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Domains of holomorphy for Fourier transforms of solutions to discrete convolution equations

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Dedicated to the memory of Lu Qikeng,
a great mathematician

Abstract We study solutions to convolution equations for functions with discrete support in $\mathbf{R}^n$, a special case being functions with support in the integer points. The Fourier transform of a solution can be extended to a holomorphic function in some domains in $\mathbf{C}^n$, and we determine possible domains in terms of the properties of the convolution operator.

Keywords discrete convolution, domain of holomorphy, Fourier transformation

Mathematics Subject Classification (2010) 32D10, 39A05, 39A12, 42B05, 42A85

Contents

1. Introduction

If f is a function defined in $\mathbf{R}^n$ satisfying an exponential growth condition in a cone and vanishing outside this cone, then its Fourier transform can be extended to a holomorphic function in a certain domain in $\mathbf{C}^n$. This is the first consideration in this paper.

Next, if μ and f are given functions, we look for solutions u to the convolution equation $\mu * u = f$. If we take the Fourier transforms of the functions involved, we obtain $\hat{\mu}\hat{u} = \hat{f}$ with pointwise multiplication in the left-hand side. So if $\hat{\mu}$ and $\hat{u}$ are holomorphic in a domain, then $\hat{f}$ is holomorphic in the same domain.

We also see that $\hat{u}$ is a solution to a division problem: $\hat{u} = \hat{f}/\hat{\mu}$. So if $\hat{f}$ and $\hat{\mu}$ are both holomorphic in an open set Ω, then $\hat{u}$ will be holomorphic in the set $\Omega \smallsetminus Z(\hat{\mu})$, where $Z(\hat{\mu})$ is the set of zeros of $\hat{\mu}$. Such division problems have been studied a long time in the framework of partial differential equations, going back to at least Leon Ehrenpreis's paper (1954). We investigate possible holomorphic extensions of the Fourier transform of u in terms of properties of μ.

The results of this paper were presented at a conference in Liverpool on 2013 December 16. Those of them that do not use the Fourier transformation were published in my paper (2015); now the Fourier transformation will come in as an essential tool.

The plan of the paper is as follows. Sections 2, 3 and 4 set the stage by recalling notions about discrete convolution, the Fourier transformation, and tropicalization. Section 5 introduces measures of growth at infinity of functions. Then extensions into the complex domain of Fourier transforms of solutions to convolution equations are studied, first when the functions have support in $\mathbf{Z}^n$ (Section 6), then for more general supports (Section 7). Results on the growth of solutions are given in Section 8. Some examples are mentioned in Section 9. Finally, Section 10 presents a conclusion and some hints for further work.

Notation

The boldface letters $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$ have their usual meaning according to Bourbaki (1961); thus for instance $\mathbf{N} = \{0, 1, 2, \dots\}$ is the set of natural numbers.

We shall use $\mathbf{R}_+$ to denote the set of all positive real numbers and

$$\mathbf{R}_! = [-\infty, +\infty] = \mathbf{R} \cup \{-\infty, +\infty\}$$

to denote the set of extended real numbers, adding two infinities.

The *ceiling function* $\mathbf{R} \ni t \mapsto \lceil t \rceil \in \mathbf{Z}$ and the *floor function* $\mathbf{R} \ni t \mapsto \lfloor t \rfloor \in \mathbf{Z}$ are defined by the inequalities

$$t \leqslant \lceil t \rceil < t + 1 \text{ and } t - 1 < \lfloor t \rfloor \leqslant t.$$

For brevity we shall write $x \ll 0$ when a condition holds for all real numbers x which are negative and for which $|x|$ is sufficiently large.

Addition $\mathbf{R}^2 \ni (x, y) \mapsto x + y \in \mathbf{R}$ can be extended in two different ways to operations $(\mathbf{R}_!)^2 \to \mathbf{R}_!$: the *upper sum* $x \dotplus y$ is defined as $+\infty$ if one of the terms is equal to $+\infty$, and the *lower sum* $x \mathbin{\underset{\cdot}{+}} y$ is defined as $-\infty$ if one of the terms is equal to $-\infty$. We use $x \wedge y$ for the minimum of x and y; $x \vee y$ for the maximum. Under these operations $\mathbf{Z}$ and $\mathbf{R}$ are lattices, and $\mathbf{Z}_!$ and $\mathbf{R}_!$ are complete lattices.

The *indicator function* $\operatorname{ind}_A = -\log \chi_A$, where χ_A is the characteristic function, takes the value 0 in A and $+\infty$ in its complement. We shall write $\operatorname{card}(A)$ for the cardinality of a set A. Thus $\operatorname{card}(\varnothing) = 0$, $\operatorname{card}(\mathbf{N}) = \aleph_0$, $\operatorname{card}(\mathbf{R}) = 2^{\aleph_0}$.

We shall use the l^p-norm $\|x\|_p = (\sum_j |x_j|^p)^{1/p}$, $1 \leqslant p < +\infty$, and the l^∞-norm $\|x\|_\infty = \sup_j |x_j|$ for $x \in \mathbf{R}^n$. We shall use these norms also for functions. When any norm can serve, we write just $\|x\|$. The inner product is written $\xi \cdot x = \xi_1 x_1 + \cdots + \xi_n x_n$, $(\xi, x) \in \mathbf{R}^n \times \mathbf{R}^n$.

In a metric space X with metric d we shall denote by $B_<(c,r)$ and $B_\leqslant(c,r)$ the *strict ball* and the *non-strict ball* with center at $c \in X$ and radius $r \in \mathbf{R}$, respectively, thus

$$B_<(c,r) = \{x \in X;\ d(c,x) < r\} \text{ and } B_\leqslant(c,r) = \{x \in X;\ d(c,x) \leqslant r\}.$$

The closure and interior of a subset A of a topological space will be denoted by $\overline{A}$ and A°, respectively. Thus in $\mathbf{R}^n$, $\overline{B_<(c,r)} = B_\leqslant(c,r)$ if r is positive, and $B_\leqslant(c,r)^\circ = B_<(c,r)$ for all real r.

2. Convolution

Let G be an abelian group—most of the time we shall take $G = \mathbf{Z}^n$ or $G = \mathbf{R}^n$. We define the *convolution product* $h = f * g$ of two functions $f, g\colon G \to \mathbf{C}$ by

$$h(x) = (f * g)(x) = \sum_{y+z=x} f(y)g(z), \qquad x \in G, \tag{2.1}$$

provided the sum is convergent in a suitable sense. An obvious such condition is that the *support* of f, by which we mean just the set $\operatorname{supp} f = \{x \in G;\ f(x) \neq 0\}$, is finite. If G is the space $\mathbf{R}^n$, we can assume that the functions tend to zero sufficiently rapidly at infinity. See my paper (2015) for several kinds of algebras satisfying this provision, and also for other situations when the convolution can be defined.

The *Kronecker delta* δ_a, defined by $\delta_a(a) = 1$ and $\delta_a(x) = 0$ for $x \neq a$, satisfies $\delta_a * \delta_b = \delta_{a+b}$. Taking $a = 0$, we see that δ_0 is a neutral element for convolution: $f * \delta_0 = f$ for all functions f.

