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Euclid's straight lines

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Contents

Abstract
We raise two questions on Euclid's *Elements*: How to explain that Propositions 16 and 27 in his first book do not follow, strictly speaking, from his postulates (or are perhaps meaningless)? and: What are the mathematical consequences of the meanings of the term *eutheia* which we today often prefer to consider as different?

The answer to the first question is that orientability is a tacit assumption. The answer to the second is rather a discussion on efforts to avoid actual infinity, and having to (in some sense or another) construct equivalence classes of segments to achieve uniqueness.

Résumé. — *Les droites d'Euclide*
Deux questions sur les *Éléments* d'Euclide sont soulevées : Comment comprendre que les propositions 16 et 27 dans son premier livre ne sont pas des conséquences strictement dit de ses postulats (ou peut-être sont dénuées de sens) ? et : Quelles sont les conséquences mathématiques du fait que le terme *eutheia* a des sens que nous préferons souvent aujourd'hui à considérer comme divers ?

La réponse à la première question est que l'orientabilité est une hypothèse tacite. La réponse à la deuxième question est plutôt une discussion sur les efforts faits pour éviter l'infini actuel et sur la construction d'une classe d'équivalence de segments (dans un sens ou l'autre) pour obtenir l'unicité d'une droite.

1. Two questions

Stoikheia (Στοιχεῖα) by Euclid (Εὐκλείδης) is the most successful work on geometry ever written. Its translation into Latin, *Elementa* 'Elements', became better known in Western Europe. It can still be read, analyzed—and understood. Nevertheless, I experienced a difficulty when trying to understand some results.

The First Question. Euclid's Proposition 27 in the first book of his Στοιχεῖα does not follow, strictly speaking, from his postulates (axioms)—or is possibly

> meaningless. Its proof relies on Proposition 16, which suffers from the same difficulty. There must to be a hidden assumption. What can this hidden assumption be?

Proposition 27 says:

> *If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another.* (Heath 1926a:307)

Proposition 16 says:

> *In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.* (Heath 1926a:279)

Some subsequent results will also be affected.

In this note I shall try to save Euclid by reexamining the notions of straight line and triangle, and expose a possible hidden assumption.

I shall also prove that if we limit the size of the triangles suitably, Proposition 16 does become valid even in the projective plane (see Proposition 7.1).

> **The Second Question.** What does the word εὐθεῖα (*eutheia*) mean? It is often translated as 'straight line', which in English is usually understood as an infinite straight line, but in fact it must often mean instead 'rectilinear segment, straight line segment'. Which are the mathematical consequences of these meanings, which we nowadays often prefer to perceive as different?

Michel Federspiel observes:

> La définition de la droite est l'un des énoncés mathématiques grecs qui ont suscité le plus de recherches et de commentaires chez les mathématiciens et chez les historiens. (Federspiel 1991:116)

For a thorough linguistic and philosophical discussion of this term, I refer to his article. He does not discuss there—maybe because the answer is all too evident for him—whether *eutheia* means an infinite straight line, a ray, or a rectilinear segment, meanings that Charles Mugler records in his dictionary:

> 1° Ligne droite indéfinie ; aussi demi-droite. [. . .] 2° Segment de droite. (Mugler 1958–1959:201–202)

This is what I will discuss in Section 4. Before that, however, I shall fix the terminology concerning two models for Euclid's axioms, the Euclidean plane and the projective plane. I will discuss the determination of triangular domains in the two models in Section 6, the proof of Proposition 16 in Section 7, and the notion of orientability in Section 9.

2. Approaches to this paper

The following convictions have been driving forces behind this paper.

(1) Geometry is fascinating, especially its logical content—I owe this to Bertil Broström, my first mathematics teacher.

(2) Languages are fascinating—I owe this to Karl Axnäs, my teacher of German and my most inspiring teacher all categories. Much later I wanted to understand Euclid and studied Classical Greek for Ove Strid.

(3) History is fascinating—I owe this to my history teacher Nils Forssell.

This means that the present text might be difficult to classify: I combine
(A) verbatim quotes from Euclid's text to show exactly how the terms were used; with
(B) a critical look at the logic, where I feel free to use the knowledge I have now, without implying anything about what Euclid could have known.

To prove that a statement, like that of Proposition 16, does not follow from certain axioms, a standard method is to exhibit a model where the axioms are true while the statement is not. The nature of the model is not important: it can come from any time and any place, and does not allow any conclusions relevant for history. This argument should be compared with the proof by Lobačevskiĭ, Bolyai and Gauss that the Postulate of Parallels is independent of the other axioms.

As Ulf Persson remarked, history shares with mathematics the fact that its subject does not exist (any longer), while the subject of mathematics has never existed, except perhaps in some world where Plato lives. For other thoughts comparing history and mathematics, see his essay (2007) on Robin George Collingwood's book *The idea of history* (1966). The present study combines history and mathematics, hopefully so that both perspectives are discernable.

3. The Euclidean plane and the projective plane

3.1. Straight lines and rectilinear segments in the Euclidean plane

In this paper I shall use E_2 to denote what is now known as the *Euclidean plane.* This is an affine space which can be equipped with coordinates which are pairs of real numbers, in other words elements on $\mathbf{R}^2$. More precisely, given three points $a, b, c \in E_2$ which do not lie on a straight line, we can give a point $p \in E_2$ the coordinates $(x, y) \in \mathbf{R}^2$ if $p = a + x(b-a) + y(c-a)$. (Note that in an affine space, where there is no origin, a linear combination $\lambda a + \mu b + \rho c$ has a good meaning if $\lambda + \mu + \rho = 1$, which is the case here.) In order to be able to speak about angles and areas, we need to equip the associated vector space with an inner product.

In the sequel I shall use the following terms.

A *straight line* is given by $\{(1-t)a + tb \in \mathbf{R}^2;\ t \in \mathbf{R}\}$, were $a \neq b$; it is infinite in both directions.[1]

A *rectilinear segment* is given by $\{(1-t)a + tb \in \mathbf{R}^2;\ t \in \mathbf{R},\ 0 \leqslant t \leqslant 1\}$. Since I want to avoid a point being declared as a rectilinear segment, I require that $a \neq b$.

A *ray* is given by $\{(1-t)a + tb \in \mathbf{R}^2;\ t \in \mathbf{R},\ 0 \leqslant t\}$, where $a \neq b$; it is infinite in one direction.

We note in passing that the same distinctions are made in Contemporary Greek: ευθεία γραμμή (f) 'straight line'; ευθύγραμμο τμήμα (n) 'rectilinear segment'; ακτίνα (f) 'ray'; 'radius' (Petros Maragos, personal communication 2007-10-12; Takis Konstantopoulos, personal communication 2012-01-20).

Given two points a, b on a straight line L in E_2, the complement $L \smallsetminus \{a, b\}$ has three components, one of which is bounded. So the rectilinear segment with a and b as endpoints can be recognized as the union of $\{a, b\}$ with the bounded component of $L \smallsetminus \{a, b\}$.

[1] Heath (1926a) uses *straight line* and Fitzpatrick (2011) *straight-line* as hypernyms for three currently used terms: *straight line* in the sense just defined, which is the sense I shall use, *rectilinear segment*, and *ray*.

3.2. Straight lines and rectilinear segments in the projective plane

The *projective plane*, which I shall denote by P_2, is a two-dimensional manifold which can be obtained from the Euclidean plane by adding a line, called the line at infinity, thus adding to each line a point at infinity. For a brief history of projective geometry see Torretti (1984:110–116). Johannes Kepler was, according to Torretti (1984:111), the first in modern times to add, in 1604, an ideal point to a line.

There are no distinct parallel lines in P_2. Still I shall consider that it satisfies Postulate 5:

> ε′.[2] *That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.* (Book I, Postulate 5; Heath 1926a:202)

This postulate, of course, must be subject to interpretation in the new structure, and therefore the statement that P_2 is a model is not an absolute truth.[3]

The projective plane can be given coordinates from points in $\mathbf{R}^3$ as follows. A point $p \in P_2$ is represented by a triple $(x, y, z) \neq (0, 0, 0)$, where two triples (x, y, z) and (x', y', z') denote the same point if and only if $(x', y', z') = t(x, y, z)$ for some real number $t \neq 0$. In other words, we may identify P_2 with $(\mathbf{R}^3 \smallsetminus \{(0, 0, 0)\})/\!\sim$, where $\sim$ is the equivalence relation just defined.

We can also say, equivalently, that a point in P_2 is a straight line through the origin in $\mathbf{R}^3$ and that a straight line in P_2 is a plane through the origin in $\mathbf{R}^3$.

Alternatively, we can think of P_2 as the sphere

$$S^2 = \{(x, y, z) \in \mathbf{R}^3;\ x^2 + y^2 + z^2 = 1\},$$

with *point* meaning 'a pair of antipodal points' and *straight line* meaning 'a great circle with opposite points identified'. Thus with this representation, $P_2 = S^2/\!\sim$. As pointed out by Ulf Persson, we can construct the projective plane also as the union of a disk and a Möbius strip, identifying their boundaries.

