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Comments to Real analysis MN1

Contents

1 Real numbers

The first part of the course will cover topological properties of $\mathbb{R}^n$. R and V below stands for Rudin and Vretblad, respectively. Vx.y means chapter

x, section y in Vretblad, and Rz means chapter z in Rudin.
Most of the content of Vretblads compendium have been covered in previous courses and will therefore take little time during the lectures.

V1.1 **Important definitions:** Neighbourhood, inner point and interior, boundary point, open set, closed set and closure.
Important results: Arbitrary unions (and finite intersections) of open sets are open (which is equivalent to saying that finite unions and arbitrary intersections of closed sets are closed; exercise 1.6).
Comment: Theorem 1.4 will later on be used as a definition of *topology*.
Exercises: 1.1–1.9.

V1.2 **Important definitions:** Convergent sequence.
Important results: Theorem 1.5, saying that a set M is closed if and only if every sequence in M that converges has its limit point also in M, is often useful to prove that a given set is closed.
Comment: The result stated in exercise 1.14 will later on be used as a definition of continuous mappings.
Exercises: 1.13–1.14

V2.1 **Important definitions:** Limit point, subsequence.
Exercises: 2.1, 2.2 (some), 2.4, 2.5.

V2.2 **Important definitions:** Bounded set, compact set, limit superior and limit inferior.
Important results: Bolzano–Weierstrass' theorem.
Comment: The definition of compactness in $\mathbb{R}^n$ *will not* be used in arbitrary topological spaces. "Bounded and closed" is *not* equivalent to "compact" in arbitrary metric spaces.
Exercises: 2.6–2.10.

V2.3 **Important definitions:** Cauchy-sequence.
Important results: A sequence (in $\mathbb{R}^n$) is convergent if and only if it is a Cauchy-sequence.
Comment: In general topological spaces any convergent sequence is a Cauchy-sequence. There are however, in general topological spaces, examples of Cauchy-sequences that are *not* convergent, as we shall see later.
Exercises: 2.14, 2.16, 2.17, 2.19.

R1 We shall not cover the whole chapter during the lectures. Pages 12–21 are not included and some the rest is left for self-studies.
Important definitions: Ordered set, upper (and lower) bound, infimum and supremum, least-upper-bound property, ordered field.
Important results: $\mathbb{R}$ is an ordered field with the least-upper-bound property, $\mathbb{R}$ has the archemidean property.
Exercises: 1, 4, 5.

R3. The first pages (47–54) is left for the time being, until we have defined metric spaces. Much of the material in this chapter has already been covered in earlier courses in analysis (Analys MN1) and therefore we shall not spend much time during the lectures talking about it.
Important definitions: Convergent series, absolutely convergent, radius of convergence.
Important results: Comparison test, root tes, ratio test.
Exercises: 1, 3 (here it should say "... and that $s_n < 2$ for $n = 1, 2, 3, \dots$"), 5, 9, 11, 14, 16.

2 Basic topology

Here we study chapter 2 in Rudins book. Some of the theorems need to be reformulated for topological spaces.
First we need to add some definitions and examples.

2.1 Topological spaces

We begin by giving the definition of a topological space, which has been motivated in previous lectures:

Definition 1. (Topological space)
Let X be a set and $\mathcal{U}$ a collection of subsets of X satisfying the following conditions:

T1) $X, \emptyset \in \mathcal{U}$,

T2) if $U_1, \dots, U_k \in \mathcal{U}$, *then* $\cap_{j=1}^{k} U_j \in \mathcal{U}$,

T3) if $U_\alpha \in \mathcal{U}$, *for* $\alpha \in A$, *then* $\cup_{\alpha \in A} U_\alpha \in \mathcal{U}$.

We say that $U \subset X$ *is* open *if* $U \in \mathcal{U}$. *The collection* $\mathcal{U}$ *is called* a topology on X, *and* X *equipped with the topology* $\mathcal{U}$ *is called a* topological space *(sometimes denoted* $(X, \mathcal{U})$*, if we want to emphasize the topology in question).*

The motivation for this definition and some of its purposes has hopefully been clarified during the lectures, and we have also seen how the usual distance function on $\mathbb{R}^n$ induces a topology.
If we look at the definition of topological spaces, there are two topologies (on any set) that are 'obvious':

Example 1. a) (*The indiscrete topology*) Let X be a set and let $\mathcal{U} = \{X, \emptyset\}$. Then obviously T1)–T3) are satisfied. The set X equipped with this topology is called *an indiscrete space*.
b) (*The discrete topology*) Let X be a set and let $\mathcal{U}$ consist of *all* subsets of X. Then $X, \emptyset \in \mathcal{U}$ and since any intersection and any union of subsets of X are subsets of X, T2) and T3) are also satisfied. The set X equipped with the discrete topology is called *a discrete space*.

