UPPSALA UNIVERSITET
Matematiska institutionen
Christer Kiselman

ÖVNINGAR, Blad 6
Teori för differentialekvationer
1998 02 10

6.1. Solve the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \text{ when } t > ax; \qquad u(x,t) = 0 \text{ and } \frac{\partial u}{\partial t}(x,t) = x \text{ when } t = ax,$$

where a is a constant $\neq \pm 1/c$. *Hint:* Introduce new coordinates with the help of the Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \quad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}},$$

for a suitable v, or, even simpler,

$$x' = x - vt, \quad t' = t - vx/c^2.$$

6.2. Solve the Dirichlet problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in the square } 0 \leqslant x \leqslant \pi, \quad 0 \leqslant y \leqslant \pi,$$

with the boundary values $u(x,0) = u(0,y) = u(\pi,y) = 0$ and $u(x,\pi) = \sin mx$.

6.3. Let g be a function which is continuous on $\mathbf{R}$ and satisfies $g(x) \leqslant C/(1+x^2)$. Define a function u on $\mathbf{R}^2$ as

$$u(x) = \frac{1}{2\pi}\int_{\mathbf{R}} \log((x_1 - y_1)^2 + x_2^2) g(y_1)dy_1.$$

Prove that $\Delta u = 0$ when $x_2 > 0$ and that $u_{x_2}(x_1,x_2) \to g(x_1)$ as $x_2 \to 0$.

6.4. Decide whether the following statements are true or false.
a) If u is a harmonic function in the plane, then e^u is subharmonic.
b) If u is a harmonic function in the plane, then $\log(1+u^2)$ is subharmonic.

6.5. The potential of a mass distribution with constant surface density on a sphere is defined by

$$u(x) = \gamma \int_{|y|=r} \frac{1}{|x-y|} dS(y), \qquad x \in \mathbf{R}^3,$$

where γ is a positive constant. Determine u.

6.6. Prove that if u is a bounded solution to $\Delta u = u$ in all of $\mathbf{R}^n$, $n \geqslant 2$, then u must be zero. *Hint:* Use the translation invariance and the rotation invariance of the operator and then compare it with the solution of, e.g., $v(x) = a + b|x|^2$, where the constants a and b are chosen so that $0 \leqslant \Delta v \leqslant v$.