The projective plane can be covered by coordinate patches which are diffeomorphic to $\mathbf{R}^2$. For any open hemisphere, we can project the points on that hemisphere to the tangent plane at its center. Then all points except those on the boundary of the hemisphere are represented.

On the sphere, angles are well-defined, but not in the projective plane. To illustrate this, take for example an equilateral triangle with vertices at latitude $\varphi > 0$ and longitudes 0, $2\pi/3$ and $-2\pi/3$, respectively. Then its angles θ on the sphere can be obtained from Napier's rule, and are given by

$$\sin\varphi = \cos\Big(\frac{\pi}{2} - \varphi\Big) = \cot\frac{\pi}{3}\cot\frac{\theta}{2} = \frac{1}{\sqrt{3}}\cot\frac{\theta}{2}, \qquad 0 < \varphi < \frac{\pi}{2}.$$

Thus θ tends to π as $\varphi \to 0$ (a large triangle close to the equator). The same is true of the angle at a vertex if we use the coordinate patch centered at that very vertex.

[2]Statements are numbered by letters marked by a keraia (κεραία): α′ = 1, β′ = 2, …, ϛ′ (stigma) = 6, …, ια′ = 11, ιβ′ = 12, …, κε′ = 25, … .

[3]A better known manifold is the Möbius strip, which can be obtained from P_2 by removing a point, as Bo Göran Johansson points out (personal communication 2012-02-14). Now there are some parallel lines. However, this interesting structure does not satisfy Postulate 5 if we measure angles as described later in this subsection.

But θ tends to $\pi/3$ as $\varphi \to \pi/2$ (a small triangle close to the north pole). The projection of the triangle onto the tangent plane at $(0,0,1)$ is a usual equilateral triangle, thus with angles equal to $\pi/3$ for all values of φ, $0 < \varphi < \pi/2$. Thus we cannot measure angles in arbitrary coordinate patches, only in coordinate patches with center at the vertex of the angle; equivalently on the sphere.

It is convenient to use this way of measuring angles in the projective plane as a means of controlling the size of triangles. So, although it is meaningless to talk about angles in the projective plane itself, the sphere can serve as a kind of premodel for the projective plane, and the angles on the sphere can serve a purpose.

Given two points a, b on a straight line L in P_2, the complement $L \smallsetminus \{a, b\}$ has two components, and we cannot distinguish them. So to define a segment in P_2 we need two points a, b and one more bit of information, viz. which component of $L \smallsetminus \{a, b\}$ we shall consider. Since it seems that Euclid lets two points determine a segment without any additional information, shall we conclude already at this point that he excludes the projective plane? Anyway, in the projective plane, two distinct points determine uniquely a straight line, but not a rectilinear segment.

Explicitly, in the projective plane a point is given by the union of two rays $\mathbf{R}_+ a$ and $\mathbf{R}_- a$ in $\mathbf{R}^3$, where a is a point in $\mathbf{R}^3$ different from the origin, and where $\mathbf{R}_+$ denotes the set of positive real numbers, $\mathbf{R}_-$ the set of negative real numbers. Given two points, we can define two rectilinear segments, corresponding to two double sectors in $\mathbf{R}^3$. These are given as

$$\mathbf{cvxh}(\mathbf{R}_+ a \cup \mathbf{R}_+ b) \cup \mathbf{cvxh}(\mathbf{R}_- a \cup \mathbf{R}_- b)$$

and

$$\mathbf{cvxh}(\mathbf{R}_+ a \cup \mathbf{R}_- b) \cup \mathbf{cvxh}(\mathbf{R}_- a \cup \mathbf{R}_+ b),$$

respectively, where $\mathbf{cvxh}(A)$ denotes the convex hull of a set A. There is no way to distinguish them; to get a unique definition we must add some information as to which one we are referring to.

So the cognitive content of a segment is different in E_2 and P_2: a segment in P_2 needs one more bit of information to be defined.

4. What does *eutheia* mean?

Charles Mugler writes:

> [...] l'instrument linguistique de la géométrie grecque donne au lecteur la même impression que la géométrie elle-même, celle d'une perfection sans histoire. Cette langue sobre et élégante, avec son vocabulaire précis et différencié, invariable, à quelques changement sémantiques près, à travers mille ans de l'histoire de la pensée grecque, [...]

and continues

> la diction des *Éléments*, qui fixe l'expression de la pensée mathématique pour des siècles, se relève à l'analyse comme un résultat auquel ont contribué de nombreuses générations de géomètres. (Mugler 1958–1959:7)

May this suffice to show that we are not trying to analyze here some ephemeral choice of terms.

4.1. Lines

Euclid defines a line second in his first book:

> β′. Γραμμὴ δὲ μῆκος ἀπλατές. (Book I, Definition 2) — Une *ligne* est une longueur sans largeur (Hoüel 1883:11) — *A* line *is a breadthless length.* (Heath 1926a:158) — *Une* ligne *est une longueur sans largeur* (Vitrac 1990:152). — And a line is a length without breadth. (Fitzpatrick 2011:6)

There is no mentioning of lines of infinite length here; also Heath does not take up the subject. The lines in this definition are not necessarily straight, but in the rest of the first book, most lines, if not all, are straight, so to get sufficiently many examples we turn to these now.

4.2. Straight lines: *eutheia*

Euclid defines the concept of *eutheia* in the fourth definition in his first book thus:

> δ′. Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται. (Book I, Definition 4) — La ligne *droite* est celle qui est située semblablement par rapport à tous ses points (Hoüel 1883:11) — *A* straight line *is a line which lies evenly with the points on itself.* (Heath 1926a:165) — *Une* ligne droite *est celle qui est placée de manière égale par rapport aux points qui sont sur elle* (Vitrac 1990:154) — A straight-line is (any) one which lies evenly with points on itself. (Fitzpatrick 2011:6)

Hoüel adds that the definition is "conçue en termes assez obscurs".

Euclid's first postulate states:

> α′. Ἠιτήσθω[4] ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν. (Book I, Postulate 1) — Mener une ligne droite d'un point quelconque à un autre point quelconque ; (Hoüel 1883:14) — *Let the following be postulated : to draw a straight line from any point to any point.* (Heath 1926a:195) — *Qu'il soit demandé de mener une ligne droite de tout point à tout point.* Vitrac (1990:167) — Let it have been postulated [...] to draw a straight-line from any point to any point. (Fitzpatrick 2011:7)

The term he uses for straight line in the fourth definition and the first postulate is εὐθεῖα γραμμή (*eutheia grammē*) 'a straight line',[5] later, for instance in the second and fifth postulates, shortened to εὐθεῖα 'a straight one',[6] the feminine form of an adjective which means 'straight, direct'; 'soon, immediate'; in masculine εὐθύς; in neuter εὐθύ. This brevity is not unique; see Mugler (1958–1959:18) for other condensed expressions.

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[4]This verb form, written ᾐτήσθω in lower case letters, is in middle voice, perfect imperative, singular third person of the verb αἰτεῖν 'to demand', αἰτέω 'I demand'. Since it is in the perfect tense, Fitzpatrick's translation, "Let it have been postulated," with the alternative "let it stand as postulated," is more faithful than Heath's.

[5]Liddell & Scott (1978) gives γραμμή as '*stroke* or *line* of a pen, *line*, as in mathematical figures', and εὐθύς as '*straight, direct*, whether vertically or horizontally'. Bailly (1950) gives γραμμή as 'trait, ligne', [...] 'trait dans une figure de mathématiques', and εὐθύς as 'droit, direct'. Menge (1967) defines γραμμή as 'Strich, Linie (auch mathem.)', εὐθύς as 'gerade (gerichtet)', and εὐθεῖα (γραμμή) as 'gerade Linie'. In Millén (1853) I do not find γραμμή, only γράμμα 'bokstaf'; 'det som är skrifvet, skrift, bok, bref'; εὐθύς 'rak, rät'; 'strax'; 'snart'. Linder & Walberg (1862) translates *Linie* as 'γραμμή'; *rät l.* as 'εὐθεια'; *Rak* as 'εὐθύς'.

[6]Similarly, *une droite* is very often used for *une ligne droite* in French, and *прямая* (*pryamáya*) for *прямая линия* (*pryamáya línya*) in Russian.

Curiously, according to Frisk (1960), the adjective εὐθύς has no etymological counterpart in other languages: "Ohne außergriechische Entsprechung."

4.3. Straight lines: *ex isou keitai*

A key element in Definition 4 is the expression ἐξ ἴσου [...] κεῖται (*ex isou* [...] *keitai*). It is translated as 'située semblablement', 'lies evenly', 'placée de manière égale'. The adverbial *evenly* is a translation of the prepositional expression ἐξ ἴσου, which functions like an adverbial—or actually is an adverbial (Federspiel 1991:120).

