Accepted for publication1 in *Complex Variables and Elliptic Equations* on 2022-01-03

Generalized elementary functions

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Abstract

Ramon Edgar Moore and Alexander M. Gofen introduced a generalization of Joseph Liouville's concept of elementary functions. Gofen even defined two variants of these, viz. scalar generalized elementary functions and vector generalized elementary functions, and formulated a conjecture concerning them. We prove that, for some modified conjectures, the two classes are different.

Keywords: Ordinary differential equations, elementary functions, generalized elementary functions, Joseph Liouville, Joseph Fels Ritt, Alexander Gofen.

MSC: 34 [Ordinary differential equations],
34A12 [Initial value problems, existence, uniqueness, continuous dependence and continuation of solutions to ordinary differential equations],
34A34 [Nonlinear ordinary differential equations and systems, general theory].

1. Introduction

1.1. Liouville

Liouville[1] proved that there are elementary functions which are not the derivative of any elementary function. This is the origin of the present note.

A function $f\colon \Omega \to \mathbf{C}$, where Ω is an open subset of $\mathbf{C}$, is said to be ***elementary in the sense of Liouville*** if it is a sum, difference, product, quotient, or composition of finitely many polynomials, rational functions, trigonometric and exponential functions, and their inverses (thus including algebraic functions and logarithms).

[1] Joseph Liouville (1809–1882). For Liouville's education, his work as a teacher, journal editor, politician, and academician, as well as an analysis of his mathematical work in an historical perspective, see Jesper Lützen's book (1990), where more than seventy pages (pp. 351–422) are devoted to works on elementary functions, most of it by Liouville himself.

The rational function $\mathbf{C} \smallsetminus \{0\} \ni z \mapsto 1/z$ can be extended to all of $\mathbf{C}$ by defining it as ∞ for $z = 0$. We then obtain a continuous function with values in the Riemann sphere $\mathbf{C} \cup \{\infty\}$. But a rational functions of two variables, like $(z_1, z_2) \mapsto z_2/z_1$, typically does not have a limit as $(z_1, z_2) \to (0, 0)$, so it is problematic and deserves special attention.

For any family $\mathscr{F}$ of holomorphic functions $\Omega \to \mathbf{C}$ we define $\mathscr{F}' = \{f';\ f \in \mathscr{F}\}$ as the set of all derivatives of functions in $\mathscr{F}$. If $\mathscr{F}$ is a vector space and $\mathscr{F}'$ is contained in $\mathscr{F}$, we can form the quotient space $\mathscr{F}/\mathscr{F}'$. If $\mathscr{F}$ is the space of all polynomials of degree $\leqslant m$, then $\mathscr{F}/\mathscr{F}'$ is one-dimensional; if $\mathscr{F} = \mathscr{P}$ is the space of all polynomials, then $\mathscr{F}/\mathscr{F}'$ is of dimension zero.

Liouville proved that the function L defined by

$$L'(z) = \mathrm{e}^{z^2}, \quad z \in \mathbf{C}; \quad L(0) = 0, \tag{1.1}$$

is not elementary, thus that any antiderivative of $z \mapsto \mathrm{e}^{z^2}$ is not elementary. So if we denote by $\mathscr{L}$ the family of all elementary functions, the function $t \mapsto \mathrm{e}^{z^2}$ does not belong to $\mathscr{L}'$. During the years 1833 through 1841 he published eleven papers on this theme (Lützen 1990:351).

The theorem which is most important for us here was proved by Liouville in 1833 and was stated as follows by Ritt[2] (1948:40). Let us say that u is ***algebraic in functions*** $v_1, \dots, v_p$ if $u\colon \Omega \to \mathbf{C}$ solves an ordinary differential equation

$$a_m u^{(m)} + \cdots + a_1 u' + a_0 u = 0,$$

where the coefficients a_j are polynomials in the v_j with constant coefficients. If the v_j satisfy differential equations

$$v_j' = f_j(v_1, \dots, v_p), \qquad j = 1, \dots, p,$$

where the f_j are algebraic functions of the v_j, and if u is algebraic in the v_j and has an antiderivative U which is elementary in the v_j, then U has the form

$$U = w_0 + \sum_{j=1}^{q} c_j \log w_j$$

for some contants c_j and some functions $w_0, \dots, w_q$ which are algebraic in the v_j. Clearly this is so for an antiderivative L of L'; L' is algebraic in $v_1 = L'$. It is then proved that L cannot be of this form.

1.2. Work before Liouville

Both Laplace[3] and Abel[4] worked on the problem of antiderivatives of elementary functions before Liouville.

Lützen reports on Laplace's efforts as follows:

> Laplace's sketch anticipated Liouville's discoveries, and if Laplace had published rigorous proofs of the theorems that he claimed to have found, he would certainly have been acknowledged as the founder of the theory of integration in finite form. However, the loose way in which Laplace formulated and proved that an integral can only contain

[2] Joseph Fels Ritt (1893–1951).
[3] Pierre-Simon Laplace (1749–1827).
[4] Niels Henrik Abel (1802–1829).

