OPTIONS WRITTEN ON STOCKS WITH KNOWN DIVIDENDS

ERIK EKSTRÖM and JOHAN TYSK
Department of Mathematics
Uppsala University
Box 480, SE-751 06 UPPSALA
SWEDEN

Abstract

There are two common methods for pricing European call options on a stock with known dividends. The market practice is to use the Black-Scholes formula with the stock price reduced by the present value of the dividends. An alternative approach is to increase the strike price with the dividends compounded to expiry at the risk-free rate. These methods correspond to different stock price models and thus in general give different option prices. In the present paper we genereralize these methods to time and level dependent volatilities and to arbitrary contract functions. We show, for convex contract functions and under very general conditions on the volatility, that the method which is market practice gives the lower option price. For call options and some other common contracts we find bounds for the difference between the two prices in the case of constant volatility.

1 Introduction

In this article we consider two common methods for pricing European call options on a stock with known dividends. In both methods we follow the approach of Heath and Jarrow, compare [4], that in modelling the stock price we will separate the capital gains process from the impact of dividends. We will here essentially follow the presentation of this material that can be found in [7], pages 147-149. The capital gains process G_t is assumed to follow the usual stochastic differential equation

$$dG_t = \mu G_t + \sigma G_t \, d\tilde{W}_t, \tag{1.1}$$

where $\tilde{W}$ is a Brownian motion. One then finds the equivalent measure under which the discounted gains process $G_t B_t^{-1}$ is a martingale, where B_t is a bank account. We will in this article assume that the interest rate is deterministic and in this context we can then just as well assume that the short rate is some constant r so that $B_t = B_0 e^{rt}$. Since it is the discounted gains process that follows a martingale and not the stock process, we can use risk neutral valuation if we express the data of the Black-Scholes formula in terms of G_t instead of the stock price process S_t. Of course, G_t is not directly observable. Thus we have to express G_t in terms of S_t and the dividends. The relationship between S_t and G_t can be modelled in essentially two different ways. Firstly, the process S, representing the price of a stock which pays dividends $\kappa_1, ..., \kappa_m$ at times $T_1, ..., T_m$ before expiration T may be introduced by setting

$$S_t = G_t + \sum_{j=1}^{m} \kappa_j e^{-r(T_j - t)} \mathrm{I}_{[0,T_j]}(t), \tag{1.2}$$

where I denotes the indicator function. Let D_0 denote the sum above for $t = 0$, i.e. the dividends discounted to time 0. In this model we notice that $G_T = S_T$ but $G_0 = S_0 - D_0$. Thus, to price options in this model we should subtract the present value of the dividends from the stock price before using, for instance, Black-Scholes formula. This, in fact, is standard market practice.

Another, perhaps more theoretically satisfying model, is to put

$$S_t = G_t - \sum_{j=1}^{m} \kappa_j e^{r(t - T_j)} \mathrm{I}_{[T_j,T]}(t). \tag{1.3}$$

Letting C_T denote the second term above for $t = T$, i.e. the dividends compounded at the risk free rate to time T, we note that in this model $G_0 = S_0$ but $G_T = S_T + C_T$. Evaluating a call option we should thus increase the strike price by C_T.

In the next section we first extend these two methods to other contracts and models for stock processes. We then show that for convex contract functions and for very general volatilities, the method which is market practice gives the lower option price, compare Theorem 2.3. When the gains process G_t is modelled by geometric Brownian motion we estimate the difference between the two prices, compare Theorem 2.6, in the case of call options. Finally, in Section 3 we show that the inequality does not always hold for options written on several underlying assets.

2 Comparisons of option prices

In this section we work with the bank account as a numeraire, primarily because in the notation of the introduction, $D_0/B_0 = C_T/B_T$. This quantity, representing the discounted dividends, we call D. By abuse of notation, we henceforth denote by G_t and S_t the discounted gains process and the discounted stock price process, respectively. In the notation used in the introduction these processes are given by $B_t^{-1}G_t$ and $B_t^{-1}S_t$. We then choose a risk-neutral measure under which

$$dG_t^g = \alpha(G_t^g, t)\, dW_t, \qquad G_0^g = g, \tag{2.1}$$

where W is a Wiener process. The diffusion factor $\alpha(g,t)$ is given by $\alpha(g,t) = \sigma(g,t)g$, thus generalizing our study to the case of level- and time-dependent volatilities.

