$$\phi_i(x,t,z) = x_i \gamma_i(t,z) \tag{12}$$

for almost all z.

Proof. First note that according to Theorem 4.2 both the operators $\mathcal{A}$ and $\mathcal{B}$, corresponding to the diffusion part and the jump part of X, respectively, are LCP. Now note that if the total volatility is strictly decreasing in some interval, then either a "diffusion volatility"

$$\sqrt{\frac{\beta_{i1}^2(x_i,t) + \cdots + \beta_{in}^2(x_i,t)}{x_i^2}}$$

or a "jump volatility"

$$\sqrt{\frac{\lambda^2(t)\int_0^1 \phi_i^2(x,t,z)\,dz}{x_i^2}}$$

is strictly decreasing in some interval. However, the proof of Theorem 2.3 in [4] implies that all diffusion volatilities are increasing, and Theorem 6.2 above implies that all jump volatilities are increasing. Consequently, all diffusion volatilities and all jump volatilities are constant in x_i.

It then follows from the proof of Theorem 2.3 in [4] that all β_{ij} are linear in x_i. Moreover, if the ith jump volatility is constant in x_i, then the corresponding inequalities in (10) reduce to equalities. Since Jensen's inequality reduces to an equality if and only if the integrand is constant, we find that for almost all z the function ϕ_i has to be linear in x_i. It thus only remains to show that ϕ_i does not depend on x_j, $j \neq i$. To do this we fix $j \neq i$, and we plug $v = e_{x_i} + s e_{x_j}$ and $\phi_i = \gamma_i(\bar{x},t,z)x_i$ into inequality (9), where $s \in \mathbb{R}$ and $\bar{x} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. We find that

$$0 \leqslant \int_0^1 \phi_i(\phi_i)_{vv}\,dz = 2sx_i \int_0^1 \gamma_i(\gamma_i)_{x_j}\,dz + s^2 x_i^2 \int_0^1 \gamma_i(\gamma_i)_{x_j x_j}\,dz.$$

Since this expression is non-negative for any choice of $s \in \mathbb{R}$, we must have

$$\int_0^1 \gamma_i(\gamma_i)_{x_j}\,dz = 0. \tag{13}$$

Now let $x'_j \in \mathbb{R}_+$. Performing similar calculations as in (10), but with $v = e_{x_j}$, we find that

$$0 \leqslant \left(x_j - x'_j\right)^2 \int_0^1 \left(\phi_i(x,t,z)(\phi_i)_{x_j}(x,t,z) - \phi_i(x,t,z)(\phi_i)_{x_j}(x',t,z)\right) dz$$
$$- \left|x_j - x'_j\right| \int_0^1 \left(\phi_i(x,t,z) - \phi_i(x',t,z)\right)^2 dz,$$

where $x' = (x_1, \dots, x_{j-1}, x'_j, x_{j+1}, \dots, x_n)$. In view of (13), the first integral vanishes. Consequently, ϕ_i is z-almost surely constant in x_j, which finishes the proof. □

Remark. Note that the models satisfying (11) and (12) are all convexity preserving. Indeed, by explicit solution formulas,

$$X_i(T_0) = X_i(0) \exp\left\{-\frac{1}{2}\sum_{j=1}^n \int_0^{T_0} \gamma_{ij}^2(t)\,dt + \sum_{j=1}^n \int_0^{T_0} \gamma_{ij}(t)\,dW_j\right\} J_i(T_0),$$

where

$$J_i(T_0) = \exp\left\{-\int_0^{T_0}\int_0^1 \lambda(t)\gamma_i(t,z)\,dz\,dt\right\} \prod_{0\leqslant z\leqslant 1}\prod_{0\leqslant t\leqslant T_0} \left(1+\gamma_i(t)\right)^{v(z,t)},$$

so $X_i(T_0)$ is linear in the starting value $X_i(0)$. Consequently,

$$u(x,0) = E_{x,0}\, g\left(X(T_0)\right)$$

is convex in $X(0)$ provided g is convex.

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