> the same radicals and exponentials as the integrand sounds more like Fontaine and Condorcet than like a rigorous statement of the 19th century. (Lützen 1990:358)

Nevertheless, Laplace's considerations, although vague, led Liouville to establish his results, "and Liouville claimed for his method of proof the merit of following these intuitive ideas." (Ritt 1948:21).

Also Abel made important contributions to the theory (Ritt 1948:28, Lützen 1990: 358-369). Lützen concludes:

> [...] it was not the inspiration left by Abel that made Liouville interested in his theory, but having learned of Abel's contribution he made ample use of it. (Lützen 1990:369)

1.3. Work after Liouville

Among the literature after Liouville in this field of research, a most important book is that by Ritt (1948), where he mentions five articles which he published in the years 1923–1929. Of other publications on this subject, let me mention the following.

- Alexander M. Ostrowski's[5] article (1946), which generalizes Liouville's result to any field of meromorphic functions;
- Maxwell Rosenlicht's[6] papers (1968, 1972, 1976), of which the first is a purely algebraic proof of the main theorem;
- R. H. Risch's[7] algorithm (1969, 1970, 1976, 1979) which is used to execute the integration of elementary functions on a computer;
- Toni Kasper's book (1980) with an historical account but no proofs;
- Manuel Bronstein's[8] book (1997), where he generalizes and extends the algorithm due to Risch;
- The book by Marius van der Put and Michael F. Singer[9] (2003), which is a general survey, very well received according the reviewer in *MathSciNet*, Pedro Fortuny Ayuso;
- Brian David Conrad's notes (2005), where he establishes a criterion for proving the impossibility result in special cases;
- two important papers by Alexander M. Gofen (2008, 2009), which inspired me to begin the present study;
- the book by Teresa Crespo and Zbigniew Hajto (2011), who study ordinary differential equations from an algebraic-geometric standpoint; and finally
- Askold Georgievich Khovanskii's two papers (2019a, 2019b). In the first, he comments extensively on Ritt's book (1948). In (2019b), he proves a generalization of Liouville's theorem—this article contains all the algebraic background necessary for understanding.

[5]Alexander M. Ostrowski (1893–1986).
[6]Maxwell Rosenlicht (1924–1999).
[7]Robert Henry Risch, PhD 1968, a student of Rosenlicht.
[8]Manuel Bronstein (1963–2005), PhD 1987, a student of Rosenlicht.
[9]Michael F. Singer, PhD 1974, a student of Rosenlicht.

1.4. Generalizations of elementariness

Moore[10] (1966:108) widened the definition of elementary functions by accepting solutions to systems of ordinary differential equations of order one where the derivatives of the unknown functions are rational functions of their values; see Definition 2.5.

Alexander Gofen (2008:642, 2009:826) distinguished among the solutions to systems of ordinary differential equations of order m those functions that are solutions to a scalar differential equation of order m or higher; see Definition 2.3. He also introduced the condition on nonzero denominators, which Moore did not impose.

We shall study both scalar ordinary differential equations and systems of first-order equations.

The scalar equations are of the form

$$u^{(m)}(z) = f(z, u(z), u'(z), \dots, u^{(m-1)}(z)), \qquad z \in \Omega \subset \mathbf{C}, \quad u\colon \Omega \to \mathbf{C}, \tag{1.2}$$

where f is a given function, $f\colon \Omega \times \mathbf{C}^m \to \mathbf{C}$.

The systems of n first-order equations are vector-valued ordinary differential equations:

$$v'(z) = g(z, v(z)), \quad z \in \Omega \subset \mathbf{C}, \quad v\colon \Omega \to \mathbf{C}^n, \tag{1.3}$$

where $g\colon \Omega \times \mathbf{C}^n \to \mathbf{C}^n$.

Here the derivatives are to be understood as in classical complex analysis:

$$f' = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y}\right), \qquad z = x + \mathrm{i}y \in \mathbf{C}, \quad x, y \in \mathbf{R}.$$

In particular, for $f(z) = z^m$, $m \in \mathbf{N}$, we have $f'(z) = mz^{m-1}$; with $f(z) = \mathrm{e}^{\lambda z}$, we have $f' = \lambda f$.

If f is a function of n complex variables $z_1, \dots, z_n$, we shall write f_{z_j} for $\partial f/\partial z_j$.

Occasionally we consider a real independent variable t. Then f' denotes the usual derivative $\mathrm{d}f/\mathrm{d}t$.

The function f in (1.2) can be a polynomial, rational function, a holomorphic or meromorphic function satisfying certain conditions, or, in the real case, the restriction of such functions to some real subspace. The same is true for g in (1.3), although with vector values. As mentioned above, Moore (1966) studied only the family of rational functions of complex variables.