3. The Fourier transformation

We define the Fourier transform $\hat{f}$ of a function $f\colon \mathbf{R}^n \to \mathbf{C}$ by

$$\hat{f}(\zeta) = \sum_{x \in \mathbf{R}^n} f(x) e^{i\zeta\cdot x}, \qquad \zeta \in \Omega \subset \mathbf{C}^n, \tag{3.1}$$

for those $\zeta \in \mathbf{C}^n$ for which the sum has a good sense. We may take f with support in $\mathbf{Z}^n$, but shall allow also functions defined in $\mathbf{R}^n$ with more general support.

A Fourier transform is actually a convolution product: $\hat{f}(\xi) = (f * g)(0)$, where $g(x) = e^{-i\xi\cdot x}$.

The Fourier transform of a convolution product is given by $(f * g)\hat{} = \hat{f}\,\hat{g}$ under suitable conditions on f and g.

We have adapted the signs in (3.1) to the usual conventions concerning Fourier series. For functions with support in $\mathbf{Z}^n$, the Fourier inversion formula therefore becomes the formula for retrieving the coefficients of the Fourier series, i.e., when $\operatorname{supp} f \subset \mathbf{Z}^n$,

$$f(x) = (2\pi)^{-n} \int_0^{2\pi} \cdots \int_0^{2\pi} \hat{f}(\xi) e^{-i\xi\cdot x}\, d\xi_1 \cdots d\xi_n, \qquad x \in \mathbf{R}^n. \tag{3.2}$$

Here $\xi = (\xi_1, \ldots, \xi_n)$ are n real variables.

4. Tropicalization: Infimal convolution and the Fenchel transformation

Tropicalization means, roughly speaking, to replace a sum or integral by a supremum. A simple example is the l^p-norm,

$$\|x\|_p = \Big(\sum_{j=1}^n |x_j|^p\Big)^{1/p}, \qquad x \in \mathbf{R}^n, \quad 1 \leqslant p < +\infty,$$

which becomes

$$\|x\|_\infty = \Big(\sup_{j=1,\dots,n} |x_j|^p\Big)^{1/p} = \sup_{j=1,\dots,n} |x_j|$$

when the sum is replaced by the supremum.

4.1. Infimal convolution

Let us consider a convolution product

$$e^{-h(x)} = \int_{\mathbf{R}^n} e^{-f(x-y)} e^{-g(y)} dy, \qquad x \in \mathbf{R}^n,$$

which is well defined if f and g tend to infinity fast enough, e.g., if they satisfy $f(x), g(x) \geqslant \varepsilon\|x\| - C$ for some $\varepsilon > 0$ and some constant C. If we replace the integral by the supremum, we obtain

$$e^{-h(x)} = \sup_{y \in \mathbf{R}^n} e^{-f(x-y)} e^{-g(y)}, \qquad x \in \mathbf{R}^n;$$

more conveniently written as

$$h(x) = \inf_{y \in \mathbf{R}^n} \big(f(x-y) + g(y)\big), \qquad x \in \mathbf{R}^n.$$

We define generally the *infimal convolution* of f and g as $h = f \sqcap g$, where

$$h(x) = (f \sqcap g)(x) = \inf_{y \in \mathbf{R}^n} \big(f(x-y) \dotplus g(y)\big), \qquad x \in \mathbf{R}^n.$$

Here the upper addition allows us to admit functions with infinite values.

4.2. The Fenchel transformation

The *Fenchel transform* $\tilde{f}$ of a function $f\colon \mathbf{R}^n \to \mathbf{R}_!$ is defined as

$$\tilde{f}(\xi) = \sup_{x \in \mathbf{R}^n} \big(\xi \cdot x - f(x)\big), \qquad \xi \in \mathbf{R}^n.$$

Clearly $\xi \cdot x - f(x) \leqslant \tilde{f}(\xi)$, which can be written as

$$\xi \cdot x \leqslant f(x) \dotplus \tilde{f}(\xi), \qquad (\xi, x) \in \mathbf{R}^n \times \mathbf{R}^n,$$

called *Fenchel's inequality.* It follows that the second transform $\tilde{\tilde{f}}$ satisfies $\tilde{\tilde{f}} \leqslant f$. We have equality here if and only if f is convex, lower semicontinuous, and takes the value $-\infty$ only if it is $-\infty$ everywhere.

A Fenchel transform is actually an infimal convolution: $\tilde{f}(\xi) = -(f \sqcap g)(0)$, where $g(x) = \xi \cdot x$.

The Fenchel transformation $f \mapsto \tilde{f}$, named for Werner Fenchel (1905–1988), is a tropical counterpart of the Fourier transformation. To avoid complex numbers, it is more convenient to look at the Laplace transform of a function g:

$$(\mathscr{L} g)(\xi) = \int_0^\infty g(x) e^{-\xi x} dx, \qquad \xi \in \mathbf{R}.$$

When we replace the integral by a supremum and take the logarithm, we get

$$\log(\mathscr{L}_{\mathrm{trop}}\, g)(\xi) = \sup_x (\log g(x) - \xi x) = \tilde{f}(-\xi), \qquad \xi \in \mathbf{R}, \quad f = -\log g.$$

A special case of the Fenchel transform is obtained when the function is the indicator function of some set: $f = \mathrm{ind}_A$. Then $\tilde{f}$ is the *supporting function* H_A of A,

$$H_A(\xi) = \sup_{x \in A} \xi \cdot x, \qquad \xi \in \mathbf{R}^n.$$

We have

$$(f \sqcap g)\tilde{} = \tilde{f} \underset{\cdot}{+} \tilde{g} \leqslant \tilde{f} \dot{+} \tilde{g},$$

in analogy with the formula $(f * g)\hat{} = \hat{f}\hat{g}$. If φ and ψ are convex, then $\varphi \dot{+} \psi$ is convex, but not always $\varphi \underset{\cdot}{+} \psi$. However, when $\varphi = \tilde{f}$ and $\psi = \tilde{g}$, this is true: $\tilde{f} \underset{\cdot}{+} \tilde{g}$ is always convex, and is often equal to $\tilde{f} \dot{+} \tilde{g}$. In fact equality holds except for a few special cases.

5. Measuring the growth: The radial indicators

Definition 5.1. Given any subset A of $\mathbf{R}^n$ we define its *asymptotic cone*, to be denoted by A_∞, as the union of $\{0\}$ and the set of all $x \in \mathbf{R}^n \smallsetminus \{0\}$ such that there exists a sequence $(a^{(j)})_j$ of points in A with $\|a^{(j)}\|$ tending to $+\infty$ and $a^{(j)}/\|a^{(j)}\| \to x/\|x\|$. □

The asymptotic cone of $\mathbf{Z}^n$ is equal to all of $\mathbf{R}^n$.