Michel Federspiel would like to create ("j'aimerais créer") an adjective *isothétique* in analogy with *homothétique*—he argues that *homothétique* corresponds to the Greek ὁμοίως κεῖσθαι[7] "être placé semblablement", and that *isothétique* would correspond to the Greek ἐξ ἴσου κεῖται,[8] which occurs in Definition 4, and gives the translation (which he calls a «translation» within quotation marks)

> La droite est la ligne qui est *isothétique* de ses points. (Federspiel 1991:120)

He does not offer a mathematical definition of the new term, and it probably does not mean the same thing as in the expression *isothetic polygon*. Perhaps it is intended to preserve the vagueness of the original.

4.4. Straight lines: *sēmeion*

Vitrac (1990:189–190) points out that Euclid treats points as marks which one can place on straight lines or in relation to straight lines. That points are actually marks is further developed in two papers by Federspiel, who discusses in detail the meaning of the word σημείοις in Definition 4, plural dative of σημεῖον. He had expected the word πέρασι 'extrémités' at the place of σημείοις here (1992:387), and argues that, although in general σημεῖον certainly means 'point', in this particular definition it has a pre-Euclidean meaning, viz. 'repère,[9] extrémité' (1992:388), 'signe distinctif' (1992:389), or 'marque, repère' (1998:67) (perhaps to be rendered as *reference mark, guide mark, landmark, benchmark, extremity, mark, distinctive sign* in English). The word σημεῖα has the meaning (*sens*) 'repères' and the referent 'les extrémités' (1998:56). The referent is almost always the vertex of an angle in a polygon or a polyhedron, and there is, curiously, no *explicit* occurrence of the word σημεῖα with the endpoints of a rectilinear segment (1998:67). It seems that the only occurrence is in Definition 4 (1992:388), but it is not explicit there, since it is in a definition without explanation.

In fact, we are dealing with "un véritable archaïsme" (1998:61), whose meaning 'extremity' later disappeared (1998:62). However, in spite of this, the word σημεῖον was still understood in Euclid's time—if Euclid had found σημείοις to be incomprehensible in that sense, he would have replaced it by the contemporary πέρασι 'extrémités' (1998:62).

[7]The verb form κεῖσθαι means 'to be placed'; middle or passive voice (here most likely passive), present infinitive.

[8]The verb form κεῖται means 'it lies, it is lying' or perhaps 'it is laid, placed'; middle or passive voice, present indicative, singular, third person.

[9]"Toute marque servant à signaler un point, un enplacement à des fins précises" (*Grand Larousse* 1977).

The argument is supported by the use of σημεῖον in the sister science astronomy (1998:391–395), where it designates stars which delineate a constellation, in other words are in extreme positions relative to the constellation, essentially like the vertices of a polygon (1992:395), in particular a pentagon (1998:58), a cube (1998:58), or an icosahedron (1998:59). On the other hand, it is not necessary to consider astronomy as an intermediary; the meaning can appear directly in mathematics (1992:396); there is no reason to consider astronomy as a mother science.

The word σημεῖον was, according to Federspiel (1992:400) adopted very early in mathematics in the concrete sense of 'marque', and at any rate before the creation of the concept of point.

At this point comes to mind the statement by Reviel Netz that the lettered diagram is a combination of the continuous (the diagram itself) and the discrete (the letters) as well as a combination of visual resources (the diagram) and finite, manageable models (the letters) (Netz 1999:67).

Federspiel therefore modifies his translation from 1991 quoted above in Subsection 4.3 to the following.

> La ligne droite est la ligne qui est isothétique de ses extrémités. (Federspiel 1992:404)

And then to:

> La ligne droite est la ligne qui est isothétique de ses repères. (Federspiel 1998:56)[10]

In his argument, a straight line thus lies evenly between its extremities. This presupposes that a straight line does have two endpoints, which is a possible interpretation of Definition 3 (which is actually a proposition rather than a definition):

> γʹ. Γραμμῆς δὲ πέρατα σημεῖα. (Book I, Definition 3) — Les extrémités d'une ligne sont des points. (Hoüel 1883:11) — The extremities of a line are points. (Heath 1926a:165) — *Les* limites d'une ligne *sont des points* (Vitrac 1990:153) — And the extremities of a line are points. (Fitzpatrick 2011:6)

However, there are lines which do not have endpoints (circles, ellipses, and infinite straight lines). Heath therefore argues that Definition 3 "is really no more than an explanation that, if a line *has* extremities, those extremities are points." (1926a:165). Vitrac agrees (1990:153): "Il faut certainement comprendre que la présente définition signifie simplement : lorsqu'une ligne a des limites, ce sont des points."

It seems plausible that the definition was primarily thought of as defining a rectilinear segment, but that later, a wider use of the term εὐθεῖα forced mathematicians to accept a broader interpretation.

[10]Note the indefinite article in the two English translations and the definite article in four of the five French translations of Definition 4; in the Greek original there is no article. Federspiel (1995:252; 2005:105, note 29) explains that at the first occurrence of a mathematical term, it is given without article; at the second occurrence and later, it appears with the article. He calls this the *Loi fondamentale* for the use of the article in Classical Greek mathematical texts. When it comes to translations into French, Vitrac (1990:194, footnote 1) says with reference to his translation of Proposition 1 quoted in Subsubsection 4.9.4 below: "L'habitude française moderne est d'utiliser l'article indéfini pour souligner la validité universelle de la proposition."

4.5. Discretization

Zeno of Elea (Ζήνων ὁ Ἐλεάτης) formulated four paradoxes about motion, discussed in detail by Segelberg (1945) and Ferber (1981). The first of these is called the Dichotomy paradox since it uses division into halves. It says, according to Aristotle (Ἀριστοτέλης):

> ...πρῶτος μὲν ὁ (*scil.*[11] λόγος) περὶ τοῦ μὴ κινεῖσθαι διὰ τὸ πρότερον εἰς τὸ ἥμισυ δεῖν ἀφικέσθαι τὸ φερόμενον ἢ πρὸς τὸ τέλος,... — The first says that motion is impossible, because an object in motion must reach the half-way point before it gets to the end. (Quoted after Segelberg 1945:16)

By repeating the argument, we conclude that the object, if we agree that it is supposed to move from 0 to 1, must reach $\frac{1}{4}$ before reaching $\frac{1}{2}$, and $\frac{1}{8}$ before $\frac{1}{4}$, and so on. We see that the object must in fact reach all points with a binary coordinate $k/2^m$, $k = 1, \dots, 2^m - 1$, $m = 1, 2, \dots$, thus infinitely many. Euclid does construct the midpoint of a segment (Book I, Proposition 10, quoted in Subsubsection 4.9.4), so also for him there are infinitely many points on any given segment. We can think of these points as forming a potential infinity, because we can find the finitely many points $k/2^m$ for a certain m and then proceed to $m+1$, but the object cannot move in this order; for the object, the points represent an actual infinity—hence the alleged impossibility of motion (see, e.g., White (1992:147)).

In his third paradox, on the arrow which cannot move, Zeno can be seen as a precursor of a discretization of time, and therefore also of the line.

It would be interesting to know what Euclid thought about this paradox. As I understand it, his lines are neutral with respect to the consequences that Zeno's discretized time or line lead to. The points are without parts and thus are atoms:

> αʹ. Σημεῖόν ἐστιν, οὗ μέρος οὐθέν. (Book I, Definition 1) — Un *point* est ce qui n'a pas de parties. (Hoüel 1883:11) — *A* point *is that which has no part.* (Heath 1926a:155) — *Un* point[...] *est ce dont il n'y a aucune partie* (Vitrac 1990:151) — A point is that of which there is no part. (Fitzpatrick 2011:6)

A line does not *consist* of points; the points are, as we have seen in Subsection 4.4, special marks, *repères*, on the line. And in a construction we can hardly have an infinity of *repères*, like all those with coordinates $k/2^m$.

The two ideas—that the line is infinitely divisible while time consists of moments which cannot be further divided—are not easy to reconcile: we cannot arrive at the atoms by subdividing a segment. White (1992) discusses this difficulty; see in particular the section "The Quantum Model: Spatial Magnitude." Islamic thinkers in the middle ages resolved the conflict by making time divisible to a high degree while giving up infinite divisibility. As a prominent example of these ideas, Mosheh ben Maimon, a Sephardic Jewish philosopher who was born in Córdoba in 1135 or 1138 and died in Egypt in 1204, and who is now better known under his Greek name Maimonides, wrote that an hour is divisible by 60 ten times or more: "at last after ten or more successive divisions by sixty, time-elements are obtained which are not subject to division, and in fact are indivisible" (Whitrow 1990:79). So we can arrive at the time atoms! Now 60^{-10} hours is about 6 femtoseconds, 60^{-11} hours is about 100 attoseconds, and we are then down at the time scale of some chemical reactions studied nowadays in femtochemistry.

[11] Abbreviation for *scilicet* 'it is permitted to know'.