2. Definitions

Definition 2.1. Let $f\colon \Omega \times \mathbf{C}^m \to \mathbf{C}$ be any function. We shall denote by $\mathscr{U}(f)$ the set of all solutions $u\colon \Omega \to \mathbf{C}$ to the equation (1.2) (allowing for all possible initial values), and, given a family $\mathscr{F}$ of functions, by $\mathscr{U}(\mathscr{F})$ the union of all $\mathscr{U}(f)$ with $f \in \mathscr{F}$. Similarly for real-valued functions. We also define $\mathscr{U}_0(f)$ as the solution (or possibly the family of solutions) with initial values $u^{(k-1)}(0) = 0$, $k = 1, \dots, m$. □

Definition 2.2. We shall denote by $\mathscr{V}(g)$ the set of all solutions $v\colon \Omega \to \mathbf{C}^n$ to the vector equation (1.3) (allowing for arbitrary initial values), and by $\mathscr{V}(\mathscr{G}_1 \times \cdots \times \mathscr{G}_n)$

[10] Ramon Edgar (Ray) Moore (1929–2015).

the union of all $\mathscr{V}(g) = \mathscr{V}(g_1, \dots, g_n)$ for $g_j \in \mathscr{G}_j$, $j = 1, \dots, n$. We also define $\mathscr{V}_0(g)$ as the family of solutions with initial values $v_k(0) = 0$, $k = 1, \dots, n$. □

As we see in Remark 2.7 below it may happen that we have non-uniqueness in the two initial-value problems studied.

Definition 2.3. We shall say that a function $u\colon \Omega \to \mathbf{C}$ is ***scalar generalized elementary with respect to a family $\mathscr{F}$ of functions*** if it belongs to $\mathscr{U}(\mathscr{F})$. □

Example 2.4. The function L defined by (1.1) satisfies $L''(z) = 2z\mathrm{e}^{z^2} = 2zL'(z)$, thus $L''(z) = f(z, L(z), L'(z))$ with f as the polynomial defined by $f(s_1, s_2, s_3) = 2s_1s_3$. So it is scalar generalized elementary in the sense of Definition 2.3 with $\mathscr{F} = \mathscr{P}$, the family of polynomials. □

Definition 2.5. We shall say that a vector-valued function $v\colon \Omega \to \mathbf{C}^n$ is ***vector generalized elementary with respect to a family $\mathscr{G}_1 \times \cdots \times \mathscr{G}_n$ of n-tuples of functions*** if it belongs to $\mathscr{V}(\mathscr{G}_1 \times \cdots \times \mathscr{G}_n)$. □

Definition 2.6. We shall say that a function $v_1\colon \Omega \to \mathbf{C}$ is ***vector generalized elementary with respect to a family $\mathscr{G}_1 \times \cdots \times \mathscr{G}_n$ of n-tuples of functions*** if there exists a function $(v_2, \dots, v_n)\colon \Omega \to \mathbf{C}^{n-1}$ such that the n-tuple $(v_1, v_2, \dots, v_n)$ is vector generalized elementary with respect to $\mathscr{G}_1 \times \cdots \times \mathscr{G}_n$ in the sense of Definition 2.5. □

Moore and Gofen usually take $\mathscr{F} = \mathscr{G}_j = \mathscr{R}$, the family of rational functions.

Remark 2.7. Given initial conditions $u(0)$ and $u'(0)$, there may exist several solutions to the equation (1.2). A simple example is to define, given any $a \geqslant 0$, $u(t) = 0$ for $t \leqslant a$ and $u(t) = (t-a)^3$ for $t > a$. This function is of class $C^2(\mathbf{R})$ and satisfies

$$u' = 3u^{2/3}, \quad u'' = 6u^{1/3}, \quad u'' = 2\sqrt{3}(u')^{1/2}, \quad u(0) = 0, \quad u'(0) = 0.$$

So here u satisfies

$$\begin{aligned} u'(t) &= f_1(t, u(t)) && \text{with } f_1(s_1, s_2) = 3s_2^{2/3}, \\ u''(t) &= f_2(t, u(t), u'(t)) && \text{with } f_2(s_1, s_2, s_3) = 6s_2^{1/3}, \text{ as well as} \\ u''(t) &= f_3(t, u(t), u'(t)) && \text{with } f_3(s_1, s_2, s_3) = 2\sqrt{3}\, s_3^{1/2}, \end{aligned}$$

where $(s_1, s_2, s_3) \in \mathbf{R}^3$. (There are similar examples with u of class C^∞.)

For complex z we can take $a \geqslant 0$ and $u(z) = 0$ for $\operatorname{Re} z \leqslant a$; $u(z) = (\operatorname{Re} z - a)^3$ for $\operatorname{Re} z > a$, yielding $u'(z) = \frac{3}{2}u^{2/3}$ and $u''(z) = \frac{3}{2}u^{1/3}$. So also here we can have non-uniqueness. □

We note that in this example the functions f_1, f_2 and f_3 are not Lipschitz continuous. Well-known theorems guarantee that a Lipschitz condition, even a local Lipschitz condition, implies uniqueness.