These examples represents the two extremes for topologies on a set X. The indiscrete topology only satisfies the minimum requirement for a topology and in the discrete topology *all* subsets are considered as open. In between them there is a whole spectrum of topologies, and we shall pay attention to some of these.

Example 2. Metric spaces, and among them $\mathbb{R}^n$, are also examples of topological spaces (see definition 2.15 and theorem 2.24 in Rudins book).

Some more definitions (see Rudin p. 32, Definition 2.18 for the case of metric spaces):

Definition 2. *Let X be a topological space.*

a) *A* neighbourhood *of a point $p \in X$ is an open subset of X containing p.*

b) *A point p is called a* limit point *of the set $E \subset X$ if every neighbourhood of p contains at least one point $q \neq p$ such that $q \in E$. The set of limit points of E is denoted E'.*

c) *If $p \in E$ and if p is not a limit point of E, then p is called an* isolated point *of E.*

d) *A subset F of X is called* closed *if $F' \subset F$.*

e) *A point $p \in E$ is called* interior *if there is a neighbourhood of p which is contained in E. The set of interior points of E is denoted E^o.*

f) *A subset E of X is called* perfect *if it is closed and contains no isolated points.*

g) A subset E of X is called dense *if every point of X is a limit point of E, or a point of E (or both).*

h) The closure *of $E \subset X$ is the set $E \cup E' = \overline{E}$.*

Theorem 1. *A subset F of a topological space X is closed if and only if its complement F^c is open.*

Proof . First suppose that F is closed and let $x \in F^c$. Then of course $x \notin F$ and hence it is not a limit point of F, by Definition 2 d). Therefore there exists a neighbourhood V of x such that $V \cap F = \emptyset$, i e such that $V \subset F^c$, which shows that F^c is open.
Assume conversely that F^c is open and let $x \in F'$. Then every neighbourhood of x contains points from F, by Definition 2 b). Hence, x is not an interior point of F^c and since F^c is open this shows that $x \in F$ and, since x was arbitrarily choosen, this proves that $F' \subset F$, i e that F is closed.

From this we get immediately the following result:

Corollary 1. *A subset E of a topological space is open if and only if its complement E^c is closed.*

From the definition of closed sets and the definition of topology, it follows that:

Theorem 2. *If X is a topological space, then*

C1) X and $\emptyset$ are closed subset of X,

C2) if $F_1, \ldots, F_k$, $k \in \mathbb{Z}_+$, are closed subsets of X, then $\cup_{j=1}^{k} F_j$ is a closed subset of X,

C3) if $\{F_\alpha\}_{\alpha \in A}$ is an arbitrary collection of closed subsets of X, then $\cap_{\alpha \in A} F_\alpha$ is a closed subset of X.

Proof . Here items *C2)* and *C3)* follows from the equalities

$$(E \cap F)^c = E^c \cup F^c \text{ and } (E \cup F)^c = E^c \cap F^c$$

and *C1)* follows from the fact that $X^c = \emptyset$ and $\emptyset^c = X$.

Theorem 3. *Let X be a topological space and $E \subset X$. Then:*

a) $\overline{E}$ is closed,

b) $E = \overline{E}$ if and only if E is closed, *c)* $\overline{E} \subset F$ *for each closed subset* $F \subset X$ *such that* $E \subset F$.

Proof . The proof is the same as the one given for Theorem 2.27, p. 35, in Rudins book.

Next we introduce the so-called *relative topology* on subsets of a topological space:

Definition 3. *Suppose* $E \subset Y \subset X$, *where* X *is a topological space.*
We say that E *is* open relative to Y, *if for each* $p \in E$ *there is a neighbourhood* V *of* p *in* X, *such that* $V \cap Y \subset E$.
If $\mathcal{U} = \{U_i\}_{i \in I}$ *is the topology on* X *and if* $Y \subset X$, *we name* $\mathcal{U}_Y = \{U_i \cap Y\}_{i \in I}$ *the* relative topology *(or* induced topology*) on* Y.