Definition 2.1. A function $\alpha = \alpha(g,t)$ defined on $(0,\infty) \times [0,T]$ is said to be *locally Hölder(1/2)* in the g-variable if for every $K > 1$ there exists a constant C_K such that

$$|\alpha(x,t) - \alpha(y,t)| \leq C_K |x-y|^{1/2}$$

for all $x, y \in [K^{-1}, K]$ and $t \in [0,T]$.

Definition 2.2. The function $\alpha = \alpha(g,t)$ is said to be *admissible* if

- it is measurable and locally Hölder(1/2) on $(0,\infty) \times [0,T]$;
- $\alpha(g,t) = 0$ for $g \leq 0$;
- there is a constant C such that $|\alpha(g,t)| \leq C(1+g)$ for all g and $t \in [0,T]$;
- for any fixed $t \in [0,T]$ the function $|\alpha(g,t)|$ is non-decreasing in g.

The first three conditions ensure existence of a unique solution, absorbed in 0, to equation (2.1). Note that we have no assumption of continuity at $g = 0$. The non-decreasing property of $\alpha(g,t)$ is very natural from an economical point of view. It allows the volatility $\sigma(g,t)$ (or rather $|\sigma(g,t)|$) to decrease in g, but the absolute fluctuations of G_t, measured by $\alpha(g,t)$, are assumed to increase in g.

Now, consider the two different models for the value of an option written on a dividend-paying stock considered in the introduction. For stock prices $s \geq D$, the price of an option with pay-off $\psi(S_T)$ in the first model is, using our new notation,

$$P_1(s) = E\psi(G_T^{s-D}) \tag{2.2}$$

where we to avoid negative stock prices assume that 0 is an absorbing barrier for G_t^{s-D}. In the second model the price is

$$P_2(s) = E\psi(G_T^s - D), \tag{2.3}$$

where, again to avoid negative stock prices, it is assumed that D is an absorbing barrier for the process G_t^s.

Theorem 2.3. *Assume that $\alpha = \alpha(g, t)$ is admissible in the sense of Definition 2.2 and that the contract function ψ is convex. Then the option prices P_1 and P_2 defined above satisfy the inequality*

$$P_1(s) \le P_2(s)$$

for all $s \ge D$.

Proof. Let G^{s-D} be the solution to (2.1) starting at $s - D$, and let G^s be the solution to the stochastic differential equation

$$dG_t^s = \alpha(G_t^s, t) 1_{\{G_t^s > D\}} \, dW_t$$

starting at s. Observe that G^{s-D} is absorbed at 0 and G^s is absorbed at D. Define the stochastic processes $X_t := G_t^{s-D}$ and $Y_t := G_t^s - D$, and let $\beta(g, t) := \alpha(g + D, t) 1_{\{g>0\}}$. Then, since α is admissible,

$$|\beta(g, t)| \ge |\alpha(g, t)| \text{ for all } g \ge 0 \text{ and for all } t \in [0, T]. \tag{2.4}$$

Moreover,

$$dX_t = dG_t^{s-D} = \alpha(G_t^{s-D}, t) \, dW_t = \alpha(X_t, t) \, dW_t, \qquad X_0 = s - D,$$

and

$$dY_t = dG_t^s = \alpha(G_t^s, t) 1_{\{G_t^s > D\}} \, dW_t = \beta(Y_t, t) \, dW_t, \qquad Y_0 = s - D,$$

and 0 is an absorbing barrier both for X_t and for Y_t. Since

$$P_1(s) = E\psi(X_T)$$

and

$$P_2(s) = E\psi(Y_T),$$

and since the price of an option with convex pay-off is increasing in the volatility (Janson and Tysk (2002), Theorem 7, see also [1], [5] and [3]), it follows that (2.4) implies that $P_1(s) \le P_2(s)$. □

Remark 2.4. Note that if the diffusion factor $\alpha = \alpha(t)$ is a function of t alone, then the two option prices are the same. Also note that we cannot remove the condition that α should be non-decreasing as the next example shows.

Example 2.5. Let $M > 0$ and $\sigma > 0$ and define the volatility function α by

$$\alpha(g, t) := \begin{cases} M - g & \text{if } g \le M; \\ 0 & \text{if } g > M. \end{cases}$$

Choose positive numbers s, K, D such that $s > D$ and $K < M < s < K + D$, and consider a call option with strike price K. Then

$$P_2(s) = E(G_T^s - K - D)^+ = 0 < E(G_T^{s-D} - K)^+ = P_1(s).$$

□

Now that we know that $P_1(s) \le P_2(s)$ it is natural to look for bounds on the difference $P_2(s) - P_1(s)$. The following result establishes such a bound for the call option when the gains process follows a geometric Brownian motion.