4.6. The chord property in the sense of Euclid

A property which is relevant for this discussion is what I called the *chord property in the sense of Euclid* (2011:359): for any two points a, b in the set A considered, the rectilinear segment (chord) $[a, b]$ is contained in A. This agrees with the translations of Definition 4 given in Subsections 4.2 and 4.3. To reconcile it with Federspiel's later translations quoted in Subsection 4.4, one has to note that, for every two points p, q belonging to a chord $[a, b]$, the segment $[p, q]$ is contained in $[a, b]$.

In fact, the strongest chord property is obtained when we start with the two endpoints of a rectilinear segment. However, on a straight line one can start quite naturally with any pair of points as *repères* and consider for these two points the segment determined by them using the chord property.

The chord property in the sense of Euclid has a counterpart in digital geometry, viz. the *chord property in the sense of Rosenfeld* introduced by Azriel Rosenfeld in 1974 and mentioned in my paper (2011:359). Moses Maimonides would have liked it.

4.7. The mathematical meaning of *eutheia*

What does *eutheia* mean mathematically? Proclus (Πρόκλος ὁ Διάδοχος), in his commentary to Euclid's first book (Proclus 1948:92, 1992:83) notes that *eutheia* has what we now usually perceive as three different meanings: a straight line; a rectilinear segment; and a ray. "La ligne est donc prise de trois manières par Euclide" (Proclus 1948:92); "our geometer makes a threefold use of it" (Proclus 1992:83). Thus already Proclus writes about three different meanings.

Euclid often refers to extension of straight lines, for instance in the famous Postulate 5, the Axiom of Parallels, quoted in Subsection 3.2, which was to keep mathematicians busy for more than two millennia. The postulate implies that the two straight lines do not necessarily meet initially, so he must be talking about rectilinear segments. We may conclude that, here at least, *eutheia* means a rectilinear segment, not an infinite straight line.

The Greek original has ἐκβαλλομένας[12] [. . .] ἐπ' ἄπειρον, which Heath translates as 'produced indefinitely'. Similarly, Definition 23 has ἐκαλλόμεναι[13] εἰς ἄπειρον, translated in the same way. Fitzpatrick (2011:7) translates both as 'being produced to infinity'. However, Heath (1926a:190) explicitly warns against that interpretation. Similarly, Vitrac (1990:166) makes the distinction between being extended "indéfiniment" and being extended "à l'infini" and maintains that the expressions εἰς ἄπειρον and ἐπ' ἄπειρον refer to the former.

4.8. Infinitely long lines vs. equivalence classes of segments

On the other hand, when two points are given, they determine uniquely a straight line. Actually, Postulate 1 does not explicitly say so, but the discussion in Heath (1926a:195), which leads to the conclusion that this is what is meant, is quite

[12]Middle or passive voice, present participle, plural, feminine, accusative. Of the many meanings of the verb ἐκβάλλειν (*ekballein*; active voice, present, infinitive), the basic one is 'to throw out'. Liddell & Scott (1978) and Menge (1967) explicitly mention the mathematical sense of extending a line.

[13]Middle or passive voice, present participle, plural, feminine, nominative.

convincing. Here it would be natural for us in the twenty-first century to think about an infinite straight line, but it is also possible to limit the consideration to rectilinear segments by forming the family of all segments which contain the two given points—or at least a family of rectilinear segments which go out arbitrarily far in both directions. If so, we can avoid here actual infinity, and work only with potential infinity by looking at one segment at a time rather than at an infinitely long line. Vitrac (1990:169) mentions this possibility: "la droite peut être envisagée comme indéfinie ou potentiellement infinie."

Michel Federspiel states quite categorically: "Il n'y a pas d'infini actuel dans la géométrie grecque." (1991:118, Note 10). This should be contrasted with an assertion by Reviel Netz: "[...] Archimedes [Ἀρχιμήδης] calculated with actual infinities — in direct opposition to everything historians of mathematics have always believed about their discipline." The quotation refers to the calculation of a volume in the palimpsest now at the Walters Art Museum in Baltimore, MD, USA (Netz & Noel 2007:199). It seems the basis for this assertion is not very firm. More to the point is Euclid's own statement in his Book X: γ′. [...] ὑπάρχουσιν εὐθεῖαι πλήθει ἄπειροι [...] (Book X, Definition 3) — [...] there exist an infinite multitude of straight-lines [...] (Fitzpatrick 2011:282).

We may note that Proclus makes the distinction between "partie infinies en acte" (actual infinity) and "en puissance seulement" (potential infinity) (1948:140); "The latter statement [an infinite number of parts] makes an infinite number actual, the former [a magnitude is infinitely divisible] only potential; the latter assigns existence to the infinite, the other only genesis" (1992:125).

However, if we act like this—whether under the pressure of Aristotle or not—there will be a lot of rectilinear segments that contain the two given points: perhaps one with a length of one hemiplethron, then one with a length of one plethron, one stadion, one hippikon, then one with a length of a parasang, and one with a length of one stathmos, and so on—it does not stop. But all of these segments represent the *same* line: there has to be only one line. That the segments all *represent* the same line is today conveniently expressed in the parlance of equivalence classes. The formation of an equivalence class is a means of obtaining uniqueness—to unite the many segments into one single entity.

Let me emphasize again that two points determine a straight line segment if we are in E_2, and that, conversely, a straight line segment uniquely determines two points, viz. its endpoints. If this were all there is to it, we would have perfect uniqueness in both directions. But if we extend a segment to a longer segment, we have two different segments, which, however, *represent the same* straight line. What does then *represent* mean? And what does *the same* mean? If we nowadays can speak about equivalence classes, this is a convenient way to understand the verb *represent*, but it is only there as a help to the modern reader. I do not know how Euclid thought, but he must have been aware of this problem of nonuniqueness.

As for actual vs. potential infinity, we may compare with prime numbers: it is sometimes said that Euclid proved that there are infinitely many prime numbers, but actually he proved in his ninth book, Proposition 20, that, given three prime numbers, he can find a fourth. Clearly the proof works for any finite set of primes: with the idea of the proof we can go from n primes to $n + 1$ primes for any n. All prime numbers need not exist at once. So this is an instructive example of potential infinity; we need not believe in the existence of an actual infinity.

Aristotle expressed a very clear opinion on the need to consider infinite straight lines:

> I have argued that there is no such ting as an actual infinite which is untraversable, but this position does not rob mathematicians from their study. Even as things are, they do not need the infinite, because they make no use of it. All they need is a finite line of any desired length. (*Physics*, Book III, Part 7, quoted here from Aristotle 1996:75–76)

The uniqueness requirement then leads to the need of forming an equivalence class of all these segments.

Not only is an actual infinity unnecessary for geometry; it is even impossible in the physical world:

> [...] there can be no magnitude which exceeds every specified magnitude: that would mean that there was something larger than the universe. (*Physics*, Book II, Part 7, quoted from Aristotle 1996:75)

However, as Rosenfeld (1988:183) points out, Aristotle's doctrine "that mathematical concepts are obtained by abstracting from objects of the real world enables one to disengage oneself from the finiteness of physical magnitudes." Ibn Rushd (Averroes) wrote that a geometer can admit "an arbitrarily large magnitude—something a physicist cannot do [...]".

We should also add that on the sphere, a straight line in the plane corresponds to a great circle, μέγιστος κύκλος (*megistos kuklos*; Mugler 1958–1959:19). Certainly Aristotle would not object to considering a circle on a sphere as a complete, existing entity.[14] But I guess he did not see a great circle as a compactification of a straight line as we now do quite easily—after so many years.

Since every rectilinear segment determines a unique straight line, it might appear that there is no big difference whether we say that two distinct points determine a straight line or that two distinct points determine a rectilinear segment. However, the latter assertion is untenable (if we keep ourselves strictly to the axioms) in view of the fact that, as noted in Subsection 3.2, two points in the projective plane determine not one segment but two.

4.9. Examples

4.9.1. *Eutheia* bounded

That the English term *straight line* or *straight-line* can denote a rectilinear segment is explicitly mentioned by Heath "*if two straight lines* ('rectilinear segments' as Veronese would call them) *have the same extremities* [...]" (1926a:195); "what modern Italian geometers aptly call *rectilinear segment*, that is, a straight line having *two* extremities." (1926a:196). For both the Greek term and the English term, this is clear as well from several examples, e.g., the first few propositions in Book I:

> β′. Πρὸς τῷ δοθέντι σημείῳ τῇ δοθείσῃ εὐθείᾳ ἴσην εὐθεῖαν θέσθαι. (Book I, Proposition 2) — *A partir d'un point donné* A [...], *placer une droite égale à une droite donnée* BC (Hoüel 1883:16) — *To place at a given point* (*as an extremity*) *a straight line equal to a given straight line.* (Heath 1926a:244) — *Placer, en un point donné,*

[14] For the history of spherical geometry, see Rosenfeld (1988: Chapter 1).

> *une droite égale à une droite donnée.* (Vitrac 1990:197) — To place a straight-line equal to a given straight-line at a given point (as an extremity). (Fitzpatrick 2011:8)

Equality of lines here means equality of their lengths.