Remark . If $Y \subset X$, where X is a topological space and Y is equipped with the relative topology, then we can forget about the space X and just look at Y, if we are merely interested in subsets of Y. I e any subset of a topological space is a topological space in itself, when equipped with the relative topology.

The following concepts can also be useful sometimes (see also exercises 22–29, p. 45, in Rudins book):

Definition 4. (Base and subbase) *Let* $(X, \mathcal{U})$ *be a topological space and let* $\mathcal{V}$ *be a subset of* $\mathcal{U}$. *We say that*

a) $\mathcal{V}$ *is an* open base *for the topology* $\mathcal{U}$, *if every member of* $\mathcal{U}$ *can be written as a union of elements in* $\mathcal{V}$. *The elements of* $\mathcal{V}$ *are referred to as* basic open sets.

b) $\mathcal{V}$ *is an* open subbase *for the topology* $\mathcal{U}$, *if the collection of all finite intersections of elements in* $\mathcal{V}$ *form a base for* $\mathcal{U}$, *i e if any member of* $\mathcal{U}$ *can be written as a union of finite intersections of elements in* $\mathcal{V}$.

Example 3. The family of sets of the form $\{x \in \mathbb{R}^n : \|x - a\| < r\}$, where $a \in \mathbb{R}^n$ and $r > 0$, is an open base for the (usual) topology on $\mathbb{R}^n$, and the family of sets of the form $V_i = \{x \in \mathbb{R}^n : |x_i| < r_i\}$, $1 \leq i \leq n$, forms an open subbase for the same topology.

We end this section by defining *dense subsets* and *separability* (see also exercise 22–29, p. 45, in Rudins book):

Definition 5. *Let* X *be a topological space and* A *a subset of* X. *We say that* A *is* dense *(or* everywhere dense*) in* X, *if* $\overline{A} = X$. *If* X *has a countable, dense subset, then we say that* X *is* separable.

Example 4. Since $\mathbb{Q}$ is countable and $\overline{\mathbb{Q}} = \mathbb{R}$, we see that $\mathbb{R}$ is separable. This example can easily be extended to proving that $\mathbb{R}^n$ (and $\mathbb{C}^n$) is separable, $n \in \mathbb{Z}_+$.

2.2 Compact sets

Let X be a topological space and let $\{V_i\}_{i\in I}$ be a collection of open subsets of X.

Definition 6. *We say that $\{V_i\}_{i\in I}$ is an* open cover *of $E \subset X$, if $E \subset \cup_{i\in I} V_i$.*

Example 5. The open intervals $V_i =]-1/i, 1+1/i[$, $i = 1, 2, \ldots$, is an open cover of the (closed) interval $E_1 = [0,1] \subset \mathbb{R}$ and the open intervals $]1/i, 1-1/i[$, $i = 1, 2, \ldots$, is an open cover of the (open) interval $E_2 =]0,1[$.

Next we define the concept of *compact subset*:

Definition 7. *A subset K of a topological space X is said to be* compact *if every open cover of K has a finite subcover, i e if $K \subset \cup_{i\in I} V_i$, where each $V_i \subset X$ is open, then there exist a finite number of the V_i's, say $V_{i_1}, \ldots, V_{i_m}$ such that $K \subset \cup_{k=1}^m V_{i_k}$.*

Example 6. Finite subsets of topological spaces are always compact. The empty set is always compact.

Compact topological spaces plays an important rôle, especially in connection with the study of continuous functions.

Example 7. a) If X is equipped with the indiscrete topology and if $K \subset X$ is non-empty, then the only open subsets of X are X itself and $\emptyset$. So, if $\{V_i\}_{i\in I}$ is an open cover of K, then at least one of the V_i's has to be equal to X, and therefore there is a finite subcover. Hence, any subset of this space is compact.
b) If X is equipped with the discrete topology and if $K \subset X$, then $\{V_x\}_{x\in K}$, where $V_x = \{x\}$, is an open cover of K and the only chance for this cover to have a finite subcover is that K itself is finite. Hence, only finite subsets are compact.
c) If $K = \{x_n\}_{n=1}^\infty$ is a convergent sequence in a topological space X, converging to say x, then $K \cup \{x\}$ is compact. To see this we choose an open cover $\{V_i\}_{i\in I}$ of $K \cup \{x\}$. Then we choose $i_0 \in I$ so that $x \in V_{i_0}$. From the definition of convergence it follows that there exist N such that $x_n \in V_{i_0}$ if $n > N$. Then we choose $i_1, \ldots, i_N \in I$ such that $x_j \in V_{i_j}$ for $1 \le j \le N$ and we get that $V_{i_0}, V_{i_1}, \ldots, V_{i_N}$ covers $K \cup \{x\}$.