Theorem 2.6. *Let G_t be a geometric Brownian motion and consider a call option with strike price K. In other words,*

$$G_t = g\exp\{-\frac{\sigma^2}{2}t + \sigma W_t\}$$

and $\psi(s) = (s-K)^+$ for some positive constants σ and K. Then

$$0 \le P_2(s) - P_1(s) \le \frac{D\sigma\sqrt{T}}{\sqrt{2\pi}}.$$

Proof. First we introduce the auxiliary process $H_t := \exp\{-\frac{\sigma^2}{2}T + \sigma W_T\}$ and the stopping time

$$\tau_D := \inf\{t \ge 0; sH_t \le D\}.$$

Then the inequality $a^+ - b^+ \le (a-b)^+$ yields that

$$\begin{aligned}
0 &\le P_2(s) - P_1(s) \\
&= E(sH_{T\wedge\tau_D} - D - K)^+ - E((s-D)H_T - K)^+ \\
&\le E(sH_{T\wedge\tau_D} - D - (s-D)H_T)^+ \\
&= E1_{\{\tau_D < T\}}(-(s-D)H_T)^+ + DE1_{\{\tau_D \ge T\}}(H_T - 1)^+ \\
&\le DE(H_T - 1)^+ \\
&= \frac{D}{\sqrt{2\pi}}\int_{\sigma\sqrt{T}/2}^{\infty}(\exp\{-\frac{\sigma^2 T}{2} + \sigma\sqrt{T}y\} - 1)\exp\{-\frac{y^2}{2}\}\,dy \\
&= \frac{D}{\sqrt{2\pi}}\int_{\sigma\sqrt{T}/2}^{\infty}\exp\{-\frac{(y-\sigma\sqrt{T})^2}{2}\}\,dy - \frac{D}{\sqrt{2\pi}}\int_{\sigma\sqrt{T}/2}^{\infty}\exp\{-\frac{y^2}{2}\}\,dy \\
&= \frac{D}{\sqrt{2\pi}}\int_{-\sigma\sqrt{T}/2}^{\sigma\sqrt{T}/2}\exp\{-\frac{y^2}{2}\}\,dy \le \frac{D\sigma\sqrt{T}}{\sqrt{2\pi}}.
\end{aligned}$$

□

3 Options on several underlying assets

We now consider options written on several underlying assets. Thus we consider a market with an n-dimensional gains process which follows the dynamics

$$dG_t^g = \alpha(G_t^g, t)\,dW_t, \qquad G_0^g = g \tag{3.1}$$

where W is an n-dimensional Brownian motion and $\alpha = (\alpha_{ij}(g,t))_{i,j=1}^n$ is an $n \times n$-matrix which is non-singular for all g and t. If $D = (D_1, .., D_n)$ is the vector of dividends, $s = (s_1, .., s_n)$ is the vector of initial stock prices and $\psi = \psi(s_1, ..s_n)$ is the contract function, then the two option prices are given by

$$P_1(s) = E\psi(G_T^{s-D})$$

and

$$P_2(s) = E\psi(G_T^s - D),$$

respectively. In this section we show that if the ith asset only depends on the ith component of W, then the same conclusion as in Theorem 2.3 can be drawn. However, if we allow the assets to be dependent, then counter-examples can be constructed.

Theorem 3.1. *Assume that the volatility matrix is diagonal with diagonal elements $\alpha_{ii}(g,t) = \alpha_{ii}(g_i,t)$, i.e. the volatility of the i:th asset only depends upon the current value of that asset and time, where α_{ii} is admissible. Also assume that the contract function ψ is convex. Then*

$$P_1(s) \leq P_2(s).$$

Proof. Let the gains process G be as in (3.1) where W is an n-dimensional Brownian motion. Define $\beta_{ii}(g_i,t) := \alpha_{ii}(g_i + D_i, t)1_{\{g_i>0\}}$, where D_i is the dividends of the ith asset, and let $X_t := G_t^{s-D}$ and $Y_t := G_t^s - D$, where the ith component of G_t^s is assumed to be absorbed at D_i. Then

