How can we measure deviations from convexity?

Christer Oscar Kiselman

Contents:

Abstract

We ask questions about possible measures of deviation from convexity for sets in $\mathbf{R}^n$ and $\mathbf{Z}^n$ as well as for functions defined on $\mathbf{R}^n$ and $\mathbf{Z}^n$.

1. Introduction

If a subset of $\mathbf{R}^n$ is not convex, it can still be more or less close to a convex set, in particular to its convex hull. We are thus led to study a kind of distance from a set A to its convex hull, and, more generally its distance to the family of all convex sets.

We shall start with some examples of measures of deviation from convexity and ask which properties they have. We formulate these properties as axioms. Then we ask about all measures of deviation from convexity which satisfy these axioms. Maybe the fauna of all these examples will lead us to a revision of the set of axioms. With the new set of axioms, we ask again for examples. This defines an algorithm between families of measures and families of axioms, a kind of dynamic duality between the two. We might hope that the process stabilizes after a finite number of iterations. If so, we have arrived at a satisfying family of axioms defining a satisfying family of measures.

The present version contains two new sections, Sections 4 and 5, which have been added following important suggestions by Gerd Brandell.

Questions on minimal coverings of hyperplanes by dilations of its discretizations—a related problem—are studied by Adama Arouna Koné in his thesis (2016) and his article (2017).

2. Measures for subsets of a vector space

Let us denote by $\mathbf{cvxh}(A)$ the convex hull of a set A in a vector space, and by $\lambda(A)$ the Lebesgue measure of a subset A of $\mathbf{R}^n$.

Of the possible distances between sets, an example which comes to mind easily is

$$\mu_1(A) = \frac{\lambda(\mathbf{cvxh}(A) \smallsetminus A)}{\lambda(\mathbf{cvxh}(A))}, \qquad A \subset \mathbf{R}^n,$$

assuming that $0 < \lambda(\mathbf{cvxh}(A)) < +\infty$. This functional measures how far A is from its convex hull. If we take the Lebesgue measure in the affine subspace $E_{\mathbf{cvxh}(A)}$ of $\mathbf{R}^n$ spanned by $\mathbf{cvxh}(A)$, then $\lambda(\mathbf{cvxh}(A)) > 0$. It is clear that $0 \leqslant \mu_1(A) \leqslant 1$. If A is convex, then $\mu_1(A) = 0$, but the converse does not hold unless A is a singleton, equivalently $\dim E_{\mathbf{cvxh}(A)} = 0$.

We may also study

$$\mu_2(A) = \inf_C \frac{\lambda((A \smallsetminus C) \cup (C \smallsetminus A))}{\lambda(C)} = \inf_C \frac{\lambda((A \cup C) \smallsetminus (A \cap C))}{\lambda(C)}, \qquad A \subset \mathbf{R}^n,$$

taking the infimum over all convex sets C, not just $C = \mathbf{cvxh}(A)$.

Question 2.1. *Which properties of μ_2 differ from those of μ_1?* □

Example 2.2. Let us take A as a thin crescent

$$A = \left\{(x, y) \in \mathbf{R}^2;\ -1 \leqslant x \leqslant 1 \text{ and } b\sqrt{1 - x^2} \leqslant y \leqslant \sqrt{1 - x^2}\right\}$$

for a number b just a little smaller than 1. The convex hull of A is a half-disk. A good exercise is to calculate $\mu_1(A)$ and $\mu_2(A)$. □

Question 2.3. *A natural question is whether μ_1 is the restriction to $B = \mathbf{cvxh}(A)$ of some functional $\mathrm{M}_1\colon (A, B) \mapsto \mathrm{M}_1(A, B)$ defined for all sets A and B. Do the measures that admit a sufficiently interesting extension have other properties than those who do not?* □

It is clear that the functionals μ_1 and μ_2 satisfy the following four axioms.

Axiom 2.4. *$\mu(A + c) = \mu(A)$, $c \in \mathbf{R}^n$: μ is invariant under translations by a vector in $\mathbf{R}^n$.*

Axiom 2.5. *$\mu(sA) = \mu(A)$, $s > 0$: μ is invariant under homotheties by a positive scalar.*

Axiom 2.6. *$\mu(T_*(A)) = \mu(A)$, T an isometry: μ is invariant under isometries of $\mathbf{R}^n$. Here $T_*(A) = \{T(x);\ x \in A\}$.*

Axiom 2.7. *$\mu(T_*(A)) = \mu(A)$, T a bijective affine mapping: μ is invariant under affine bijections.*

It is clear that Axiom 2.7 implies Axiom 2.6, which in turn implies Axioms 2.5 and 2.4.

Example 2.8. Let us take the unit disk $D = \{(x, y) \in \mathbf{R}^2;\ x^2 + y^2 \leqslant 1\}$ and remove an ellipsoid $\{(x, y);\ (x/a)^2 + (y/b)^2 < 1\}$, where $0 < a < b < 1$ with a small and b close to 1. Then the set of points in A where $x = 0$ is close to the convex hull $[-1, 1]$, whereas the set of points with $y = 0$ is far from $[-1, 1]$. So the distance to the intersections of convex sets with affine subspaces can vary a lot.

The observation made in this example leads us to two more convenient and hopefully useful definitions:

Definition 2.9.

$$\mu_3(A) = \inf_F \inf_C \frac{\lambda((A \smallsetminus C) \cap F)}{\lambda(A \cap F)},$$

where the second infimum is taken over all subspaces F of $\mathbf{R}^n$.