Theorem 4. *Suppose that $K \subset Y \subset X$, where X is a topological space and Y is equipped with the relative topology. Then K is compact relative Y if and only if K is compact relative X.*

Proof . The proof is the same as for Theorem 2.33, p. 37, in Rudins book.

Remark . a) If we let $K = Y$ in the above theorem, we see that K is a *compact topological space* when equipped with the topology inherited from X.

b) It is *not* true for general topological spaces that compact sets are closed. Let e g $X = \mathbb{R}$ with the topology $\mathcal{U} = \{]\alpha, \infty[\}_{\alpha \in \mathbb{R} \cup \{\pm\infty\}}$, i e the open sets are of the form $]\alpha, \infty[$, where $\alpha \in \mathbb{R} \cup \{\pm\infty\}$. If $K = \{x\}$, where $x \in \mathbb{R}$, then any open cover of K has a finite subcover (in fact it has a subcover consisting of *one* set). So, in this topology $K = \{x\}$ is compact, but it is not closed - the closure of K is $]-\infty, x]$ – prove this! ($\mathbb{R}$ with this topology is a so-called T_0-space – see the next definition below.)

Remark b) calls for further investigation. Under what conditions on the topology are compact sets closed? If we look at the proof of the statement that *compact subsets of metric spaces are closed* as presented in Rudins book (Theorem 2.34, p. 37), we see that it uses the fact that *any two different points can be separated by two disjoint open sets*, i e if x and y are different points in a metric space, then there are open subsets U and V of the metric space, such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. This motivates the following definition:

Definition 8. (Separation of points) *Let $(X, \mathcal{U})$ be a topological space.*

a) We say that $\mathcal{U}$ separates points *in X, if for each pair of different points x and y in X, there exists $U, V \in \mathcal{U}$, such that $U \cap V = \emptyset$ and $x \in U$, $y \in V$. If $\mathcal{U}$ separates points in X, then we say that $(X, \mathcal{U})$ a* Hausdorff space *(or* T_2-space*).*

b) If $\mathcal{U}$ is such that for any pair of different points x and y in X, there exists an open set $U \in \mathcal{U}$ containing x but not y, then we say that X is a T_1-space.

c) If $\mathcal{U}$ is such that for any pair of different points x and y in X, there exists one open set $U \in \mathcal{U}$ containing one of x and y, but not the other, then we say that X is a T_0-space.

Remark . a) Metric spaces are Hausdorff spaces, as is seen from the proof of theorem 2.34 in Rudins book.

b) $T_2 \subset T_1 \subset T_0$ as classes of topological spaces.

c) The difference between T_0 and T_1 is perhaps a bit difficult to see from the definition. For T_0 we only require that there is an open set containing *one* of the points x or y. For T_1 on the other hand, we require that there is one set containing x but not y *and* one set containing y but not x. (See also the next example below.)

Example 8. a) The space $X = \mathbb{R}$ with the topology $\{]\alpha, \infty[\}_{\alpha \in \mathbb{R} \cup \{\pm\infty\}}$ is T_0 but not T_1, since given x and y, with say $x < y$, there is an open set $]\alpha, \infty[$ with $x < \alpha < y$ containing y but not x, but there is no open set containing x but not y.

b) If we let $X = \mathbb{R}^n$ (or $\mathbb{C}^n$) and define a subbase for a topology to be sets of the form

$$\{x \in X : p(x) = 0\}^c$$

where p is a polynomial. I e the open sets are arbitrary unions of finite intersections of complements of zero-sets of polynomials. Then X is a T_1 space, but not Hausdorff. This topology is called *the Zariski-topology* on $\mathbb{R}^n$ (or $\mathbb{C}^n$) and is important in algebraic geometry, where one studies properties of polynomials among other things.