$$dX_t = \alpha(X_t,t)\,dW_t, \qquad X_0 = s - D,$$

and

$$dY_t = \beta(Y_t,t)\,dW_t, \qquad Y_0 = s - D.$$

Since

$$P_1(s) = E\psi(X_T),$$

and

$$P_2(s) = E\psi(Y_T),$$

and since $|\beta_{ii}(g_i,t)| \leq |\alpha_{ii}(g_i,t)|$ for all g_i and t, it follows from Proposition 4.9 in [2] that

$$P_1(s) \leq P_2(s).$$

□

Example 3.2. (Dependent assets.) Consider an option written on two stocks, the first of which has a gains process modelled by the process Y^1 and pays known dividends $D = D_1 > 0$. Let

$$dY_t^1 = \alpha(Y_t^1,t)\,dW_t, \qquad Y_0^1 = s,$$

with D as an absorbing barrier and W is a (one-dimensional) Brownian motion. Next, let $Y_t^2 := Y_t^1 - D$ model the price of a stock paying no dividends. Then the volatility matrix is singular (which is usually not permitted), but this does not make a significant difference in this example; to get a non-singular volatility matrix we can slightly modify the system by adding a small disturbance ϵV_t to Y^2. Let $\psi = \psi(s_1, s_2)$ be convex and such that $\psi(s_1, s_2) \geq 0$ and $\psi(s, s) = 0$. Then

$$P_2(s, s - D) = E\psi(Y_T^1 - D, Y_T^2) = E\psi(Y_T^2, Y_T^2) = 0.$$

However, defining the gains processes X^1 and X^2 by

$$dX_t^1 = \alpha(X_t^1, t)\, dW_t, \qquad X_0^1 = s - D,$$

and $X_t^2 = Y_t^2$, we get

$$P_1(s, s - D) = E\psi(X_T^1, X_T^2) \geq 0.$$

It is easy to see that the last inequality can be made strict if ψ and α are chosen properly. $\square$

As in the previous section, in certain cases it is easy to find bounds on the difference of the prices in the two different models.

Theorem 3.3. *Consider the call option on the maximum of n assets. Assume that the gains processes follow independent geometric Brownian motions, i.e. $G_t^i = G_0^i H_t^i$ where*

$$H_t^i := \exp\{-\frac{\sigma_i^2}{2} + \sigma_i W_t^i\}.$$

Then the option prices satisfy the following inequalities:

$$0 \leq P_2(s_1, .., s_n) - P_1(s_1, .., s_n) \leq \frac{\sqrt{T}}{\sqrt{2\pi}}(D_1\sigma_1 + ... + D_n\sigma_n).$$

Proof. The left inequality is a direct consequence of Theorem 3.1. As for the second one, introducing

$$\tau_i := \inf\{t \geq 0; s_i H_t^i \leq D_i\}$$

and using the inequality

$$(\max_{1\leq i\leq n}\{a_i\} - K)^+ - (\max_{1\leq i\leq n}\{b_i\} - K)^+ \leq \Sigma_{i=1}^n (a_i - b_i)^+$$

we get

$$\begin{aligned} P_2(s) - P_1(s) &= E(\max_{1\leq i\leq n}\{s_i H_{T\wedge\tau_i}^i - D_i\} - K)^+ \\ &\quad -E(\max_{1\leq i\leq n}\{(s_i - D_i)H_T^i\} - K)^+ \\ &\leq \Sigma_{i=1}^n D_i E(H_T^i - 1)^+, \end{aligned}$$

and so the result follows in the same way as in the proof of Theorem 2.6. $\square$

References

[1] Y. Bergman, B. Grundy and Z. Wiener, *General properties of option prices*, J. Finance **51** (1996), 1573-1610.

[2] E. Ekström, *Properties of American option prices*, Preprint.

[3] N. El Karoui, M. Jeanblanc-Picque and S. Shreve, *Robustness of the Black and Scholes Formula,* Math. Finance **8** (1998), 93-126.

[4] D. C. Heath and R. A. Jarrow, Ex-dividend stock price behavior and arbitrage opportunities. *J. Business* **61** (1988), 95-108.

[5] D. Hobson, *Volatility misspecification, option pricing and superreplication via coupling*, Ann. Appl. Probab. **8** (1998), 193-205.

[6] S. Janson and J. Tysk, *Volatility time and properties of option prices.* To appear in Ann. Appl. Probab.

[7] M. Musiela and M. Rutkowski, *Martingale Methods in Financial Modelling*, Springer Verlag Berlin Heidelberg New York (1998).