Definition 5.2. Given a function $f \colon A \to \mathbf{C}$ we define its *upper radial indicator* as

$$p_f(x) = \limsup \frac{\|x\|}{\|a\|} \log |f(a)|, \qquad x \in A_\infty \smallsetminus \{0\},$$

where the limit superior is taken over all $a \in A$ such that $\|a\| \to +\infty$ and $a/\|a\| \to x/\|x\|$. Similarly, we define its *lower radial indicator* as

$$q^f(x) = \liminf \frac{\|x\|}{\|a\|} \log |f(a)|, \qquad x \in A_\infty \smallsetminus \{0\}.$$

Finally, we define $p_f(0) = q^f(0) = 0$. □

Proposition 5.3. *If $f\colon \mathbf{R}^n \to \mathbf{C}$ has finite support, then for a dense open set M in $\mathbf{R}^n$, which depends only on* $\operatorname{supp} f$*, we have*

$$p_{\hat f}(\zeta) = q^{\hat f}(\zeta) = H_{\operatorname{supp} f}(-\operatorname{Im}\zeta), \qquad \zeta \in \mathbf{C}^n,\ -\operatorname{Im}\zeta \in M,$$

the supporting function of the support of f evaluated at $-\operatorname{Im}\zeta$. In particular

$$p_{\hat f}(-i\eta) = q^{\hat f}(-i\eta) = H_{\operatorname{supp} f}(\eta), \qquad \eta \in M \subset \mathbf{R}^n.$$

Thus both radial indicators are equal and depend only on the convex hull of the support of the function except for a small closed set.

Proof. We have

$$|\hat f(\theta)| \leqslant \sum_x |f(x)| e^{-\operatorname{Im}\theta\cdot x} \leqslant m \sup_{x \in \operatorname{supp} f} |f(x)| e^{-\operatorname{Im}\theta\cdot x}, \qquad \theta \in \mathbf{C}^n,$$

where m is the cardinality of $\operatorname{supp} f$. From this we get

$$\frac{\|\zeta\|}{\|\theta\|}\log|\hat f(\theta)| \leqslant \frac{\|\zeta\|}{\|\theta\|}\log m + \sup_{x\,\in\,\operatorname{supp} f} \frac{\|\zeta\|}{\|\theta\|}\left(\log|f(x)| - \operatorname{Im}\theta\cdot x\right), \qquad \theta, \zeta \in \mathbf{C}^n.$$

As $\|\theta\| \to +\infty$ and $\theta/\|\theta\| \to \zeta/\|\zeta\|$, the right-hand side converges to

$$\sup_{x\,\in\,\operatorname{supp} f}(-\operatorname{Im}\zeta\cdot x) = H_{\operatorname{supp} f}(-\operatorname{Im}\zeta).$$

Thus we have $p_{\hat f}(-i\eta) = q^{\hat f}(-i\eta) \leqslant H_{\operatorname{supp} f}(\eta)$ everywhere.

To prove equality we consider a vector $\zeta \neq 0$ such that there is a point b in $\operatorname{supp} f$ satisfying

$$\frac{-\operatorname{Im}\zeta\cdot x}{\|\zeta\|} < \frac{-\operatorname{Im}\zeta\cdot b}{\|\zeta\|} \text{ for all points } x \in \operatorname{supp} f \text{ different from } b.$$

Then for all vectors $\theta \neq 0$ such that $\theta/\|\theta\|$ is sufficiently close to $\zeta/\|\zeta\|$, there is a positive number ε such that we have

$$\frac{-\operatorname{Im}\theta\cdot x}{\|\theta\|} \leqslant \frac{-\operatorname{Im}\theta\cdot b}{\|\theta\|} - \varepsilon \text{ for all points } x \in \operatorname{supp} f \text{ different from } b.$$

We get

$$|\hat f(\theta)| \geqslant |f(b)| e^{-\operatorname{Im}\theta\cdot b} - \sum_{x\,\neq\,b} |f(x)| e^{-\operatorname{Im}\theta\cdot x}.$$

For large enough $\|\theta\|$ the latter sum is at most half of the first term. Indeed,

$$\sum_{x\,\neq\,b} |f(x)| e^{-\operatorname{Im}\theta\cdot x} \leqslant \sum_{x\,\neq\,b} |f(x)| e^{-\operatorname{Im}\theta\cdot b - \varepsilon\|\theta\|} \leqslant \tfrac{1}{2}|f(b)| e^{-\operatorname{Im}\theta\cdot b}$$

if

$$\sum_{x\,\neq\,b} |f(x)| e^{-\varepsilon\|\theta\|} \leqslant \tfrac{1}{2}|f(b)|,$$

which is true for $\|\theta\|$ large enough. This implies that

$$\log|\hat{f}(\theta)| \geqslant -\mathrm{Im}\,\theta\cdot b + \log(\tfrac{1}{2}|f(b)|).$$

Thus

$$\frac{\|\zeta\|}{\|\theta\|}\log|\hat{f}(\theta)| \geqslant \frac{\|\zeta\|}{\|\theta\|}\Big(-\mathrm{Im}\,\theta\cdot b + \log(\tfrac{1}{2}|f(b)|)\Big) \to -\mathrm{Im}\,\zeta\cdot b = H_{\mathrm{supp}\,f}(-\mathrm{Im}\,\zeta)$$

as $\|\theta\| \to +\infty$ and $\theta/\|\theta\|$ converges to $\zeta/\|\zeta\|$. Therefore equality holds for all directions $-\eta$ such that the supporting hyperplane $\{x;\ -\eta\cdot x = -\eta\cdot b\}$ at a point b intersects $\mathrm{supp}\,f$ only in this point. This is a dense open set of vectors. □

6. Domains of holomorphy for transforms of functions with support in $\mathbf{Z}^n$

For functions with support contained in $\mathbf{Z}^n$ we can profit from the arithmetic of integers to get easy estimates.

If f has its support in $\mathbf{N}^n$ and is of exponential growth, say $|f(x)| \leqslant Ce^{\sigma\cdot x}$, $x \in \mathbf{N}^n$, for some real vector σ, then $\hat{f}$ and $|f|\hat{}$ are well defined and holomorphic in the domain defined by $\mathrm{Im}\,\zeta_j > \sigma_j$, $j = 1, \dots, n$, and can be estimated by

$$|\hat{f}(\zeta)| \leqslant ||f|\hat{}(\zeta)| \leqslant C\prod_{j=1}^{n}\frac{1}{1-e^{\sigma_j-\mathrm{Im}\,\zeta_j}}, \qquad \zeta \in \mathbf{C}^n,\ \mathrm{Im}\,\zeta_j > \sigma_j. \tag{6.1}$$

If all the σ_j are negative, the Fourier transform is defined in $\mathbf{R}^n$, but otherwise we have to go out into complex space.

If f has its support in $\mathbf{N}^n$ and grows exponentially, we cannot apply the inversion formula (3.2) to $\hat{f}$, but to $\hat{f}_\theta$, the Fourier transform of $f_\theta(x) = f(x)e^{\theta\cdot x}$, for a real vector θ satisfying $\theta_j + \sigma_j < 0$, where the σ_j are chosen so that $|f(x)| \leqslant Ce^{\sigma\cdot x}$. We obtain

$$f_\theta(x) = f(x)e^{\theta\cdot x} = (2\pi)^{-n}\int_0^{2\pi}\cdots\int_0^{2\pi}\hat{f}_\theta(\xi)e^{-i\xi\cdot x}\,d\xi_1\cdots d\xi_n, \qquad x \in \mathbf{Z}^n,$$

where $\hat{f}_\theta(\zeta) = \hat{f}(\zeta - i\theta)$, which means that for $\hat{f}$, the integral goes over a cube in $\mathbf{R}^n$ translated in $\mathbf{C}^n$ by the imaginary vector $-i\theta$.

We can generalize (6.1) to the following.