Back to the original question: When are compact sets closed? The next theorem gives a sufficient condition on the topology for this to be the case:

Theorem 5. *Compact subsets of Hausdorff spaces are closed.*

Proof . The proof follows that given for Theorem 2.34 in Rudins book: let $K \subset X$ be compact, where X is a Hausdorff space, and let $p \in K^c$. If $q \in K$, then there exists disjoint, open sets V_q and W_q containing p and q, respectively. Furthermore, $\{W_q\}_{q \in K}$ is an open cover of K, so there exists a finite subcover $\{W_{q_1}, \ldots, W_{q_n}\}$. If we let $V = \cap_{k=1}^n V_{q_k}$, then V is a neighbourhood of p that does not intersect $W = \cup_{k=1}^n W_{q_k}$, and hence p is an interior point of K^c, which proves that K^c is open.

The next two theorems are also valid in Hausdorff spaces:

Theorem 6. *Let $F \subset K \subset X$, where X is is a topological space, K is compact and F is closed in X. Then F is compact (i e closed subsets of compact sets are compact).*

Proof . The proof is the same as for Theorem 2.35, p. 37, in Rudins book.

Corollary 2. *If X is a Hausdorff topological space, K is compact in X and F is closed in X, then $F \cap K$ is compact.*

Proof . By Theorem 5 we know that K is closed, so $F \cap K$ is closed. Furthermore, $F \cap K \subset K$, so Theorem 6 shows that $F \cap K$ is compact.

Theorem 2.36, the corollary and Theorem 2.37, p. 38, in Rudins book are true also in Hausdorff spaces, with the same proofs as in Rudins book.
We end this discussion of compactness by giving some definitions and results that are not presented in Rudins book. The first definition and result will be needed when we study products of topological spaces:

Definition 9. (Finite intersection property) *Let $\{F_\alpha\}_{\alpha\in A}$ be a family of subsets of a topological space X. We say that this family has* the finite intersection property *if any finite subfamily $\{F_{\alpha_1}, \dots F_{\alpha_k}\}$ has non-empty intersection.*

An easy example:

Example 9. In $\mathbb{R}$ the family $F_n = [0, 1/n]$ has the finite intersection property: any finite subfamily $\{F_{n_k}\}_{k=1}^N$ has intersection $\cap_{k=1}^N F_{n_k} = [0, 1/n_N]$, if $n_1 < n_2 < \cdots < n_N$.

Theorem 7. *A topological space X is compact if and only if, for any family $\{F_\alpha\}_{\alpha\in A}$ of closed subsets of X having the finite intersection property, the total intersection $\cap_{\alpha\in A} F_\alpha$ is non-empty.*

Proof . Assume that X is compact and let $\{F_\alpha\}_{\alpha\in A}$ be a family of closed subsets of X having the finite intersection property. Suppose that $\cap_{\alpha\in A} F_\alpha$ is empty. The the complements F_α^c forms an open covering of X and therefore we can find a finite subcovering $X = \cup_{i=1}^k F_{\alpha_k}^c$. We conclude that $X^c = \cap_{i=1}^k F_{\alpha_k}^c$ is empty, which contradicts the fact that the original family has the finite intersection property. So, $\cap_{\alpha\in A} F_\alpha$ is non-empty.
Conversely, suppose that for each family $\{F_\alpha\}_{\alpha\in A}$ of closed subsets of X having the finite intersection property, $\cap_{\alpha\in A} F_\alpha$ is non-empty. Suppose that X is non-compact and let $\{U_\alpha\}_{\alpha\in A}$ be an open covering of X with no finite subcovering. Then the sets $F_\alpha = U_\alpha^c$, $\alpha \in A$, are closed and has the finite intersection property, so by assumption $\cap_{\alpha\in A} F_\alpha = \cap_{\alpha\in A} U_\alpha^c$ is non-empty, which contradicts the assumption that $\{U_\alpha\}_{\alpha\in A}$ is an open covering of X.

Example 10. In the above example we have $\cap_{n=1}^\infty F_n = \{0\}$.

A sometimes useful characterization of compactness (in metric spaces) is given by the following definition and theorem:

Definition 10. (Sequential compactness)
Let X be a topological space. We say that X is sequentially compact *(or that X has* the Bolzano-Weierstrass property*) if every sequence $\{x_n\}_{n=1}^{\infty}$ in X has a limit point in X.*

Theorem 8. *Let S be a subset of a metric space X. Then S is compact if and only if it is sequentially compact.*

For the proof of this we need two lemmas:

Lemma 1. *Let $\{U_i\}_{i \in I}$ be an open covering of a sequentially compact subset S in a metric space. Then there is an $r > 0$ such that for each $y \in S$, $D(y, r) = \{x \in X : d(y, x) < r\} \subset U_i$ for some i.*