It is easy to see that scalar elementariness implies vector elementariness (Lemma 7.1). Can we go in the opposite direction? The answer depends of course on which families of functions we consider.

3. Alexander Gofen's conjecture

Alexander Gofen published a conjecture in an article (2008:642). The reader is kindly asked to consult the original formulation in this article. See also his web site (2020). Here I state the conjecture with my notation and how I have understood it.

Conjecture 3.1. *Let a system of first-order ordinary differential equations* (1.3) *be given with a vector-valued rational function* $g = (g_1, \dots, g_n)$*. Fix a* $(1+n)$*-tuple*

$$(z_0, a_1, a_2, \dots, a_n) \in \mathbf{C} \times \mathbf{C}^n$$

and assume that the problem satisfies the following condition with respect to this element of $\mathbf{C} \times \mathbf{C}^n$*.*

> **Condition** (g)**.** *The functions* $g_j = p_j/q_j$ *are quotients of polynomials* p_j *and* q_j*. The denominators* q_j *are all nonzero at* $(z_0, a_1, \dots, a_n)$*.*[11]

Then the first component v_1 *of the vector which solves equation* (1.3) *satisfies an ordinary differential equation* (1.2) *with* $m = n+1$*, where* $f = p/q$ *is a quotient of polynomials* p *and* q*, where the denominator* q *is nonzero at the point* $(z_0, a_1, \dots, a_{n+1})$*, and where* v_1 *has the initial values* $v_1^{(k-1)}(z_0) = a_k$*,* $k = 1, \dots, n+1$*.* □

Example 3.2. The system of type (1.3) with $n = 2$

$$v_1'(z) = v_2(z), \qquad v_2'(z) = \frac{v_2(z)}{z},$$

thus with $g(s_1, s_2, s_3) = (s_3, s_3/s_1)$, has for $z_0 \neq 0$ the solution

$$v_1(z) = a_1 - \frac{a_2 z_0}{2} + \frac{a_2 z^2}{2z_0}, \qquad v_2(z) = \frac{a_2 z}{z_0},$$

with prescribed initial values $v_j(z_0) = a_j$. This is an example of a legitimate situation for the conjecture.

For $z_0 = 0$ the solution is

$$v_1(z) = a_1 + \gamma z^2, \qquad v_2(z) = 2\gamma z,$$

thus as before a family with two parameters a_1 and γ. But the initial values are now $v_1(0) = a_1$, $v_2(0) = 0$; we can no longer prescribe the initial value for v_2. This situation is not allowed in the formulation of the conjecture.

The system mentioned here corresponds to the differential equation $u''(z) = u'(z)/z$ or $zu''(z) - u'(z) = 0$, thus an equation of the type (1.2) with the problematic rational function $f(s_1, s_2, s_3) = s_3/s_1$. □

Example 3.3. The function E defined as $E(z) = (\mathrm{e}^z - 1)/z$ for $z \in \mathbf{C} \smallsetminus \{0\}$ and $E(0) = 1$, satisfies the equation

$$E'(z) = E(z) - \frac{E(z) - 1}{z}, \qquad z \in \mathbf{C} \smallsetminus \{0\}, \quad E'(0) = \tfrac{1}{2}. \tag{3.1}$$

In his article (2008) Alexander Gofen studies in detail this function, also briefly mentioned in (2009:847). It satisfies differential equations but only with denominators vanishing for $z = 0$.

[11] As is well known, this implies that the problem has a unique solution at least in some neighborhood of z_0.

Other functions worth of study are $z \mapsto \cos\sqrt{z}$ and $z \mapsto z^{-1}\sin z$.

See also (Flanders 2007) for similar results. □

4. Modified conjectures

Since Alexander Gofen's conjecture is not yet proved or disproved, it might be of interest to study some modifications of it. Such modified conventions could lead to ideas about what can occur.

So we take g in a class $\mathscr{G}_1 \times \cdots \times \mathscr{G}_n$ of functions and ask whether a solution v solves a scalar ordinary differential equation (1.2) with $f \in \mathscr{G}_1$. In such situations, Condition (g) could be replaced by a suitable condition guaranteeing the existence of a unique solution—or we can just drop it.

We can for instance weaken the conditions by removing the requirement that the denominators be nonzero. In this situation, Gofen (2020: Appendix 1) proved that this weakened kind of vector elementariness implies the weakened property of scalar elementariness.