Theorem 6.1. *Let $f\colon \mathbf{Z}^n \to \mathbf{C}$ have support in a cone K and satisfy an estimate $|f(x)| \leqslant Ce^{\sigma\cdot x}$ for $x \in K$. Then the Fourier transform $\hat{f}(\zeta)$ is holomorphic for* $\mathrm{Im}\,\zeta$ *in the interior of $\sigma + K^{\mathrm{dual}}$, where K^{dual} is the dual of K, defined as*

$$K^{\mathrm{dual}} = \{\eta \in \mathbf{R}^n;\ \eta\cdot x \geqslant 0 \text{ for all } x \in K\}. \tag{6.2}$$

Proof. By considering $f(x)e^{-\sigma\cdot x}/C$ we are reduced to the case $\sigma = 0$, $C = 1$. We shall thus prove that, if $\|f\|_\infty \leqslant 1$, then $\hat{f}$ is well defined and holomorphic in $\mathbf{R}^n \times i\Lambda$, where Λ is the interior of K^{dual}. We have

$$|\hat{f}(\zeta)| \leqslant \sum_{x\in K\cap\mathbf{Z}^n} e^{-\eta\cdot x}, \qquad \zeta \in \mathbf{C}^n,\ \eta = \mathrm{Im}\,\zeta,$$

for certain values of ζ to be determined now. Define cones

$$\Lambda_\tau = \{\eta \in \mathbf{R}^n;\ \eta \cdot x \geqslant \tau\|\eta\|\,\|x\|_1 \ \text{ for all } x \in K\}, \qquad \tau > 0. \tag{6.3}$$

The union of all the $\Lambda_\tau \smallsetminus B_<(0,\rho)$, $\tau > 0$, $\rho > 0$, is equal to Λ. Fix τ and ρ. Then, for $\operatorname{Im}\zeta = \eta \in \Lambda_\tau$, we obtain

$$|\hat f(\zeta)| \leqslant \sum_{x\,\in\, K\cap\mathbf{Z}^n} e^{-\tau\|\eta\|\,\|x\|_1} \leqslant \sum_{x\,\in\,\mathbf{Z}^n} e^{-\tau\|\eta\|\,\|x\|_1} = \sum_{x\in\mathbf{Z}^n}\prod_{j=1}^n e^{-\tau\|\eta\|\,|x_j|}. \tag{6.4}$$

When $\tau, \rho > 0$ and $\|\eta\| \geqslant \rho$, the last expression is equal to

$$\prod_{j=1}^n \frac{1+e^{-\tau\|\eta\|}}{1-e^{-\tau\|\eta\|}} = \left(\frac{1+e^{-\tau\|\eta\|}}{1-e^{-\tau\|\eta\|}}\right)^n \leqslant \left(\frac{1+e^{-\tau\rho}}{1-e^{-\tau\rho}}\right)^n < +\infty.$$

Thus $\hat f$ is bounded in $\mathbf{R}^n + i(\Lambda_\tau \smallsetminus B_<(0,\rho))$ for every positive τ and ρ, and holomorphic in the interior, hence also holomorphic in the union $\mathbf{R}^n + i\Lambda$ as claimed. $\square$

Remark 6.2. Any norm can be used for $\|\eta\|$ in the definition of Λ. We have used the same norm to define $B_<(0,\rho)$ in the calculation, but actually also here any norm can be used. On the other hand, the use of the l^1 norm $\|x\|_1$ in (6.4) is essential for the precise result in the calculation. $\square$

Theorem 6.3. *Given a strict closed convex cone K, assume that a function $f\colon \mathbf{Z}^n \to \mathbf{C}$ with support in K satisfies a family of estimates*

$$|f(x)| \leqslant C_\sigma e^{\sigma\cdot x}, \qquad x \in K,\ \sigma \in \Sigma,$$

for some subset Σ of $\mathbf{R}^n$. Then the Fourier transform of f is holomorphic in the union of all the sets $\mathbf{R}^n + i(\sigma + (K^{\mathrm{dual}})^\circ)$, $\sigma \in \Sigma$.

Proof. For any two elements ρ, σ of Σ there is a point τ in $\mathbf{R}^n$ such that

$$\tau \in (\rho + \Lambda) \cap (\sigma + \Lambda),$$

where again Λ stands for $(K^{\mathrm{dual}})^\circ$. To see this, pick a point η in the open cone Λ, which is nonempty by assumption. Then for large positive t we have

$$\frac{1}{t}(\rho - \sigma) + \eta \in \Lambda.$$

Multiplying by t we get

$$\rho + t\eta \in \sigma + t\Lambda = \sigma + \Lambda,$$

so that $\tau = \rho + t\eta$ belongs to both $\rho + \Lambda$ and $\sigma + \Lambda$. Moreover $\tau + \Lambda$ is contained in $(\rho + \Lambda) \cap (\sigma + \Lambda)$.

We thus have a holomorphic function in each of the convex open sets $\mathbf{R}^n + i(\sigma + \Lambda)$, $\sigma \in \Sigma$. These open sets all intersect, and in all intersections they agree. So in the union we have a well-defined holomorphic function. $\square$

If f has finite support, or more generally bounded support and if $\|f\|_1$ is finite, its Fourier transform is an entire function. When studying holomorphy of a transform, such functions do not influence the domain. This fact we now use to improve the result in Theorem 6.1.

Theorem 6.4. *Let $f\colon \mathbf{Z}^n \to \mathbf{C}$ be given and define K_r as the smallest closed convex cone containing $\{a \in \operatorname{supp} f;\ \|a\| \geqslant r\}$, $r \geqslant 0$, and K_∞ as the intersection of all the K_r, $0 \leqslant r < +\infty$. Given a vector σ such that, for some cone L such that $L^\circ \supset K_\infty \smallsetminus \{0\}$, the estimate $|f(x)| \leqslant Ce^{\sigma\cdot x}$ holds for all $x \in L$, the Fourier transform $\hat{f}$ of f is holomorphic in $\Omega = \mathbf{R}^n + i(\sigma + (K_\infty^{\mathrm{dual}})^\circ)$.*

Proof. Define f_r as f in K_r and zero elsewhere. The transforms $\hat{f}_r$ and $\hat{f}$ are holomorphic in the same open set. So applying Theorem 6.1 to f_r, we see that its transform is holomorphic in

$$\Omega_r = \mathbf{R}^n + i(\sigma + (K_r^{\mathrm{dual}})^\circ).$$

The union of all the Ω_r is equal to the set Ω defined in the statement of the theorem. In fact, given any compact subset Γ of $(K_r^{\mathrm{dual}})^\circ$, there exists a number r such that K_r is contained in L. So $\hat{f}_r$, hence $\hat{f}$ as well, is holomorphic in $\mathbf{R}^n + i(\sigma + \Gamma^\circ)$, therefore in all of Ω. □

Theorem 6.5. *Given a function $f\colon \mathbf{Z}^n \to \mathbf{C}$, define K_r, $0 \leqslant r \leqslant \infty$, as in Theorem 6.4 and assume that f satisfies a family of estimates*

$$|f(x)| \leqslant C_\sigma e^{\sigma\cdot x}, \qquad x \in L_\sigma,\ \sigma \in \Sigma,$$

for some subset Σ of $\mathbf{R}^n$, where, for each $\sigma \in \Sigma$, L_σ is a cone such that $L_\sigma^\circ \supset K_\infty \smallsetminus \{0\}$. Then the Fourier transform of f is holomorphic in the union of all the sets

$$\mathbf{R}^n + i(\sigma + (K_\infty^{\mathrm{dual}})^\circ), \qquad \sigma \in \Sigma.$$

