The Bulletin of the Swedish Mathematical Society, 2019 May 15, pp. 20–23.

Hans Rådström and how to define smooth functions on any set

Christer Oscar Kiselman

Hans Rådström was born in Stockholm on 1919 March 26, a little more than a century ago. He died in Linköping on 1970 November 05. He was Associate Professor (in Swedish *laborator*) at the Royal Institute of Technology from 1954 through 1964.

During his tenure at the Royal Institute of Technology Hans advised two students to the PhD (at the time called *Tekn Dr*) in mathematics. Lars Ingelstam defended his thesis *Studies in Real Normed Algebras* in 1964 (he later became Professor of Technology and Social Change at Linköping University). In 1967, Johan Philip presented his thesis *Reconstruction of intensity distributions*.[1]

Hans Rådström then moved to a similar position at Stockholm University, where he stayed from 1964 up to 1969. It was there that I came to know him.

Part of Hans' wide interests in different branches of mathematics and applications led him to approach non-zero-sum games in a novel way. His results, however, were never published. For a similar situation, but with a different outcome, see the section on smooth functions below.

Hans acted for the creation of a single Department of Mathematics in Stockholm, joint for the Royal Institute of Technology and Stockholm University. He based his arguments on his experience from both institutions. That this was seriously considered by Stockholm University is illustrated by the fact that Göran Björck and several others visited Copenhagen, Aarhus and Oslo to see how such a "Mathematical Center" could function. The proposal was on the agenda for decades. Mikael Passare (1959–2011), when he chaired the Math Department at Stockholm University some fifty years later, acted in the same direction.

In the spring of 1969 Rådström was appointed Professor of Applied Mathematics at the Linköping Institute of Technology, later to become part of Linköping University (founded in 1975). He contributed with rare enthusiasm to the quality of research and education in this newly established institution. Gerd Brandell, who was with Hans in Stockholm and moved with him to Linköping as an assistant, remarks that he enjoyed creating something new and in accordance with his own ideas.

In addition to the two doctors mentioned above, he advised two more students to a doctoral degree: Per Enflo (PhD 1970) with thesis title *Investigations on Hilbert's fifth problem for non locally compact groups* and Martin Ribe (PhD 1972) with thesis title *On spaces which are not supposed to be locally convex*. Martin had also a second advisor: Edgar Asplund (1931–1974).

[1] These two theses are not mentioned in the *Mathematics Genealogy Project* (checked 2019-05-14). I sincerely hope that they will appear there soon.

Lars Ingelstam and Hans kept a close friendship even after Lars left mathematics for other pursuits. After Hans' death in 1970 Lars went to Linköping to clean up his affairs.

Afternoon tea at Stockholm University

Every day, at 15:00, there was an afternoon tea at the Math Department of Stockholm University. Hans was there and often initiated discussions on a wide range of topics. Mathematics, of course, generally with a philosophical touch, but he also often made provocative statements. He smoked all the time and filled the room with smoke. Those were the days.

Advising interrupted

Hans was the advisor of Gerd Becker (now Gerd Brandell), who was writing a thesis for the Licentiate Degree. She did not finish before the death of her advisor and asked me to take over. All the mathematical ideas were there before this happened.

Gerd's thesis has the title *On the relation between the real and complex structures in some categories of complex linear spaces* (1971), and it was approved at Uppsala University.

Can you define smooth functions on an irregular set?

On a C^∞ manifold you can define C^∞ functions, but what can you do without such a structure? Hans Rådström had a nice idea how to define smooth functions in a very general setting. He presented it to Jan Boman and me. We attacked his claims like hawks (his wording). His statements were not quite correct, which we discovered. However, the main idea was great and could be developed. Jan took this up in his paper (1967).

Let X be any set and define for a fixed $m \in \mathbf{N} \cup \{\infty\}$ two mappings F_m and G_m as follows. Let Γ be any set of mappings from the real line $\mathbf{R}$ into X, and Φ any set of real-valued functions on X. Then we define $F_m(\Gamma)$ as the set of all functions φ from X into $\mathbf{R}$ such that the composition $\varphi \circ \gamma$ is of class C^m for all $\gamma \in \Gamma$. We define $G_m(\Phi)$ as the set of all mappings γ from $\mathbf{R}$ into X such that $\varphi \circ \gamma$ is of class C^m for all functions $\varphi \in \Phi$.

If we write $\mathscr{P}(Y)$ for the family of all subsets of a set Y and $\mathscr{F}(Y, Z)$ for the family of all mappings from Y into Z, then we can express this briefly as

$$F_m(\Gamma) = \{\varphi;\ \varphi \circ \gamma \in C^m(\mathbf{R}, \mathbf{R}) \text{ for all } \gamma \in \Gamma\}, \quad \Gamma \in \mathscr{P}(\mathscr{F}(\mathbf{R}, X));$$
$$G_m(\Phi) = \{\gamma;\ \varphi \circ \gamma \in C^m(\mathbf{R}, \mathbf{R}) \text{ for all } \varphi \in \Phi\}, \quad \Phi \in \mathscr{P}(\mathscr{F}(X, \mathbf{R})).$$

Thus F_m and G_m are mappings

$$F_m \colon \mathscr{P}(\mathscr{F}(\mathbf{R}, X)) \to \mathscr{P}(\mathscr{F}(X, \mathbf{R}));$$
$$G_m \colon \mathscr{P}(\mathscr{F}(X, \mathbf{R})) \to \mathscr{P}(\mathscr{F}(\mathbf{R}, X)).$$

The pair (F_m, G_m) is a Galois connection, which means that F_m and G_m are decreasing and that $F_m \circ G_m$ and $G_m \circ F_m$ are larger than the identity. It follows that they are idempotent. All this dates back by almost two centuries: to the work of Évariste Galois (1811–1832) on the group of automorphisms of a field.