> γʹ. Δύο δοθεισῶν εὐθειῶν ἀνίσων ἀπὸ τῆς μείζονος τῇ ἐλάσσονι ἴσην εὐθεῖαν ἀφελεῖν. (Book I, Proposition 3) — *Étant données deux droites inégales,* AB, C [...], *retrancher de la plus grande* AB *une droite égale à la plus petite* C (Hoüel 1883:17) — *Given two unequal straight lines, to cut off from the greater a straight line equal to the less.* (Heath 1926a:246) — *De deux droites inégales données, retrancher de la plus grande, une droite égale à la plus petite.* (Vitrac 1990:199) — For two given unequal straight-lines, to cut off from the greater a straight-line equal to the lesser. (Fitzpatrick 2011:9)

> δʹ. Ἐὰν δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δυσὶ πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ καὶ τὴν γωνίαν τῇ γωνίᾳ ἴσην ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην, [...] (Book I, Proposition 4) — *Si deux triangles* ABC, DEF [...] *ont les deux côtés* AB, AC *respectivement égaux aux deux côtés* DE, DF*, et si les angles* BAC, EDF*, compris entre les côtés égaux, sont égaux,* [...] (Hoüel 1883:18) — *If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal,* [...] (Heath 1926a:247) — *Si deux triangles ont deux côtés égaux à deux côtés, chacun à chachun* [...], *et s'ils ont un angle égal à un angle, celui contenu par les droites égales,* [...] (Vitrac 1990:200) — If two triangles have two sides equal to two sides, respectively, and have the angle(s) enclosed by the equal straight-lines equal, [...] (Fitzpatrick 2011:10)

We note that here the sides of a triangle are sometimes called *sides, cotés*; sometimes *straight lines, straight-lines, droites.*

> εʹ. Τῶν ἰσοσκελῶν τριγώνων αἱ πρὸς τῇ βάσει γωνίαι ἴσαι ἀλλήλαις εἰσίν, καὶ προσεκβληθεισῶν τῶν ἴσων εὐθειῶν αἱ ὑπό τὴν βάσιν γωνίαι ἴσαι ἀλλήλαις ἔσονται. (Book I, Proposition 5) — *Dans tout triangle isoscèle* ABC [...], 1° *les angles à la base* ABC, ACB *sont égaux entre eux ;* 2° *si l'on prolonge les côtés égaux* AB, AC*, les angles formés au-dessous de la base,* DBC, ECB*, seront aussi égaux entre eux.* (Hoüel 1883:18–19) — *In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another.* (Heath 1926a:251) — *Les angles à* [...] *la base des triangles isoscèles sont égaux entre eux, et si les droites égales sont prolongées au-delà, les angles sous la base seront égaux entre eux.* (Vitrac 1990:204) — For isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Fitzpatrick 2011:11)

In Book I, Proposition 12, εὐθεῖα receives the attribute ἄπειρος (*apeiros*) 'unbounded, infinite':

> ιβʹ. Ἐπὶ τὴν δοθεῖσαν εὐθεῖαν ἄπειρον ἀπὸ τοῦ δοθὲντος σημείου, ὃ μή ἐστιν ἐπ' αὐτῆς, κάθετον εὐθεῖαν γραμμὴν ἀγαγεῖν. (Book I, Proposition 12) — *D'un point donné* C [...], *abaisser une perpendiculaire sur une droite indéfinie donnée* AB. (Hoüel 1883:24) — *To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line.* (Heath 1926a:270) — *Mener une ligne droite perpendiculaire à une droite indéfinie* [...] *donnée à partir d'un point donné qui n'est pas sur celle-ci.* (Vitrac 1990:219) — To draw a straight-line perpendicular to a given infinite straight-line from a point which is not on it. (Fitzpatrick 2011:17)

Here the qualification ἄπειρος would not be necessary if an εὐθεῖα were always something unbounded in both directions.

Apollonius (Ἀπολλώνιος) mentions an εὐθεῖα in a context that clearly indicates that it refers to a segment; he needs to extend it in both directions:

> Ἐὰν ἀπό τινος σημείου πρὸς κύκλου περιφέρειαν, ὃς οὐκ ἔστιν ἐν τῷ αὐτῷ ἐπιπέδῳ τῷ σημείῳ, εὐθεῖα ἐπιζευχθεῖσα ἐφ᾽ ἑκάτερα προσεκβληθῇ, [...] (Ἀπολλώνιος, Κωνικῶν α΄. Ὅροι πρῶτοι. Apollonius, *Conics*, Book 1, First definitions) — If a point is joined by a straight line with a point in the circumference of a circle which is not in the same plane with the point, and the line is continued in both directions, [...] (Rosenfeld 2012:3)

4.9.2. Segment

The Classical Greek word τμῆμα (n) (*tmēma*) is translated by Liddell & Scott (1978) as '*part cut off, section, piece*'; '*segment* of a line, of a circle (i.e. portion cut off by a chord), also of the portion cut off by radii, *sector*' [...] 'of *segments* of other figures cut off by straight lines or planes; and of segments bounded by a circle and circumscribed polygon'. Bailly (1950) translates it as 'morceau coupé, section, part, segment de cercle', and Menge (1967) as 'Schnitt'; 'Abschnitt'.

In all cases it is about some part cut out from a given object. This object could be a disk or a rectilinear segment, viz. when a rectilinear segment is given, and one then cuts out a part of it (Book II, Propositions 3 and 4). As I understand it, the term is not used for a rectilinear segment *per se*, only for a certain part cut out from something else in the course of a construction (in Section 5 we shall take a look at how the Greek viewed geometric constructions). So in general an εὐθεῖα is not thought of as being cut out from a straight line.

The term τμῆμα is used for a segment of a circle[15] in Book III:

> κε΄. Κύκλου τμήματος δοθέντος προσαναγράψαι τὸν κύκλον, οὗπέρ ἐστι τμῆμα. (Book III, Proposition 25) — *Given a segment of a circle, to describe the complete circle of which it is a segment.* (Heath 1926b:54) — *Etant donné un segment de cercle, décrire complètement* [...] *le cercle duquel il est un segment.* (Vitrac 1990:440) — For a given segment of a circle, to complete the circle, the very one of which it is a segment. (Fitzpatrick 2011:94)

The meaning 'segment of a disk' occurs, e.g., in Definition 6 in Book III:

> ϛ΄. Τμῆμα κύκλου ἐστὶ τὸ περιεχόμενον σχῆμα ὑπό τε εὐθείας καὶ κύκλου περιφερείας (Book III, Definition 6) — A **segment of a circle** is that contained by a straight line and a circumference of a circle. (Heath 1926b:1) — Un segment de cercle *est la figure contenue par une droite et une circonférence de cercle* (Vitrac 1990:388) — A segment of a circle is the figure contained by a straight-line and a circumference of a circle. (Fitzpatrick 2011:70)

A definition of *segment* has also been "interpolated" after Definition 18 in Book I; see Definition 19 in Euclid (1573:39), Hoüel (1883:12), and the remark on Definition 18 in Heath (1926a:187). It seems that the term is not used for a chord.

In conclusion, τμῆμα is related to the verb τέμνειν 'to cut', τέμνω 'I cut', and is firmly attached to the act of cutting. Therefore it is not used for rectilinear segments in general, which are just there, not being the result of any cutting.

The English word *segment*, from the Latin *segmentum* 'a piece cut out', formed from *secare* 'to cut', also carries this connotation, like the Russian *прямолинейный*

[15] Here it does not really matter whether κύκλος means 'circle' or 'circular disk'.

отрезок (*pryamolinéĭnyĭ otrézok*) 'rectilinear segment', from *резать* (*rézat′*) 'to cut'. This connotation is completely absent in the German *Strecke*, the Esperanto *streko*, and the Swedish *sträcka*.

4.9.3. Radius and chord

In a circle there are rectilinear segments which have received special names in many languages: radii and chords.

> The Greeks had no distinct word for *radius*, which is with them [...] *the* (*straight line drawn*) *from the centre* ἡ ἐκ τοῦ κέντρου (εὐθεῖα) [*hē ek tou kentrou* (*eutheia*)] (Book III, Definition 1; Heath 1926b:2)

Mugler (1958–1959:17) gives the full expression for radius as ἡ ἐκ τοῦ κέντρου (sc.[16] πρὸς τὴν περιφέρειαν ἠγμένη εὐθεῖα γραμμή).

There is also a word διάστημα (n) (*diastēma*) used for 'radius', or often for 'the length of a radius' (Mugler 1958–1959:17).

Federspiel (2005:98, note 5) opposes the statement by Heath quoted above: he says that the Greek had two words for 'radius', viz. the two just mentioned.

He explains that the first expression needs the article ἡ, and in a situation where one needs the indefinite form, it cannot be used; here the word διάστημα comes in, a fact which also explains why they are in complementary distribution (2005:105).