Proof. We just combine the proofs of Theorems 6.3 and 6.4. □

7. Domains of holomorphy for transforms of functions with more general support

By moving points in the support of a function to an integer point nearby with larger norm we can get a result for functions with arbitrary (not necessarily discrete support):

Theorem 7.1. *Given $f\colon \mathbf{R}^n \to \mathbf{C}$, define*

$$f_{\mathbf{Z}}(x) = \sum_{a\in C(x)} |f(a)|, \qquad x \in \mathbf{Z}^n,$$

where $C(x)$, $x \in \mathbf{Z}^n$, is the set of all $a \in \mathbf{R}^n$ such that $\lceil |a_j| \rceil = |x_j|$, $j = 1, \ldots, n$. If $f_{\mathbf{Z}}$ satisfies an estimate $|f_{\mathbf{Z}}(x)| \leqslant Ce^{\sigma\cdot x}$ for $x \in K \cap \mathbf{Z}^n$, where K is the smallest closed convex cone which contains all points $x \in \mathbf{Z}^n$ such that $C(x)$ is nonempty, then the Fourier transform $\hat{f}(\zeta)$ is holomorphic for $\operatorname{Im}\zeta$ in the interior of $\sigma + K^{\mathrm{dual}}$.

So $C(x)$ is a cube with a vertex at $x \in \mathbf{Z}^n$ and such that $\|x\|_\infty - 1 < \|a\|_\infty \leqslant \|x\|_\infty$ for all $a \in C(x)$. The function $f_{\mathbf{Z}}$ is obtained by balayage of f from $\mathbf{R}^n$ to $\mathbf{Z}^n$.

Proof. We apply Theorem 6.4 to $f_{\mathbf{Z}}$ and observe that all estimates for $f_{\mathbf{Z}}$ are valid also for f, as is the conclusion about holomorphy. ☐

The cone L spanned by the cubes $C(x)$ can be large, since when $\|x\|$ is small, the cube subtends a big angle as viewed from the origin. But by removing points near the origin we can again get a larger domain of holomorphy:

Theorem 7.2. *Given a function $f\colon \mathbf{R}^n \to \mathbf{C}$ such that*

$$\sum_{\substack{a\in\mathbf{R}^n\\ \|a\|<r}} |f(a)|$$

is finite for every r with $0 < r < +\infty$, define K_r and K_∞ as in Theorem 6.4 and

$$f_{\mathbf{Z},r}(x) = \sum_{a\in C_r(x)} |f(a)|, \qquad x \in \mathbf{Z}^n,$$

where for $\|x\| \geqslant r$, $C_r(x)$ is defined as $C(x)$ in Theorem 7.1, while $C_r(x) = \emptyset$ when $\|x\| < r$. If, for some r, $f_{\mathbf{Z},r}$ satisfies an estimate $|f_{\mathbf{Z},r}(x)| \leqslant Ce^{\sigma\cdot x}$ for $x \in L \cap \mathbf{Z}^n$, where L is a closed convex cone such that L° contains $K_\infty \smallsetminus \{0\}$, then the Fourier transform $\hat f(\zeta)$ is holomorphic for $\operatorname{Im}\zeta$ in the interior of $\sigma + (K_\infty)^{\mathrm{dual}}$.

Proof. Given a cone L the interior of which contains $K_\infty \smallsetminus \{0\}$, we can take r so large that the points x for which $C_r(x)$ is nonempty is contained in L. Then Theorem 6.4 will yield the result. ☐

We can also use an estimate for f itself if its support is sufficiently sparse, and we also here can remove points near the origin.

Theorem 7.3. *Assume that the support A of a function $f\colon \mathbf{R}^n \to \mathbf{C}$ is sparse in the sense that $C(x)$, $x \in \mathbf{Z}^n$, as defined in Theorem 7.1, contains a number of points in A which grows slower than every exponential function $e^{\varepsilon\|x\|}$, $\varepsilon > 0$. Define K_r, $0 \leqslant r \leqslant \infty$, as in Theorem 6.4. If f satisfies an estimate $|f(x)| \leqslant Ce^{\sigma\cdot x}$ for $x \in L$, where the interior of L contains $K_\infty \smallsetminus \{0\}$, then the Fourier transform $\hat f(\zeta)$ is holomorphic for $\operatorname{Im}\zeta$ in the interior of $\sigma + K_\infty^{\mathrm{dual}}$.*

Proof. It is clear that the series

$$\sum_{a\in K_r\cap A} e^{-\tau\|\eta\|\,\|a\|_1} \leqslant \sum_{x\in\mathbf{Z}^n} \operatorname{card}(C(x))e^{-\tau\|\eta\|\,\|x\|_1}$$

converges for $\tau > 0$, $\|\eta\| \geqslant \rho > 0$ (cf. (6.4)). In the estimate $\operatorname{card}(C(x)) \leqslant e^{\varepsilon\|x\|_1}$ we take $\varepsilon < \tau\rho$. ☐

8. Estimates for solutions to convolution equations

The study of solutions w to a convolution equation $\nu * w = f = \sum_{a\in\mathbf{R}^n} f(a)\delta_a$ can be reduced to the special equation $(\delta_0 - \mu) * u = \delta_0$ as explained in my paper (2015: Lemma 4.3). In the sequel we shall consider only this equation; all results can then be translated to the equation $\nu * w = f$.

8.1. Sufficient conditions for the growth of solutions

Theorem 8.1. *Let $\mu\colon \mathbf{R}^n \to \mathbf{C}$ be a nonzero function with finite support and $\theta \neq 0$ a given vector in $\mathbf{R}^n$. Define*

$$r = \inf_y(\theta\cdot y;\ y \in \operatorname{supp}\mu), \quad R = \sup_y(\theta\cdot y;\ y \in \operatorname{supp}\mu).$$

Assume that r is positive. Let a real vector $\sigma = (\sigma_1, \dots, \sigma_n)$ be given and define a real number γ by

$$|\mu|\hat{}\,(i\sigma) = \sum_y |\mu(y)|e^{-\sigma\cdot y} = e^\gamma. \tag{8.1}$$

Then the unique function $u\colon \mathbf{R}^n \to \mathbf{R}$ which solves $(\delta_0 - \mu) * u = \delta_0$ and is zero when $\theta\cdot x \ll 0$ can be estimated as

$$|u(x)| \leqslant e^{(\sigma+\gamma\theta/r)\cdot x}, \qquad x \in \mathbf{R}^n,\ \text{if } \gamma \geqslant 0; \tag{8.2}$$

and

$$|u(x)| \leqslant e^{(\sigma+\gamma\theta/R)\cdot x}, \qquad x \in \mathbf{R}^n,\ \text{if } \gamma \leqslant 0. \tag{8.3}$$

The same conclusions hold if μ is nonnegative, if we have $u(x) = 0$ when $\theta\cdot x \ll 0$ and u satisfies only the inequality $u \leqslant \delta_0 + u * \mu$.