Hans Rådström defined smooth functions on a set X by fixing a set of curves Γ and then defining a function $\varphi\colon X \to \mathbf{R}$ to be C^m smooth if it belongs to $F_m(\Gamma)$. As far as I know, he did not develop a theory, and he published nothing. A basic question is whether we get the usual C^m functions if X is a differential manifold and Γ is the set of C^m curves in the manifold; Rådström said that this is so, and Jan Boman (1967) proved it—for $m = \infty$.

More precisely, Jan proved that, for finite $m \geqslant 1$,

$$C^m(\mathbf{R}^n, \mathbf{R}) \subset F_m(C^\infty(\mathbf{R}, \mathbf{R}^n)) \subset C^{m-1,1}(\mathbf{R}^n, \mathbf{R}).$$

Here $C^m(\mathbf{R}^n, \mathbf{R})$ denotes the space of all functions on $\mathbf{R}^n$ with real values whose derivatives of order at most m exist and are continuous, while $C^{m-1,1}(\mathbf{R}^n, \mathbf{R})$ is the subspace of $C^{m-1}(\mathbf{R}^n, \mathbf{R})$ consisting of functions whose derivates of order $m-1$ are all Lipschitz continuous. Jan proved that the first inclusion here is strict; obviously so is the second. Hans Rådström's claim (or was it just a conjecture?) that this would work for every finite m was therefore not true, but when taking the intersection over all finite $m \geqslant 1$, the loss in differentiability is of no consequence, and we see that $F_\infty(C^\infty(\mathbf{R}, \mathbf{R}^n)) = C^\infty(\mathbf{R}^n, \mathbf{R})$ proving Rådström's claim for $m = \infty$.

Jan Boman has also obtained an explicit description of $F_m(C^m(\mathbf{R}, \mathbf{R}^n))$ for all finite $m \geqslant 1$ (personal communication 2008-09-18).

For $m = \infty$, Eike Petermann (1979) developed a formalism in the framework of category theory. Finally, Michor (1984), Kriegl & Nel (1990), and Kriegl & Michor (1997) developed a theory for global analysis using smooth curves.

I published a remark about this method to define smooth functions (2010: Example 5:14). It fits nicely into my ideas on lower and upper inverses of mappings between ordered sets, a concept generalizing Galois correspondences.

Semigroups embedded into normed vector spaces

Hans Rådström published a paper (1952) where his main goal was to prove that a semigroup can be embedded into a normed vector space when it satisfies certain conditions. This paper inspired Lars Hörmander (1931–2012) to write a paper (1955) proving similar results using the support function and the duality in convex geometry based on the pioneering work of Werner Fenchel (1905–1988).

MathSciNet, the web version of *Mathematical Reviews*, lists nineteen publications with Rådström's name in the title. Most of these are concerned with the embedding mentioned here, called the *Rådström embedding* in some titles. Several titles mention *Minkowski–Rådström–Hörmander space*, used for the space into which a semigroup is embedded.

In conclusion

Hans Rådström's ideas were received with interest and curiosity by several mathematicians. His open mind and his friendly manners inspired continued research. His generosity in sharing his conjectures was highly appreciated, and led to valuable contributions to science.

References in chronological order

1952. Rådström, Hans. An embedding theorem for spaces of convex sets. *Proc. Amer. Math. Soc.* **3**, 165–169.

1955. Hörmander, Lars. Sur la fonction d'appui des ensembles convexes dans un espace localement convexe. *Ark. mat.* **3**, 181–186.

1967. Boman, Jan. Differentiability of a function and of its compositions with functions of one variable. *Math. Scand.* **20**, 249–268.

1979. Petermann, Eike. On a method of constructing categories. *J. Pure Appl. Algebra* **15**, 271–281.

1984. Michor, Peter. A convenient setting for differential geometry and global analysis. *Cahiers Topologique Géom. Différentielle* **25**, no. 1, 63–109; no. 2, 113–178.

1990. Kriegl, A.; Nel, L. D. Convenient vector spaces of smooth functions. *Math. Nachr.* **147**, 39–45.

1997. Kriegl, Andreas; Michor, Peter W. *The convenient setting of global analysis.* Mathematical Surveys and Monographs, 53. Providence, RI: American Mathematical Society.

2010. Kiselman, Christer O. Inverses and quotients of mappings between ordered sets. *Image and Vision Computing* **28**, 1429–1442.

Uppsala University, Department of Information Technology
Paper address: P. O. Box 337, SE-751 05 Uppsala
Amber addresses: kiselman@it.uu.se, christer@kiselman.eu
URL: www.cb.uu.se/~kiselman