In Contemporary Greek, the word used for radius is ακτίνα (f) (Petros Maragos, personal communication 2007-10-12; Takis Konstantopoulos, personal communication 2012-01-20). However, this word also means 'ray'.

Similarly, they did not have a simple word for chord (in a circle): it is ἡ ἐν τῷ κύκλῳ εὐθεῖα (*hē en tō kuklō eutheia*) as used not by Euclid but later by Heron (Erik Bohlin, personal communication 2012-01-18; cf. Mugler 1958–1959:202) and by Ptolemy (1898:48), who in the heading of Table ια′ (11) writes: Κανόνιον τῶν ἐν κύκλοῳ εὐθειῶν. With Euclid, not the expression itself but the words used in referring to a chord appear in Definition 4 in Book III, see Heath (1926b:3); and in Proposition 14 in Book III, see Heath (1926b:34).

The word χορδή (f) (*khordē*) is given by Liddell & Scott (1978) as 'guts, tripe' [...] 'string of gut, 'string of musical instrument'. Bailly (1950) translates it as 'boyau', [...] 'corde à boyau, corde d'un instrument de musique'. Frisk (1960) as 'Darm, Darmsaite, Saite, Wurst' and Menge (1967) as 'Darm, Darmsaite'. Frisk (1960) states that it is "Ohne genaue Außergreich. Enstprechung". Linder & Walberg (1862) translate *Sträng på ett instrument* as 'χορδή', and *Tarm* as 'ἔντερον, χορδή'. But χορδή is missing in Millén (1853).

In Contemporary Greek the word used for chord and string is χορδή (f) (Takis Konstantopoulos, personal communication 2012-01-20).

4.9.4. *Eutheia* unbounded

However, sometimes εὐθεῖα carries another qualification:

> β′. Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ' εὐθείας ἐκβαλεῖν.[17] (Book I, Postulate 2) — Prolonger indéfiniment, suivant sa direction, une ligne droite finie ; (Hoüel 1883:14) — *To produce a finite straight line continuously in a straight line.*

[16] This abbreviation stands for *scilicet* 'it is permitted to know'.
[17] The verb form ἐκβαλεῖν is in active voice, strong aorist, infinitive.

> (Heath 1926a:196) — *Et de prolonger continûment en ligne droite une ligne droite limitée.* (Vitrac 1990:168) — And to produce a finite straight-line continuously in a straight-line. (Fitzpatrick 2011:7)

From this it is obvious that an εὐθεῖα can be explicitly qualified as bounded, which indicates that the term could refer also to an unbounded line. Or, with a potential infinity, a family of rectilinear segments! In other words, we can interpret Postulate 2 to mean that we can extend a given segment to another segment, as long as we wish, but still of finite length.

> αʹ. Ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τρίγωνον ἰσόπλευρον συστήσασθαι. (Book I, Proposition 1) — *Sur une droite finie donnée* AB [. . .], *construire un triangle équilatéral.* (Hoüel 1883:15) — *On a given finite straight line to construct an equilateral triangle.* (Heath 1926a:241) — *Sur une*[. . .] *droite limitée donnée, construire un triangle équilatéral.* (Vitrac 1990:194) — To construct an equilateral triangle on a given finite straight-line. (Fitzpatrick 2011:8)

> ιʹ. Τὴν δοθεῖσαν εὐθεῖαν πεπερασμένην δίχα τεμεῖν. (Book I, Proposition 10) — *Partager une droite finie donnée* AB [. . .] *en deux parties égales.* (Hoüel 1883:22) — To bisect a given finite straight line. (Heath 1926a:267) — *Couper en deux parties égales*[. . .] *une droite limitée donnée.* (Vitrac 1990:216) — To cut a given finite straight-line in half. (Fitzpatrick 2011:15)

The attribute πεπερασμένη 'finite, bounded' (passive voice, perfect participle, singular, feminine, nominative) would not be necessary here if εὐθεῖα always meant 'rectilinear segment'.

In the proof of Proposition 12, Euclid uses the fact that an *eutheia* divides the plane into two half planes. This of course must imply that the line is infinite in both directions.

4.9.5. *Eutheia* as ray

Finally, we note that sometimes εὐθεῖα can mean 'ray':

> Ἐκκείσθω τις εὐθεῖα ἡ ΔΕ πεπερασμένη μὲν κατὰ τὸ Δ ἄπεροις δὲ κατὰ τὸ Ε, [. . .] (Book I, Proof of Proposition 22) — Tirons une droite DE, terminée en D, indéfinie vers E. (Hoüel 1883:31) — Let there be set out a straight line DE, terminated at D but of infinite length in the direction of E, [. . .] (Heath 1926a:292) — Que soit d'abord proposée une certaine droite DE, limitée d'un côté au point D, illimitée de l'autre en E, [. . .] (Vitrac 1990:237) — Let some straight-line DE be set out, terminated at D, and infinite in the direction of E. (Fitzpatrick 2011:25)

In the statement of this proposition the lines are of finite length, but in its proof there suddenly appears a ray.

5. Constructions

The discussion on segments in Subsubsection 4.9.2 opens up the question what the Greek mathematicians could have meant when they talked about constructions.

> Hellenistic mathematics was certainly constructive (every new figure introduced by Euclid comes with a description of its construction), but in a sense much stronger than that of modern constructivism, because the construction was not just a metaphor used for providing a demonstration of existence, but the actual goal of the

> theory, just as the machine described by Heron was constructed to lift weights and not just to prove a "theorem of existence" about the machine. (Russo 2004:186)

Who is constructing?

> Le géomètre grec ne reconnait qu'exceptionnellement des constructions dans le sens que nous attachons communément à ce terme, c'est-à-dire dans le sens de la réalisation progressive d'une figure au moyen de lignes et de points ajoutés successivement aux lignes et aux points qui constituent les données primitives du problème. Pour le géomètre grec la figure, même si ses propriétés sont encore à démontrer, préexiste à toute intervention humaine [...] (Mugler 1958–1959:19)

Proclus (1992:64), Mugler (just quoted), Vitrac (1990:134) and Federspiel (2005: 106) all state that the Ancient Greek never constructed anything. The figures are already there for all eternity:

> Proclus nous avertit en effet que certains soutenaient que toutes les propositions étaient des théorèmes, en tant que propositions d'une science théorétique portant sur des objets éternels, lesquels n'admettent, en tant que tels, ni changement, ni devenir, ni production : ce qu'on appelle « construction » n'est tel, de ce point de vue, qu'au regard de la connaissance que nous prenons des choses éternelles (Vitrac 1990:134)

> [...] une thèse fondamentale de Platon et de ses successeurs [...] : en mathématiques, on ne construit pas : les figures sont en réalité déjà construites de toute éternité ; il n'y a donc pas d'avant ni d'après. (Federspiel 2005:105–106)

So any movement in time refers only to the way we learn about these things.

Christian Marinus Taisbak explains similarly:

> When mathematicians are doing geometry, describing circles, constructing triangles, producing straight lines, they are not really *creating* these items, but only *drawing pictures* of them. (Taisbak 2003:27)

Plato, in *The Republic*, asserts (as we could expect): "[...] geometry is the knowledge of the eternally existent." (Plato 1935:171, Book VII, 527B).

This Platonic idea is often reinforced by the language itself: the authors use the passive voice, without indicating an agent, and the perfect tense, i.e., a tense which indicates that something has occurred in the past and has a result remaining up to the present time (Mugler 1958–1959:20; Michel Federspiel, personal communication 2012-04-16). This is in slight contradiction to Plato's statement about the language of geometricians:

> Their language is most ludicrous,[...] though they cannot help it,[...] for they speak as if they were doing something [...] and as if all their words were directed towards action. (Plato 1935:171, Book VII, 527B)

There are, however, some exceptions to the use of the passive voice: In Euclid's *Data* (Δεδομένα), the first two definitions use the pronoun *we*. "The use of 'we' in the definitions is alien to Euclid's style; in the *Elements* no person is involved in constructions or proofs in any way [...]" (Taisbak 2003:18).

Regardless of these philosophical and linguistic considerations it is convenient for us nowadays to think of an ongoing construction, just as a way of thinking—not implying any opinion on this interesting historical question.

6. Triangular domains

A triangular domain can be given in three different ways: using points, segments, or straight lines, respectively.

6.1. Triangular domains in the Euclidean plane

E1. In E_2, three points which do not lie on a straight line determine a triangular domain: it is the convex hull of the three points. If the points are a, b, c, their convex hull is the set

$$\mathbf{cvxh}(\{a, b, c\}) = \{\lambda a + \mu b + \rho c;\ \lambda, \mu, \rho \geqslant 0,\ \lambda + \mu + \rho = 1\}.$$

This is the closed triangular domain defined by a, b, c.
E2. A triangular domain can also be given by three segments $[a, b], [b, c], [c, a]$ with pairwise common endpoints but not contained in a straight line. The complement of the union $[a, b] \cup [b, c] \cup [c, a]$ has two components, and one is bounded—this is the open triangular domain.
E3. Finally, a triangular domain in E_2 can be given by three straight lines L_1, L_2, L_3 which meet in exactly three different points. The complement of the union $L_1 \cup L_2 \cup L_3$ has seven components, and exactly one of them is bounded; this defines the open triangular domain.