Remark 8.2. For $\sigma = s\theta$, $s \in \mathbf{R}$, the estimates take the form

$$|u(x)| \leqslant e^{(s+\gamma/r)\theta\cdot x}, \qquad x \in \mathbf{R}^n, \quad \text{if } \gamma = \log|\mu|\hat{}\,(is\theta) \geqslant 0; \tag{8.4}$$

in particular for $s = 0$,

$$|u(x)| \leqslant \|\mu\|_1^{\theta\cdot x/r}, \qquad x \in \mathbf{R}^n, \quad \text{if } \|\mu\|_1 = e^\gamma \geqslant 1;$$

and

$$|u(x)| \leqslant e^{(s+\gamma/R)\theta\cdot x}, \qquad x \in \mathbf{R}^n, \quad \text{if } \gamma = \log|\mu|\hat{}\,(is\theta) \leqslant 0; \tag{8.5}$$

in particular for $s = 0$,

$$|u(x)| \leqslant \|\mu\|_1^{\theta\cdot x/R}, \qquad x \in \mathbf{R}^n, \quad \text{if } \|\mu\|_1 = e^\gamma \leqslant 1.$$

This shows that u tends to zero as $\theta\cdot x$ tends to $+\infty$ if $\|\mu\|_1 < 1$; that u is bounded if $\|\mu\|_1 = 1$; and that u grows at most exponentially if $\|\mu\|_1 > 1$. □

Corollary 8.3. *With μ, u and σ as in the theorem, the estimate $|\mu|\hat{\ }(i\sigma) \leqslant 1$ implies that $|u(x)| \leqslant e^{\sigma\cdot x}$, $x \in \mathbf{R}^n$.*

Proof of Theorem 8.1. If $u = \delta_0 + u * \mu$, we have $u(0) = 1$ and, for $x \neq 0$,

$$|u(x)| \leqslant |(u*\mu)(x)| \leqslant \sum_y |\mu(y)|\,|u(x-y)|. \tag{8.6}$$

If μ is nonnegative and u satisfies the inequality $u \leqslant \delta_0 + u * \mu$, we have $u(0) \leqslant 1$ and the same inequality holds.

Let us try to prove that

$$|u(x)| \leqslant e^{(\sigma+t\theta)\cdot x}, \qquad x \in \mathbf{R}^n, \tag{8.7}$$

where t is a real number to be determined later. Now the values of y for which $\mu(y) \neq 0$ in (8.6) must satisfy $\theta \cdot y \geqslant r$, so that $\theta \cdot (x-y) \leqslant \theta \cdot x - r$. By induction on $\theta \cdot x$ we may therefore assume that all the values of $u(x-y)$ that occur in (8.6) satisfy the estimate. We get

$$\begin{aligned} |u(x)| &\leqslant \sum_y |\mu(y)| e^{(\sigma+t\theta)\cdot(x-y)} = e^{(\sigma+t\theta)\cdot x} \sum_y |\mu(y)| e^{-(\sigma+t\theta)\cdot y} \\ &\leqslant e^{(\sigma+t\theta)\cdot x} \sum_y |\mu(y)| e^{-\sigma\cdot y} \sup_y e^{-t\theta\cdot y} \leqslant e^{(\sigma+t\theta)\cdot x} e^{\gamma} \sup_y e^{-t\theta\cdot y}. \end{aligned}$$

For $t \geqslant 0$ we have $e^{\gamma} \sup_y e^{-t\theta\cdot y} = e^{\gamma - tr}$; for $t \leqslant 0$ we have $e^{\gamma} \sup_y e^{-t\theta\cdot y} = e^{\gamma - tR}$. We now choose $t = \gamma/r$ if $\gamma \geqslant 0$ and $t = \gamma/R$ if $\gamma \leqslant 0$. With these choices of t in (8.7) we obtain the estimates (8.2) and (8.3). □

When $\sigma = s\theta$, there is an inverse relation between s and γ: the larger s is, the smaller is γ. It is therefore natural to ask which is the best estimate that can be obtained by this method. The answer is an easy one:

Corollary 8.4. *Let μ, θ, r and R be as in the theorem, take $\sigma = s\theta$, and define s_0 as the unique real number such that $|\mu|\hat{\ }(is_0\theta) = 1$. Then the best estimate of the form (8.4) or (8.5) is obtained when $\gamma = 0$, viz.*

$$|u(x)| \leqslant e^{s_0\theta\cdot x}, \qquad x \in \mathbf{R}^n.$$

Proof. The function $\gamma\colon \mathbf{R} \to \mathbf{R}$ defined by

$$e^{\gamma(s)} = |\mu|\hat{\ }(is\theta) = \sum_y |\mu(y)| e^{-s\theta\cdot y}, \qquad s \in \mathbf{R},$$

is strictly decreasing, tends to $+\infty$ when $s \to -\infty$, and tends to 0 when $s \to +\infty$. The existence and uniqueness of s_0 such that $e^{\gamma(s_0)} = 1$ is therefore clear.

We obtain

$$e^{s+\gamma(s)/r} = \Big(\sum_y |\mu(y)| e^{s(r-\theta\cdot y)}\Big)^{1/r}, \qquad s \leqslant s_0,$$

and

$$e^{s+\gamma(s)/R} = \Big(\sum_y |\mu(y)| e^{s(R-\theta\cdot y)}\Big)^{1/R}, \qquad s \geqslant s_0.$$

Since $r \leqslant \theta\cdot y \leqslant R$ for all y that occur, the first is decreasing and the second is increasing. Hence both expressions attain their minima at $s = s_0$, which proves the corollary. □

Example 8.5. For the array b of binomial coefficients (see Subsection 9.1), as well as for the array b_n of multinomial coefficients (see Subsection 9.2), we choose a θ with all components θ_j positive, and obtain $r = \min_j \theta_j$, $R = \max_j \theta_j$. If $\theta = (1, 1, \dots, 1)$, then $R = r = 1$, and in general we get $R = r$ if the support of μ is contained in a hyperplane $\{x;\ \theta\cdot x = r\}$, $r > 0$. We have $e^\gamma = \sum_j e^{-\sigma_j}$. If $\sigma = s(1, 1, \dots, 1)$, we get $s_0 = \log n$, so that $b_n(x) \leqslant n^{x_1+\dots+x_n}$, which is the best possible estimate of the form $b_n(x) \leqslant a^{x_1+\dots+x_n}$. □

Example 8.6. For the array of Delannoy numbers, we have $\|\mu\|_1 = 3$; $\hat\mu(i\sigma) = e^{-\sigma_1} + e^{-\sigma_2} + e^{-\sigma_1-\sigma_2} = e^\gamma$. For a vector θ with positive components, we have $r = \min(\theta_1, \theta_2)$, $R = \theta_1 + \theta_2$, thus $R \geqslant 2r$. We may take $\theta = (1, 1)$, so that $r = 1$ and $R = 2$. Then $\hat\mu(is\theta) = 2e^{-s} + e^{-2s}$. Thus for $\sigma = s\theta$, we have $\gamma \geqslant 0$ if and only if $2e^{-s} + e^{-2s} \geqslant 1$, and $\gamma \leqslant 0$ if and only if $2e^{-s} + e^{-2s} \leqslant 1$. The number s_0 is equal to $\log(\sqrt{2}+1)$; thus $d(x) \leqslant (\sqrt{2}+1)^{x_1+x_2}$, which is the best possible estimate of the form $d(x) \leqslant a^{x_1+x_2}$ for the Delannoy numbers. □

8.2. Necessary conditions for the growth of solutions

Conversely we have, under the extra assumption that μ is nonnegative:

Theorem 8.7. *Let $\mu\colon \mathbf{R}^n \to [0, +\infty[$ have finite support contained in a half space $\{x \in \mathbf{R}^n;\ \theta\cdot x \geqslant r\}$, $r > 0$, and let K be the smallest convex cone which contains* $\operatorname{supp}\mu$. *Let u be defined as in Theorem 8.1. If for any positive ε an estimate*

$$u(x) \leqslant C_\varepsilon e^{(\sigma+\varepsilon\theta)\cdot x}, \qquad x \in K, \tag{8.8}$$

holds for some constant C_ε, then

$$|\mu|\hat{}\,(i\sigma) = |\hat\mu(i\sigma)| = \hat\mu(i\sigma) \leqslant 1.$$

Proof. We note that $u \geqslant 0$ here since μ is nonnegative.