4.1. The case of polynomials

A special case of the conjecture is when f is a polynomial and g a vector-valued polynomial. This has the advantage that the initial-value problems satisfy Condition (g) for all initial values $(z_0, a_1, \dots, a_n)$. Such is the situation for the function L defined by (1.1): the solution with initial values a_1 and a_2 is

$$L_{a_1,a_2}(z) = A + Bz + L(z), \qquad z \in \mathbf{C},$$

where

$$A = a_1 - \left(a_2 - \mathrm{e}^{z_0^2}\right) z_0 - L(z_0) \text{ and } B = a_2 - \mathrm{e}^{z_0^2}.$$

4.2. Other modified conjectures

In Subsection 7.1 we shall look at entire functions which are bounded on the real axis, and in Subsection 7.2 on an initial-value problem on the real axis.

5. The set of solutions to an equation determines the equation

Given a function f we have defined the set of solutions $\mathscr{U}(f)$, and similary $\mathscr{V}(g)$ for n-tuples of functions. Is the equation determined by its set of solutions? The answer turns out to be in the affirmative.

Proposition 5.1. *Let us assume that solutions to* (1.2) *and* (1.3) *are unique and well posed for all complex times* z_0.

If two function f and F are given and $\mathscr{U}(f) \subset \mathscr{U}(F)$, then $f = F$.

If two vector-valued functions g and G are given and $\mathscr{V}(g)$ is a subset of $\mathscr{V}(G)$, then $g = G$.

Proof. Let $(z_0, a_1, \dots, a_m)$ be any point in $\mathbf{C} \times \mathbf{C}^m$. Then the equation (1.2) has a unique solution with initial conditions $u^{(k-1)}(z_0) = a_k$, $k = 1, \dots, m$, thus belonging to $\mathscr{U}(f)$. By hypothesis it also belongs to $\mathscr{U}(F)$, so that it solves the equation with

f replaced by F. Thus $u^{(m)}(z_0) = f(z_0, a_1, \dots, a_m) = F(z_0, a_1, \dots, a_m)$. Since the $(1+m)$-tuple $(z_0, a_1, \dots, a_m)$ is arbitrary, this means that f is determined by u.

The proof in the vector case is similar. □

So the mappings $f \mapsto \mathscr{U}(f)$ and $g \mapsto \mathscr{V}(g)$ if restricted to a suitable space of locally Lipschitz functions are injective. This is not so with $\mathscr{U}_0(f)$ and $\mathscr{V}_0(g)$ as the next examples show.

Example 5.2. The equation $u'' = u$, thus with $f(s_1, s_2, s_3) = s_2$, has the solutions $u(z) = A\mathrm{e}^z + B\mathrm{e}^{-z}$, which thus describes $\mathscr{U}(f)$. We see that $\mathscr{U}_0(f)$ consists of the function which is identically equal to zero.

The equation $u'' = u'$, thus with $f(s_1, s_2, s_3) = s_3$, has the solutions $u(z) = A + B\mathrm{e}^z$, which is different from the set of solutions of the first equation. But $\mathscr{U}_0(f)$ is equal to the $\mathscr{U}_0(f)$ of the first equation, so the two equations have the same $\mathscr{U}_0(f)$. □

Example 5.3. The function $u(t) = -\log(T - t)$, $t \in \mathbf{R}$, $t < T$, where $T > 0$ is a given time, satisfies both the equation $u''(t) = u'(t)^2$ and the equation $u''(t) = (T - t)^{-2}$. The initial values are $u(0) = -\log T$ and $u'(0) = 1/T$. If we look for general initial values $u(0) = a$ and $u'(0) = b$, we find that the general solution to the first equation is

$$u(t) = -\log(b^{-1} - t) + a - \log b, \qquad t < 1/b;$$

and

$$u(t) = -\log(T - t) + (b - 1/T)t + a + \log T, \qquad t < T,$$

to the second equation. □

A similar but more complicated example is the following.

Example 5.4. Define $f(s_1, s_2, s_3) = 1/\cos^2 s_1$ and $F(s_1, s_2, s_3) = s_3^2 + 1$. Then the function u defined by $u(z) = -\log\cos z$ satisfies $u'(z) = \tan z$ and $u''(z) = 1/\cos^2 z$ with the initial values $u(0) = u'(0) = 0$, so that $u''(z) = f(z, u(z), u'(z)) = F(z, u(z), u'(z))$, two different equations. □

6. Independence of the family of solutions of the initial values

The initial-value problem (1.2) with arbitrary initial values $u^{(k-1)}(0) = a_k$ for $k = 1, \dots, m$, is equivalent to the special case with initial values $a_k = 0$:

Proposition 6.1. *A function u solves the equation* (1.2) *with initial values $u^{(k-1)}(0) = a_k$ if and only if the function defined by*

$$U(z) = u(z) - \sum_{k=1}^{m} a_k \frac{z^{k-1}}{(k-1)!}$$

solves the equation

$$U^{(m)}(z) = F(z, U(z), \dots, U^{(m-1)}(z))$$

with initial values $U^{(k-1)}(0) = 0$, $k = 1, \dots, m$, where

$$F(s) = f(s_1, s_2 + a_1, \dots, s_{m+1} + a_m), \qquad z = (s_1, \dots, s_{m+1}) \in \Omega \times \mathbf{C}^m.$$

Similarly for the vector equations (1.3).