To be precise, if the equations of the three lines are $f_j(x, y) = 0$, $j = 1, 2, 3$, where the f_j are affine functions, and if the signs are chosen so that $f_j(p) < 0$ for some point p in the bounded component of $E_2 \smallsetminus \{L_1 \cup L_2 \cup L_3\}$, then the other six components are defined by the conditions that $f_j(q)$ shall be nonzero for all j and positive for one or two choices of j; there is no point q with $f_j(q)$ positive for all j. The set of points where the convex function $f = \max(f_1, f_2, f_3)$ is negative is the open triangular domain determined by the three lines.

To sum up, in E_2 we can define a triangular domain using indifferently points, segments or straight lines.

6.2. Triangular domains in the projective plane

In P_2 the determination of triangular domains takes on a different quality.
P1. We first look at three points in P_2 which do not lie in a straight line. They are given by three rays in $\mathbf{R}^3$,

$$R_j = \mathbf{R}_+ a^{(j)} = \{ta^{(j)};\ t > 0\}, \qquad j = 1, 2, 3,$$

where the $a^{(j)}$ are three nonzero vectors in $\mathbf{R}^3$. We can now form

$$\mathbf{cvxh}(R_1 \cup \theta_2 R_2 \cup \theta_3 R_3) \cup (-\,\mathbf{cvxh}(R_1 \cup \theta_2 R_2 \cup \theta_3 R_3)),$$

where $(\theta_2, \theta_3) = (\pm 1, \pm 1)$ (four possibilites). These are the four triangular domains that we can form in P_2 from the three points, and we see that two bits of information are needed in addition to the information contained in the three points in order to determine which domain we shall consider.
P2. The complement of the union of three segments which do not lie in a straight line and have pairwise common endpoints has two components, and they are of

equal status. A triangular domain in this case is given by three segments and the additional information which of the two components is meant. And remember that the segments also require one bit of information each in addition to the information contained in the endpoints.

P3. The complement of three lines in P_2 which meet in exactly three different points has four components, all of equal status. So a triangular domain is given by three lines plus the additional information which of the four components is meant.

Explicitly, if the lines are given by three planes in $\mathbf{R}^3$ passing through the origin with linear equations $l_k(x, y, z) = 0$, the four triangular domains are

$$\left(\bigcap_{k=1}^{3} Y_{\theta,k}\right) \cup \left(-\bigcap_{k=1}^{3} Y_{\theta,k}\right), \qquad \theta = (\theta_1, \theta_2, \theta_3) \in \{-1, 1\}^3,$$

where $Y_{\theta,k}$ is the half space

$$Y_{\theta,k} = \{(x, y, z) \in \mathbf{R}^3 \smallsetminus \{(0, 0, 0)\};\ \theta_k l_k(x, y, z) \geqslant 0\}, \qquad k = 1, 2, 3,\ \ \theta \in \{-1, 1\}^3,$$

and where $\theta = (\theta_1, \theta_2, \theta_3) = (1, \pm 1, \pm 1)$ (four possibilities).

We may conclude that, just as for segments, the notion of triangular domain comes with different cognitive content in P_2 compared with E_2.

7. Proposition 16

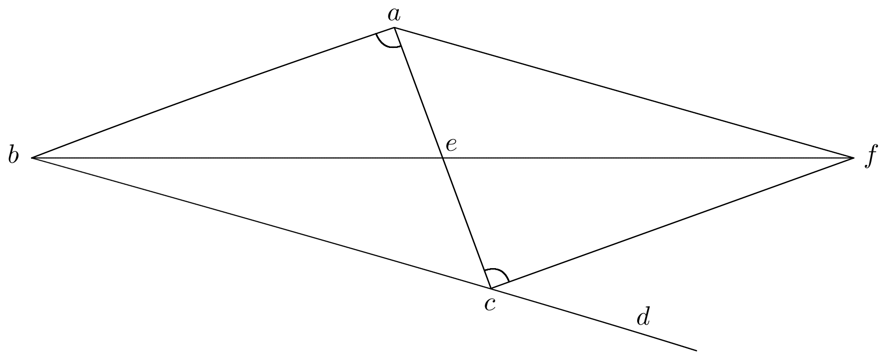
Proposition 16 says, as we have seen in Section 1, that an exterior angle in a triangle is greater than any of the two opposite interior angles. Let a triangle with vertices a, b, c be given, and let us examine the proof that the exterior angle at c is strictly larger than the interior angle $\angle bac$ at a (see the figure on page 20). Euclid extends the side $[b, c]$ beyond c to a point d such that c lies between b and d (the exact position of d is not important; it serves only to define the exterior angle $\angle acd$ at c). The problem is now to prove that the exterior angle $\angle acd$ is larger than the interior angle $\angle bac$. Euclid introduces a new point e as the midpoint of the side $[a, c]$ and extends the segment $[b, e]$ to a point f, defined so that e is the midpoint of $[b, f]$. He therefore obtains two congruent triangles $\triangle abe$ and $\triangle cfe$, where $\angle ecf = \angle eab$. Hence the angle at c in the triangle $\triangle cfe$ is equal to the angle at a in the triangle $\triangle abe$. So far everything is OK. Euclid then says:

> μείζων δέ ἐστιν ἡ ὑπὸ ΕΓΔ τῆς ὑπὸ ΕΓΖ· (Sjöstedt 1968:22; Fitzpatrick 2011:21)
> (But the angle $\angle ecd$ is greater than the angle $\angle ecf$;)

This is something we should see from a (deceptive) lettered diagram. (On the significance of the lettered diagram in Greek mathematics, see Section 8.)

At this point, it is convenient to continue the argument on a sphere. We need only look at a triangle on the sphere such that the distance $\delta(b, e)$ between b and e is $\pi/2$. (We measure as usual the length of a side by the angle subtended by it as viewed from the center of the sphere.) Then the distance between f and b is π, that is, they are antipodes and will be identified in the projective plane. Hence the great circle determined by the side $[b, c]$ and the great circle through b and e meet at f, and the exterior angle at c is equal to the interior angle at a.

This is the simplest example I have found; by perturbing it a little (taking the distance between b and e to be a little larger than $\pi/2$), we can arrange that the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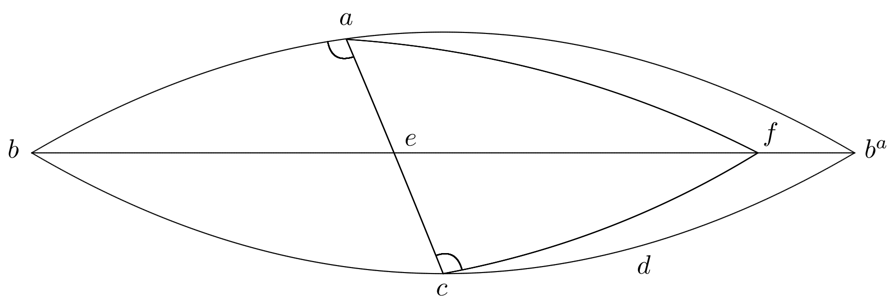
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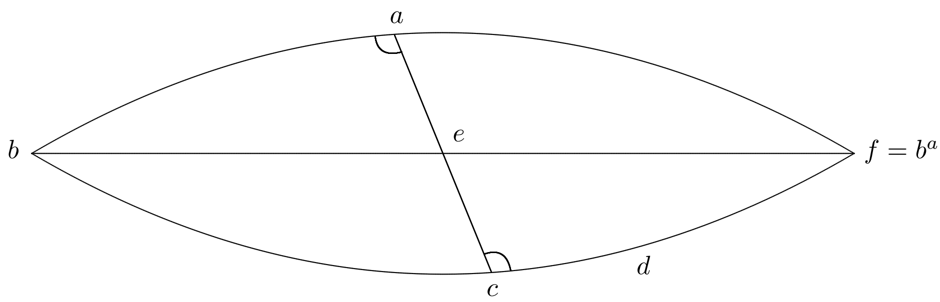
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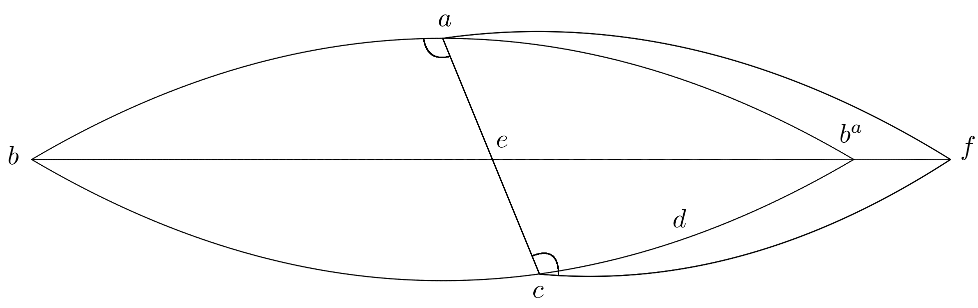
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exterior angle at c is smaller than the interior angle at a.[18] In fact, the crucial quantity here is the length of the median $[b, e]$:

Proposition 7.1. *Let a triangular domain on the sphere be given with vertices in a, b, c. We assume that all sides and all angles are less than π. Let e be the midpoint on the side $[a, c]$.*
(1) *If the distance between b and e is less than $\pi/2$, then the conclusion in Euclid's Proposition 16 holds: the exterior angle at c is larger than the interior angle at a.*
(2) *If the distance between b and e is equal to $\pi/2$, then the exterior angle at c is equal to the interior angle at a.*
(3) *If the distance between b and e is larger than $\pi/2$, then the exterior angle at c is smaller than the interior angle at a.*

It is reasonable to assume that no side or angle in the triangle is equal to π or larger—we avoid the trouble of defining the exterior angle of a concave angle.