It is enough to consider the case $\sigma = 0$. Assume that $\|\mu\|_1 > 1$:

$$\|\mu\|_1 = \hat\mu(0) = \sum_y \mu(y) > 1.$$

Then $\hat\mu(it\theta) = \sum_y \mu(y) e^{-t\theta\cdot y}$, $t \geqslant 0$, takes a value larger than 1 for $t = 0$ and tends to zero when t tends to $+\infty$ since $\theta\cdot y \geqslant r > 0$ in the support of μ. We first determine a positive number t_0 such that $\sum_y \mu(y) e^{-t_0\theta\cdot y} = 1$. Hence $\hat\mu(it\theta)$ is smaller than 1 for $t > t_0$ and equal to 1 when $t = t_0$. This implies that

$$\hat u(it\theta) = \frac{1}{1 - \hat\mu(it\theta)}, \qquad t > t_0,$$

is finite for $t > t_0$ and tends to $+\infty$ as $t \searrow t_0$.

For a small enough positive τ, θ belongs to the cone Λ_τ defined by (6.3). Fix such a τ. According to the proof of Theorem 6.1, $\hat{u}(\zeta)$ is bounded for every positive ε when $\tau\|\operatorname{Im}\zeta\| \geqslant \varepsilon$; in particular, $\hat{u}(it\theta)$ is bounded when $t\tau\|\theta\| \geqslant \varepsilon$. We can choose $\varepsilon = t_0\tau\|\theta\|$, so that $\hat{u}(it\theta)$ is bounded for all $t \geqslant t_0$, contradicting the formula above which shows that $\hat{u}(it\theta)$ tends to $+\infty$ when t tends to t_0. Hence we cannot have $\hat{\mu}(0) > 1$. □

By combining Theorem 8.3 in my paper (2015) with Theorems 8.1 and 8.7, we obtain the following result.

Theorem 8.8. *Given a function $\mu\colon \mathbf{R}^n \to [0, +\infty[$ which is nonzero only at finitely many points in a half space $\{x \in \mathbf{R}^n;\ \theta \cdot x \geqslant r\}$, $r > 0$, let u be the unique function $u\colon \mathbf{R}^n \to \mathbf{C}$ which is zero where $\theta \cdot x \ll 0$ and solves the equation $(\delta_0 - \mu) * u = \delta_0$. Then, given an arbitrary vector $\sigma \in \mathbf{R}^n$, the following four conditions are equivalent.*
(A). *For every positive ε there exists a constant C_ε such that*

$$u(x) \leqslant C_\varepsilon e^{\sigma\cdot x+\varepsilon\|x\|}, \qquad x \in \mathbf{R}^n.$$

(A′). *The upper radial indicator of u satisfies*

$$p_u(x) \leqslant \sigma \cdot x \qquad x \in \mathbf{R}^n.$$

(A″). *The Fenchel transform of $-p_u$ satisfies $(-p_u)\tilde{}(-\sigma) \leqslant 0$.*
(A‴). *$-\sigma \in M_f$, where M_f is the set such that $(-p_f)\tilde{} = \operatorname{ind}_{M_f}$.*
(B). *$u(x) \leqslant e^{\sigma\cdot x}$ for all $x \in \mathbf{R}^n$.*
(C). *$\hat{\mu}(i\sigma) \leqslant 1$.* □

Proof. Here (A), (A′), (A″) and (A‴) are the same as in Theorem 8.3 in (2015) and equivalent as already proved there, whereas (B) and (C) are from the present section. We have (A) ⇒ (C) ⇒ (B) ⇒ (A), where the first implication comes from Theorem 8.7, the second from Theorem 8.1, and the third is trivially true. □

We note in particular the implication (A) ⇒ (B), which is a kind of Liouville theorem:

Corollary 8.9. *Let u be as in the theorem. Then $\log u(x) - \varepsilon\|x\|$ is bounded from above for every positive ε if and only if u is bounded.* □

9. Examples

9.1. The binomial coefficients

Let

$$b(x,y) = \binom{x+y}{x} = \frac{(x+y)!}{x!\,y!}, \qquad (x,y) \in \mathbf{N}^2,$$

be the binomial coefficients. We define them also when $x \leqslant -1$ or $y \leqslant -1$ by taking them equal to zero there. This array satisfies the equation $(\mu_b - \delta_0) * b = \delta_0$, where $\mu_b(x,y) = \delta_{(1,0)} + \delta_{(0,1)}$.

Using Stirling's formula in the simple form

$$\log x! = x\log x - x + O(\log x), \qquad x \to +\infty,$$

we see that the radial indicators are

$$p_b(x,y) = q^b(x,y) = x\log(1+y/x) + y\log(1+x/y), \qquad (x,y) \in \mathbf{R}^2_+. \tag{9.1}$$

In particular, $p_b(x,x) = 2x\log 2 \approx 1.3863x$.

The function p_b is positively homogeneous of order 1 and concave. To prove concavity it is enough to note that $x \mapsto \log(1+x)$ is concave. This implies that the homogeneous function $(x,y) \mapsto y\log(1+x/y)$ is concave; by symmetry also $(x,y) \mapsto x\log(1+y/x)$ is concave.

The gradient of p_b is

$$\operatorname{grad} p_b(x,y) = (\log(1+y/x), \log(1+x/y)), \qquad x,y > 0.$$

We note the following special case of Theorem 8.8.

Proposition 9.1. *The Fenchel transform $(-p_b)^{\tilde{}}$ of the function $-p_b$ is equal to ind_{M_b}, where*

$$M_b = \{\eta \in \mathbf{R}^2;\ e^{\eta_1} + e^{\eta_2} \leqslant 1\}.$$

Since $-p_b$ is convex, lower semicontinuous, and does not take the value $-\infty$, we also get $q^b = p_b = -(-p_b)^{\tilde{}\tilde{}} = -(\mathrm{ind}_{M_b})^{\tilde{}}$.