Proof. We have $U(0) = u(0) - a_1$ and $U'(0) = u'(0) - a_2$ and so on. A simple calculation gives the result. □

Provided that the class $\mathscr{F}$ we consider is invariant under translations of the type used in the proof, we see that the concept of elementariness with respect to $\mathscr{F}$ is preserved. So this is in particular true if $\mathscr{F}$ is the family of all polynomials or the family of all entire functions that are bounded on the real axis.

7. Comparing scalar generalized elementariness and vector generalized elementariness

Lemma 7.1. *If u solves the equation* (1.2) *for a given function f, then*

$$v = (u, u', \dots, u^{(m-1)})$$

solves (1.3) *for an easily found vector-valued function g. Explicitly: if f belongs to $\mathscr{F}$, then*

$$(u, u', \dots, u^{(m-1)})$$

belongs to $\mathscr{G}_1 \times \cdots \times \mathscr{G}_m$, where $\mathscr{G}_j = \{\mathbf{pr}_{j+1}\}$, $j = 1, \dots, m-1$ and $\mathscr{G}_m = \{f\}$. Here $\mathbf{pr}_j$ denotes the mapping $(s_1, \dots, s_m) \mapsto s_j$.

Proof. We define $v_j = u^{(j-1)}$, $j = 1, \dots, m$. Then $v_j' = v_{j+1}$ for $j = 1, \dots, m-1$ while $v_m'(z) = u^{(m)}(z) = f(z, v_1(z), \dots, v_m(z))$, so that v solves (1.3) with $n = m$ and $g_j(s) = s_{j+1}$ for $j = 1, \dots, m-1$ and $g_m(s) = f(s)$. So g_m belongs to the same class as f while the g_j, $j = 1, \dots, m-1$ take the special form $g_j = \mathbf{pr}_{j+1}$. □

Proposition 7.2. *If the pair (v_1, v_2) solves* (1.3)*, then the function $u = v_1$ solves the scalar equation*

$$u''(z) = G(z, u(z), u'(z), v_2(z)) = H(z, u(z), u'(z), v_2(z), v_2'(z)), \tag{7.1}$$

where we have defined

$$G(s) = g_{s_1}(s_1, s_2, s_4) + g_{s_2}(s_1, s_2, s_4)s_3 + g_{s_3}(s_1, s_2, s_4)h(s_1, s_2, s_3), \tag{7.2}$$

for $s = (s_1, s_2, s_3, s_4)$, and

$$H(s) = g_{s_1}(s_1, s_2, s_4) + g_{s_2}(s_1, s_2, s_4)s_3 + g_{s_3}(s_1, s_2, s_4)s_5 \tag{7.3}$$

for $s = (s_1, s_2, s_3, s_4, s_5)$.

Proof. A simple application of the chain rule. □

We note what the conclusion of the last proposition looks like in several special cases.

Corollary 7.3. *Suppose that (v_1, v_2) satisfies the equation* (1.3) *with $n = 2$.*

(α'). *If $g = \mathbf{pr}_3$, then we are in the situation of Lemma 7.1 so that $u = v$ satisfies* (1.2).

(β'). *More generally, if $w(z) = \psi(z, v(z), v'(z))$, either globally or in a specific domain, then we can substitute the latter expression for $w(z)$ in $g(z, u(z), w(z))$ and get an expression without w and w', so that u satisfies*

$$u''(z) = G(z, u(z), u'(z), \psi(z, u(z), u'(z))).$$

(γ'). *If g is independent of s_3, then*

$$G(s_1, s_2, s_3, s_4) = H(s_1, s_2, s_3, s_4, s_5) = g_{s_1}(s_1, s_2, 0) + g_{s_2}(s_1, s_2, 0)s_3,$$

thus independent of s_4 and s_5, making the equation for v' into the equation $u'(z) = v'(z) = g(z, u(z), 0)$ with only one unknown function. This can be solved, and then u is a known function in the equation for w', viz. $w'(z) = h(z, u(z), w(z))$.

Proof. (α'). If $g(s_1, s_2, s_3) = s_3$, then $w = v'$.

(β'). Also here $w(z)$ can be expressed in terms of known quantities z, $v(z)$ and $v'(z)$.

(γ'). Clearly u is known in this case. □

So to find an example proving that vector elementariness does not imply scalar elementariness, we must avoid taking g and h as in one of the cases mentioned in Corollary 7.3. We note that in case (β'), the substitution might lead to a larger class of functions. For example, if we start with polynomials, then the inverse used in (β') can be algebraic, and in a strictly larger class.