Note that this result is a result on the geometry of the projective plane. I have chosen to formulate it for the sphere only because in this way it will be easier to visualize.

Proof. Note that we cannot speak about *the* midpoint between two non-antipodal points of the sphere, since there are two midpoints (they are antipodal). However, if a triangular domain is given, we take the midpoint which belongs to it. This is how we define e.

By the Spherical Sine Theorem applied to the triangle $\triangle bcf$ we obtain

$$\sin(\pi - \angle ecd + \angle ecf) \sin \delta(b, c) = \sin(\angle bfc) \sin \delta(b, f).$$

Now

$$\sin(\pi - \angle ecd + \angle ecf) = \sin(\angle ecd - \angle ecf) = \sin(\angle ecd - \angle bac),$$

and since $\sin \delta(b, c)$ and $\sin(\angle bfc) = \sin(\angle abc)$ are positive by assumption, the sine of the difference $\angle ecd - \angle bac$ has the same sign as $\sin \delta(b, f) = \sin 2\delta(b, e)$. The three cases (1), (2), (3) are obtained if $\delta(b, e) < \pi/2$, $= \pi/2$, and $> \pi/2$, respectively. □

Thus if all three medians in the triangle we consider are less than $\pi/2$, Euclid is all right.

8. Relying on diagrams

Reviel Netz devotes the first chapter of his book (1999:12–67) to an instructive account of the all-important role of the lettered diagram in Greek mathematics. The lettered diagram is a combination of different elements on the logical plane, the cognitive plane, the semiotic plane, and the historical plane; "the fertile intersection of different, almost antagonistic elements which is responsible for the shaping of deduction" (Netz 1999:67).

[18] Also Heath (1926a:280) remarks that in order for the proof to be valid, it is necessary that the line cf should fall within the angle $\angle acd$, and Bernard Vitrac (personal communication 2012-04-01) directs my attention to the fact that also he points this out (Vitrac 1990:228).

When I studied Euclidean geometry at Norra real in Stockholm some sixty years ago, our teacher, Bertil Broström, repeatedly emphasized that we were not allowed to draw any conclusions from the diagrams: all proofs should depend only on the axioms and the chain of logical implications. Nevertheless, the diagrams served as inspiration and mnemonic help—and perhaps a little bit more.

It is an interesting fact that we can actually draw *some* valid conclusions from a diagram—provided it is not too special (whatever that means). And it is not obvious where to draw the boundary between legitimate and forbidden use of visual information. This point was brought up in a discussion with the authors of the paper by Avigad et al. (2009). They discuss there the role of diagrams in the proofs, and the formal logical system called E which they have constructed accepts Euclid's proof considered in Section 7 without protest.[19] John Mumma explains that the system E licenses the inference that the angle $\angle ecd$ is larger that the angle $\angle ecf$.

> Similarly, one cannot generally infer, from inspecting two angles in a diagram, that one is larger than the other, but one can draw this conclusion if the diagram "shows" that the first is contained in the second. (Avigad et al. 2009:701)

So clearly the formal system E does accept some information from a diagram.

The relations of betweenness and same-sidedness are primitives in the system E. The possibility of a non-orientable plane is ruled out not by any explicit assumption but by the rules for reasoning with betweenness and same-sidedness (John Mumma, personal communication 2012-04-15). Conceivably, one could construct a similar formal system which does not have the betweenness relation for triples of points, nor the same-sidedness relation. (Cf. the *Kernsatz* of Pasch quoted in the next section.)

9. Orientability

Orientability of a manifold means, roughly speaking, that you can walk around it with a watch and the hands of the watch still go around clockwise (as viewed from the outside) when you return to the starting point after an excursion. The Euclidean plane E_2 and the sphere S_2 are both orientable. However, the sphere is not a model for Euclid's axioms (postulates), since two lines in general position will intersect in two points, not in one, and two antipodal points do not determine a great circle uniquely. This is what forces us to identify antipodes; the projective plane becomes a *bona fide* model—at least we so argued—but orientability is lost. Nevertheless, it is often convenient to conduct an argument on the sphere, as I have done in Proposition 7.1 above.

Postulate 5, the *Postulate of Parallels*, quoted in Subsection 3.2, states that two lines meet on a certain side. In the projective plane it is meaningless to talk about the side of a straight line. Given a point on a straight line, you can define two sides of the line in a neighborhood of the point, but if you go along the line and have your watch on your left wrist, you come back after a while with the watch on your right wrist (as viewed from the outside). So the very fact that Euclid talks about

[19]The system E is proved to be equivalent to an earlier formal system for Euclidean geometry due to Alfred Tarski.

"the same side" and "that side" means that he assumes the plane to be orientable. Hence projective geometry is excluded.

One can retain from Postulate 5 merely that the lines are not parallel, i.e., that they do meet somewhere, not mentioning any side. In this modified form, Postulate 5 is true also in the projective case.

Here it is of interest to note one of Pasch's axioms, viz.

> III. *Kernsatz.* — Liegt der Punkt C innerhalb der Strecke AB, so liegt der Punkt A außerhalb der Strecke BC (Pasch 1926:5). — (III. *Axiom.* If the point C lies within the segment AB, then the point A lies outside the segment BC.)

In the projective plane this can have a meaning only if we define both segments carefully; see the discussion in Subsection 3.2.

10. Conclusion

10.1. The first question

Propositions 16 and 27 become true if we suppose orientability or introduce some other hypothesis which will rule out the projective plane. And orientability is a reasonable hypothesis: Euclid in his Postulate 5 talks about the sides of a straight line, which is meaningless without orientability.

With the projective plane as a model, we can either conclude that Proposition 16 is meaningless, since we cannot compare angles, or false if we measure angles as discussed in Subsection 3.2. Proposition 27 can be interpreted as saying that the mentioned lines do not meet, and if so it is false whether we measure the angles on the sphere or not. The reasonable way out of this confusion is, again, to accept the tacit hypothesis of orientability.

If our beloved teacher, ὁ στοιχειωτής, could see my paper, he might react in one of two possible ways. Either

> α′. Sure, my boy, I do assume orientability—I just forgot to jot it down. (I was too busy thinking about Postulate Five.) In the next edition, which is now being prepared here in the Μουσεῖον, I shall include orientability as Postulate Six. Who wants to live on a Möbius strip anyway?

or

> β′. Ἰδού! — Hey, that's interesting! Seems to be a more general geometry. I shall write about it in Book Fourteen. And I like Napier's rule and the Spherical Sine Theorem which you learnt from your navigating father Sam Svensson even before you studied my geometry and plane trigonometry for Bertil Broström. We are all navigators here in Africa, aren't we? *Navigare necesse est*, as somebody will soon quip.

Can you guess which?

10.2. The second question

We have observed that the term εὐθεῖα often means a rectilinear segment. Perhaps this is its most basic meaning. In other contexts it could be interpreted as an infinite straight line, but also, if we want to avoid an actual infinity, as a family of equivalent rectilinear segments, thus as a potential infinity. However, in projective geometry, the infinite straight lines are just great circles with opposite points identified, thus hardly infinitely large. This gives us one more reason to believe that Euclid did not think about projective geometry. Finally, but rarely, it can mean 'ray'.

For straight lines in the sense of Heath that are infinite in one or both directions there appears the problem of actual infinity; if we avoid that by considering only segments, we have to obtain uniqueness by forming equivalence classes, which is certainly an anachronistic viewpoint, but maybe was exactly what Euclid did implicitly.

Let us listen to our beloved teacher once more, this time on *eutheia*:

> γ′. Ληρεῖτε! — Bah! What is straight is straight, and the wise understand. I do not waste words in my geometry. You young people use too many. Maybe you left Africa too early. I am afraid you will have to set up a Terminology Center in a futile effort to control the flood.

And on infinity:

> δ′. Aristotle and his gang of physicists are harassing us mathematicians. We must nowadays be careful when writing about infinity—potential infinity has rapidly become ΠΟ—but at night I am free to think about actual infinity. I can even see it.

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