So here $\mu_3(A) = 0$ if $A \cap F$ is convex for all subspaces F of $\mathbf{R}^n$.

Definition 2.10.

$$\mu_4(A) = \sup_F \inf_C \frac{\lambda((A \smallsetminus C) \cap F)}{\lambda(A \cap F)},$$

where the supremum is taken over all subspaces F of $\mathbf{R}^n$.

Compare μ_3 and μ_4!

3. Measures for discrete subsets

In $\mathbf{Z}^n$, we can require Axiom 2.4 only for $s \in \mathbf{N} \smallsetminus \{0\}$, and Axiom 2.5 only for $c \in \mathbf{Z}^n$. Also Axioms 2.6 and 2.7 have to be interpreted and questioned.

In $\mathbf{Z}^n$ we can define

$$\mu_4(A) = \frac{\mathbf{card}((\mathbf{cvxh}(A) \cap \mathbf{Z}^n) \smallsetminus A)}{\mathbf{card}(\mathbf{cvxh}(A) \cap \mathbf{Z}^n)}, \qquad A \subset \mathbf{Z}^n,$$

for A finite and nonempty. Also of course

$$\mu_5(A) = \inf_C \frac{\mathbf{card}((C \cap \mathbf{Z}^n) \smallsetminus A)}{\mathbf{card}(C \cap \mathbf{Z}^n)}, \qquad A \subset \mathbf{Z}^n,$$

for A finite and nonempty and the infimum ranging over all convex sets C, not just $C = \mathbf{cvxh}(A)$.

We can define

$$\mu_6(A) = \frac{\lambda((\mathbf{cvxh}(A) + S) \smallsetminus (A + S))}{\lambda(\mathbf{cvxh}(A) + S)}, \qquad A \subset \mathbf{Z}^n,$$

assuming A to be finite and nonempty, and S to be a finite structuring element which contains the origin.

We can also use the Hausdorff distance: define

$$\mu_7(A) = \inf_{r>0} \Big(r;\ \mathbf{cvxh}(A) \cap \mathbf{Z}^n \subset A + rS\Big), \qquad A \subset \mathbf{Z}^n,$$

for a structuring element S.

A variant of μ_7 is

$$\mu_8(A) = \inf_{r>0} \inf_C \Big(r;\ A + rS \subset C \cap \mathbf{Z}^n \subset A + rS)\Big), \qquad A \subset \mathbf{Z}^n,$$

for a structuring element S, the first infimum being taken over all convex sets C.

4. A measure using convex subsets of a given set

Gerd Brandell, in a personal message of 2020 July 06, proposes a measure taking into account convex subsets of a given set A in $\mathbf{R}^n$: we define

$$\mu_9(A) = \inf_C \frac{\lambda(A \smallsetminus C)}{\lambda(A)}, \qquad A \subset \mathbf{R}^n,$$

where the infimum is taken over all convex subsets C of A. If A is convex we can take $C = A$ to obtain $\mu_9(A) = 0$.

It is natural to compare this functional with μ_2 of (**??**). Obviously $\mu_2 \geqslant \mu_9$.

An exercise is to calculate $\mu_9(A)$ with A as in Example 2.2.

5. A measure using the star of a set

Gerd Brandell also proposes using the star of a set in constructing a measure of deviation from convexity. We define, given a set A in a vector space, the ***star of*** A ***at a point*** a, denoted by $\mathbf{star}_A(a)$, to be the union of all segments $[a, x]$ contained in A. Clearly A is convex if and only if $\mathbf{star}_A(a) = A$ for all $a \in A$. So we can study

$$\mu_{10}(A) = \sup_{a \in A} \frac{\lambda(A \smallsetminus \mathbf{star}_A(a))}{\lambda(A)}, \qquad A \in \mathbf{R}^n,$$

as well as

$$\mu_{11}(A) = \inf_{a \in A} \frac{\lambda(A \smallsetminus \mathbf{star}_A(a))}{\lambda(A)}, \qquad A \in \mathbf{R}^n.$$

We can of course construct also analogues of μ_9 and μ_{10} in a discrete setting.

6. Deviation of functions from convex functions

Comparing functions is equivalent to comparing sets. If we have a function $F\colon \mathbf{R}^n \to [-\infty, +\infty]$, we can compare it to its convex envelope $f = \mathbf{cvxe}(F)$ and also to any convex function. Usually it is convenient to take a weighted measure

$$\mu_{12} = \inf_f \|(F - f)w\|,$$

for a weight function w, typically tending to zero at infinity, or to use related functions, for instance

$$\mu_{13} = \inf_f \|\mathrm{e}^{-F} - \mathrm{e}^{-f}\|.$$

We may compare such deviations with the deviation of sets by introducing the characteristic functions $F = \chi_A$ and $f = \chi_C$.

References

Kiselman, Christer O. 2004. *Digital Geometry and Mathematical Morphology.* Lecture notes, 95 pages.

Kiselman, Christer Oscar. 2022. *Elements of Digital Geometry, Mathematical Morphology, and Discrete Optimization.* Singapore: World Scientific, xxiii + 461 pages.

Koné, Adama Arouna. 2016. *Géométrie digitale utilisée pour la discrétisation et le recouvrement optimal des objects euclidiens*, 114 pages. Thesis, l'Université des Sciences, des Techniques et des Technologies de Bamako (USTTB).

Koné, Adama Arouna. 2017. Covering a Euclidean line or hyperplane by dilations of its discretization, *Vietnam Journal of Mathematics*, **45**, no. 3, 351–368.