7.1. An initial-value problem for an entire function

Let us define

$$u(z) = \frac{\sin z^2}{z}, \qquad z \in \mathbf{C} \smallsetminus \{0\}, \quad u(0) = 0. \tag{7.4}$$

This is an entire function of order 2. It is bounded on the real axis: for real $z = x$ we have $|u(x)| \leqslant \min(|x|, 1/|x|) \leqslant 1$.

Also its first derivative is bounded on the real axis:

$$u'(z) = 2\cos z^2 - \frac{\sin z^2}{z^2}, \qquad z \in \mathbf{C} \smallsetminus \{0\}, \quad u'(0) = 1,$$

satisfying $|u'| \leqslant 3$ on $\mathbf{R}$.

The second derivative is

$$u''(z) = -4z\sin z^2 + \frac{2\sin z^2 - 2z^2\cos z^2}{z^3}, \qquad z \in \mathbf{C} \smallsetminus \{0\}, \quad u''(0) = 0, \tag{7.5}$$

also an entire function, but unbounded on the real axis. This implies that if $u''(z) = f(z, u(z), u'(z))$, $z \in \mathbf{C}$, then f cannot be bounded on $\mathbf{R}^3$, not even on the subset $\mathbf{R} \times [-1, 1] \times [-3, 3]$ of $\mathbf{R}^3$.

7.1.1. Prescribing initial values

We shall now check that we can prescribe initial values arbitrarily at any point z_0 for equations like (7.5), just modified a little. This means that the problem satisfies Condition (g) in Conjecture 3.1.

7.1.1.1. Prescribing at $z_0 = 0$

For $z_0 = 0$, we define

$$u_{a,b}(z) = \frac{\sin z^2}{z} + a + (b-1)\sin z, \qquad z \in \mathbf{C} \smallsetminus \{0\}, \quad u_{a,b}(0) = a.$$

Then

$$u'_{a,b}(z) = 2\cos z^2 - \frac{\sin z^2}{z^2} + (b-1)\cos z, \qquad z \in \mathbf{C} \setminus \{0\}, \quad u'_{a,b}(0) = b,$$

and

$$u''_{a,b}(z) = -4z\sin z^2 + \frac{2\sin z^2 - 2z^2\cos z^2}{z^3} - (b-1)\sin z, \qquad z \in \mathbf{C} \setminus \{0\}, \quad u''_{a,b}(0) = 0.$$

7.1.1.2. Prescribing at $z_0 \neq 0$

For $z_0 \neq 0$ we define

$$u_{a,b}(z) = \frac{\sin z^2}{z} + A + B\sin\gamma z, \qquad z \in \mathbf{C} \setminus \{0\}, \quad u_{a,b}(0) = A,$$

where the constants A and B are to be determined and where we take $\gamma = \pi/(3z_0)$, so that $\cos\gamma z_0 = \frac{1}{2}$. We see that $u_{a,b}(z_0) = a$ if we take $A = a - z_0^{-1}\sin z_0^2$. The first derivative is

$$u'_{a,b}(z) = 2\cos z^2 - \frac{\sin z^2}{z^2} + B\gamma\cos\gamma z, \qquad z \in \mathbf{C} \setminus \{0\}, \quad u_{a,b}(0) = 1 + B\gamma.$$

We have $u'_{a,b}(z_0) = b$ if we take

$$B = \frac{2b - 4\cos z_0^2 + 2z_0^{-2}\sin z_0^2}{\gamma} = \frac{2b - 2 + 2z^{-2}\sin z_0^2}{\gamma}.$$

The second derivative is

$$u''_{a,b}(z) = -4t\sin z^2 + \frac{2\sin z^2 - 2z^2\cos z^2}{z^3} - B\gamma^2\sin\gamma z, \qquad z \in \mathbf{C} \setminus \{0\}, \quad u''_{a,b}(0) = 0.$$

For $z_0 = 0$ as well as for $z_0 \neq 0$, $u_{a,b}$ and $u'_{a,b}$ are bounded on the real axis while $u''_{a,b}$ is unbounded there.

For vector-valued equations we can proceed as follows.

7.1.1.3. Vector equations

Let us now define $v(z) = u_{a,b}(z)$ and $v_2(z) = b$. With $z_0 = 0$ they satisfy a system of type (1.3), thus with

$$\begin{aligned} g_1(s_1, s_2, s_3) &= 2\cos s_1^2 - \frac{\sin s_1^2}{s_1^2} + (b-1)\cos s_1, \\ g_2(s_1, s_2, s_3) &= 0. \end{aligned} \tag{7.6}$$

These are two entire functions, both bounded for s_1 real. The initial conditions are $v_1(0) = a$, $v_2(0) = b$. Similar pairs can be defined for $t_0 \neq 0$.

7.1.1.4. Other vector equations

But we can also find other vector equations of type (1.3) with $v_1(z) = u_{a,b}(z)$ and $v_2(z) = \cos z^2$, satisfying

$$\begin{aligned} g_1(s_1, s_2, s_3) &= 2s_3 - s_1^{-2}\sin s_1^2, \\ g_2(s_1, s_2, s_3) &= -2s_1\sin s_1^2, \end{aligned} \tag{7.7}$$

also entire functions of order 2.