9.2. The multinomial coefficients

We can generalize the array b as follows. Let $x \in \mathbf{N}^n$. The number of ways of choosing n subsets with $x_1, \dots, x_n$ elements out of a set with $\sum x_j = \mathbf{1}\cdot x$ elements is

$$b_n(x) = \frac{(\mathbf{1}\cdot x)!}{\prod_{j=1}^n x_j!}, \qquad x \in \mathbf{N}^n.$$

Here we define $\mathbf{1} = (1,1,\dots,1)$. The radial indicators of b_n are

$$p_{b_n}(x) = q^{b_n}(x) = \sum_{j=1}^n x_j \log\frac{\mathbf{1}\cdot x}{x_j}, \qquad x \in \mathbf{R}^n_+,$$

generalizing (9.1): $b_2 = b$. In the formula $(\delta_0 - \mu_{b_n}) * b_n = \delta_0$ we have

$$\mu_{b_n} = \sum_{j=1}^n \delta_{e^{(j)}},$$

where $e^{(j)}$ is the unit vector with 1 at the jth place, $j = 1, \dots, n$. Its Fourier transform is

$$\hat{\mu}_{b_n}(\zeta) = \sum_{j=1}^n e^{i\zeta_j}, \qquad \zeta \in \mathbf{C}^n. \tag{9.2}$$

9.3. The Delannoy numbers

The Delannoy numbers $d(x, y)$, $(x, y) \in \mathbf{Z}^2$, are defined as 0 when $x \leqslant -1$ or when $y \leqslant -1$, as 1 when $(x, y) = (0, 0)$, and for $(x, y) \in \mathbf{N}^2 \setminus \{(0, 0)\}$ by the recursion formula

$$d(x, y) = d(x - 1, y) + d(x - 1, y - 1) + d(x, y - 1). \tag{9.3}$$

The array satisfies the equation $(\mu_d - \delta_0) * d = \delta_0$, where $\mu_d = \delta_{(1,0)} + \delta_{(1,1)} + \delta_{(0,1)}$.

This array is named for Henri-Auguste Delannoy (1833–1915) and was introduced in his paper (1895). He investigated the possible moves on a chessboard. The numbers under consideration here appear when one studies “la marche de la Reine.” For biographies of Delannoy, see Banderier & Schwer (2005) and Schwer & Autebert (2006).

It follows from (9.2) that

$$\hat{\mu}_d(\zeta_1, \zeta_2) = \hat{\mu}_{b_3}(\zeta_1, \zeta_2, \zeta_1 + \zeta_2) \qquad (\zeta_1, \zeta_2) \in \mathbf{C}^2.$$

This means that information on d can be obtained from b_3.

The Delannoy numbers appear in many problems in mathematics; see Sulanke (2003), who lists 29 different examples. To mention just one, $d(n, r) = d(r, n)$ is the cardinality of the ball of radius r in $\mathbf{Z}^n$ equipped with the l^1 metric (also known as the hyperoctahedron),

$$\Big\{x \in \mathbf{Z}^n;\ \|x\|_1 = |x_1| + \cdots + |x_n| \leqslant r\Big\};$$

Vassilev & Atanasov (1987), quoted here from Sulanke (2003, note 18). The symmetry in (n, r) is by no means obvious a priori.

To Sulanke’s examples I added a thirtieth (2008:609): for $(a, b) \in \mathbf{Z}^2$, $a + b \geqslant 0$, the number of Khalimsky-continuous functions $[0, a + b]_{\mathbf{Z}} \to \mathbf{Z}$ satisfying $f(0) = 0$ and $f(a + b) = a - b$ is equal to $d(a, b)$. For a detailed proof, see Samieinia (2010: Theorem 2.2). And then a thirty-first: a fundamental solution

$$E(x + iy) = i^{y-x} d(x, y), \qquad x + iy \in \mathbf{Z}[i],$$

for a discrete analogue of the Cauchy–Riemann operator (2008:608). Thus I came to the Delannoy numbers along two paths: digital geometry, where the Khalimsky topology is a useful structure; and discrete complex analysis.

Again, we note a special case of Theorem 8.8.

Proposition 9.2. *The Fenchel transform* $(-p_d)\tilde{}$ *of the function* $-p_d$ *is equal to* ind_{M_d}*, where*

$$M_d = \{\eta \in \mathbf{R}^2;\ e^{\eta_1} + e^{\eta_2} + e^{\eta_1 + \eta_2} \leqslant 1\}. \qquad \square$$

From this proposition we deduce the following result for the array of Delannoy numbers.

Theorem 9.3. *The upper radial indicator* p_d *of the array of Delannoy numbers is*

$$p_d(x, y) = x \log \frac{r + y}{x} + y \log \frac{r + x}{y}, \qquad (x, y) \in \mathbf{R}^2_+,$$

where $r = \sqrt{x^2 + y^2}$.

I have proved this using the methods developed here. It was proved earlier by Pemantle & Wilson (2002) using other methods.

10. Conclusion and hints for further work

We have obtained results on holomorphic extentions of the Fourier transforms of functions defined on $\mathbf{R}^n$, especially those that solve convolution equations.

The growth of a solution u to an equation $(\delta_0 - \mu) * u = \delta_0$ is related to the behavior of the Fourier transform $\hat{\mu}$ of μ. This relation is well understood when $\mu \geqslant 0$. It is not well understood when μ takes real values of both signs or non-real values. The growth of a solution u to an equation $(\delta_0 - \mu) * u = \delta_0$ can sometimes be roughly the same as that of the solution v to $(\delta_0 - |\mu|) * v = \delta_0$; sometimes v grows much faster than u.

References

Banderier, Cyril; Schwer, Sylviane. 2005. Why Delannoy numbers? *J. Statist. Plann. Inference* **135**, no. 1, 40–54; *arXiv:math/0411128* (2004).

Bourbaki, N[icolas]. 1961. *Topologie générale*, Chapter 1 and 2. Éléments de mathématique, Première partie. 3rd edition. Paris: Hermann.

Delannoy, H[enri-Auguste]. 1895. Emploi de l'échiquier pour la résolution de certains problèmes de probabilité. *Comptes rendus du congrès annuel de l'Association française pour l'avancement des sciences* **24**, Bordeaux, 70–90.

Ehrenpreis, Leon. 1954. Solution of some problems of division. I. Division by a polynomial of derivation. *Amer. J. Math.* **76**, 883–903.

Kiselman, Christer O. 2008. Functions on discrete sets holomorphic in the sense of Ferrand, or monodiffric functions of the second kind. *Science in China, Series A, Mathematics*, April 2008, **51**, No. 4, 604–619.

Kiselman, Christer O. 2015. Estimates for solutions to discrete convolution equations. *Mathematika* **61**, issue 02, pp. 295–308.

Pemantle, Robin; Wilson, Mark C. 2002. Asymptotics of multivariate sequences. I. Smooth points of the singular variety. *J. Combin. Theory Ser. A* **97**, no. 1, 129–161.

Samieinia, Shiva. 2010. The number of continuous curves in digital geometry. *Port. Math.* **67**, 75–89.

Schwer, Sylvanie R.; Autebert, Jean-Michel. 2006. Henri-Auguste Delannoy, une biographie. *Math. & Sci. hum./Mathematical Social Sciences* No. 174, 25–67.

Sulanke, Robert A. 2003. Objects counted by the central Delannoy numbers. *J. Integer Seq.* **6**, no. 1, Article 03.1.5, 19 pp.

Vassilev, Mladen; Atanassov, Krassimir. 1987. On Delanoy [sic] numbers. *Annuaire Univ. Sofia Fac. Math. Inform.* **81**, no. 1, 153–162 (1994.

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