7.1.1.5. Other scalar differential equations

The function u defined in (7.4) satisfies also other differential equations, like

$$u''(z) = -4z^2u(z) + \frac{u(z)}{z^2} - \frac{u'(z)}{z}, \qquad z \in \mathbf{C} \smallsetminus \{0\}.$$

This initial-value problem satisfies Condition (g) for $z_0 \neq 0$ but not for $(z_0, a_1, a_2) = (0, 0, 1)$. Here $f(s_1, s_2, s_3)$ contains the problematic rational functions s_2/s_1^2 and s_3/s_1; cf. Example 3.2.

7.2. An initial-value problem for a function defined on the real axis

An important property of the functions $u_{a,b}$ defined in Subsection 7.1 that $u_{a,b}$ and $u'_{a,b}$ are bounded on the real axis while $u''_{a,b}$ is unbounded. This observation leads us to a more general method of constructing examples. More precisely, we construct a sequence of waves which shrink both horizontally and vertically as we go to $+\infty$. This change of scale does not alter the slope of the wave but increases its curvature. We now formulate such a result for functions of a real variable.

Proposition 7.4. *Let $\varphi \in C^2(\mathbf{R})$ be a real-valued nonzero function with support contained in $[0, 1]$, and take a sequence $(\alpha_j)_{j\in\mathbf{N}}$ of numbers $\alpha_j \geqslant 1$ with $\limsup_{j\to+\infty} \alpha_j = +\infty$. Define*

$$u(t) = \sum \alpha_j^{-1}\varphi(\alpha_j(t - j)), \qquad t \in \mathbf{R}.$$

If u satisfies the differential equation (1.2), then f must be unbounded on

$$\mathbf{R} \times [-c_0, c_0] \times [-c_1, c_1], \text{ where } c_0 = \sup|\varphi| \text{ and } c_1 = \sup|\varphi'|.$$

Proof. We have

$$u'(t) = \sum \varphi'(\alpha_j(t - j)) \text{ and } u''(t) = \sum \alpha_j\varphi''(\alpha_j(t - j)).$$

Since the terms in the sum defining u consists at each point of a single nonzero term, we get the estimates $|u| \leqslant c_0$ and $|u'| \leqslant c_1$. But $u''(j + b/\alpha_j) = \alpha_j\varphi''(b)$, thus unbounded, when we take as b a point such that $\varphi''(b)$ is nonzero. So if u satisfies (1.2), then the sequence $j \mapsto f(j + b/\alpha_j, u(j + b/\alpha_j), u'(j + b/\alpha_j))$ must be unbounded.

On the other hand, $(v, w) = (u, 0)$ satisfies (1.3) with $n = 2$, $g_1(s_1, s_2, s_3) = u'(s_1)$ and $g_2(s_1, s_2, s_3) = 0$, both bounded for $(s_1, s_2, s_3) \in \mathbf{R}^3$. □

8. Conclusion

Theorem 8.1. *There exist pairs (v_1, v_2) of functions that are vector generalized elementary with respect to $\mathscr{G}_1 \times \mathscr{G}_2$ but the first components of which are not scalar generalized elementary with respect to $\mathscr{G}_1$. This is true both when the independent variable is complex and when it is real.*

Proof. We have seen in Subsections 7.1 and 7.2 classes that satisfy the requirements in the theorem.

There may be others ... □

Acknowledgments

Göran Björck, Jan Boman, Gerd Brandell, Alexander Gofen, and Lawrence Gruman have sent me corrections and important comments to earlier versions of the manuscript. I am most grateful to them for their contributions: they have pointed out errors and improved the presentation considerably as well.

I have received two reports: one from Alexander Gofen and one from an anonymous referee. Also these reports have led to improvements, and I am most grateful to the two persons.

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Submitted 2021-03-05
Revised 2021-05-17
Accepted 2022-01-03
Second revision 2022-01-04
`approximation/elementaryR2`

Remark added on 2022 February 18. This version is the one I submitted on 2021 March 05 and which was accepted on 2022 January 03.

I used, and always use, the Harvard system of references, indicating author name and year, like Ritt (1948). In Section 1.3, I listed work after Liouville, mentioning Alexander M. Ostrowski (1946) and several others.

In a referee report, I received praise for this list of historically important work.

The editors replaced the years with meaningless numbers, like [3], thus making the chronology invisible.

I then asked the editors to add the year, after the meaningless number, like [3] (1946). This simple addition was not implemented. To add the years "is against the journal style," according to a message from Vasudevan of 2022 February 18. To suppress useful information like an indication of the publication years is thus the "style" of this journal.

If you are aware of this "style," you might still get the years to be published by writing "In the year 1859 AD, Charles Darwin published his book *On the Origin of Species* [17]." Just might.