Proof . If this is not true, we can find $y_n \in S$ such that $D(y_n, 1/n)$ is not contained in any U_i. But $\{y_n\}_{n=1}^{\infty}$ has a convergent subsequence, by assumption, say $y_{n_k} \to y \in S$, $k \to \infty$. Then $y \in U_{i_0}$ for some $i_0 \in I$. We can choose $\varepsilon > 0$ so that $D(y, \varepsilon) \subset U_{i_0}$, and if we choose N large enough we achieve that $1/N < \varepsilon/2$ and $d(y_{n_k}, y) < \varepsilon/2$, if $k \geq N$. This imlpies that $D(y_{n_k}, 1/N) \subset U_{i_0}$, if $k \geq N$. Inparticular, if we choose $n_k \geq N$, we get $D(y_{n_k}, 1/n_k) \subset D(y_{n_k}, 1/N) \subset U_{i_0}$, which contradicts our assumption that $D(y_n, 1/n)$ is not contained in any U_i. Hence, such an $r > 0$ exists.

Lemma 2. *Let S be a sequentially compact subset of a metric space X. Then, for each $\varepsilon > 0$, there is a finite subset $\{x_1, \ldots, x_n\}$ in S, such that $S \subset \cup_{j=1}^{n} D(x_j, \varepsilon)$.*

Proof . If this is not so, then there exists an $r > 0$ such that we can find $z_1 \in S$, $z_2 \in S \setminus D(z_1, r)$ and (repeatedly) $z_n \in S \setminus D(z_1, r) \cup \cdots \cup D(z_{n-1}, r)$. This sequence satisfies $d(z_m, z_n) \geq r$ if $n \neq m$, so it cannot have a convergent subsequence, which contradicts the assumption that S is sequentially compact.

Proof . (of Theorem 8.) First assume that S is compact (then we actually only need that X is a topological space) and let E be an infinite subset of S. It is enough to prove that E has a limit point in S. Suppose that this is not so. Then, given any $x \in S$ there is a neighbourhood U_x of x containing only finitely many elements from E. The family $\{U_x\}_{x \in S}$ covers S, so it has a finite subcovering $\{U_{x_1}, \ldots, U_{x_m}\}$. Since each of these sets only contains finitely many elements in E, E itself must be finite, which is a contradiction, hence S is sequentially compact.
Conversely, assume that S is sequentially compact and let $\{U_i\}_{i \in I}$ be an open cover of S. Then, by Lemma 1 there is an $r > 0$ such that for each $y \in S$,

$D(y,r)$ is contained in U_i for some $i \in I$. Furthermore (by Lemma 2) S can be covered by finitely many such balls $D(y_j, r)$, $y_j \in S$, $j = 1, \ldots, k$, Hence, $S \subset D(y_1, r) \cup \cdots \cup D(y_k, r)$ for some $y_1, \ldots, y_k \in S$ and each $D(y_j, r) \subset U_{i_j}$ for some $i_j \in I$, so we get a finite subcovering $\{U_{i_1}, \ldots, U_{i_k}\}$.

Pages 38– 42 (starting with Theorem 2.38) we leave as they are. It should be carefully noted here that theorem 2.41 on page 40 (the Heine–Borel theorem) is *not* valid in general metric spaces – a counterexample is in given exercise 16 on page 44.

2.3 Connectedness

Definition 2.45 we take as it is, also for topological spaces and just add one theorem, which is not in Rudins book (even for metric spaces).

Theorem 9. *Let X be a topological space and Y a connected subset of X. If Z is a subset of X such that $Y \subset Z \subset \overline{Y}$, then Z is connected. In particular, $\overline{Y}$ is connected.*

Proof . Suppose that Z is not connected, i e that there exists two separated sets A and B in X such that $Z \subset A \cup B$ and $A \cap Z, B \cap Z \neq \emptyset$. Since Y is connected and $Y \subset Z \subset A \cup B$, Y is contained in either A or B and disjoint from the other. Say that $Y \subset A$. This implies that $\overline{Y} \subset \overline{A}$ and therefore $\overline{Y} \cap B \subset \overline{A} \cap B = \emptyset$ which contradicts the assumption that Z is not connected.

2.4 Exercises

On pages 43-46 in Rudins book, look at the following exercises: ("for topological spaces" means that they should be solved assuming that the underlying space is just a topological space).
2-4, 6-7 (for topological spaces), 8, 9 (for topological spaces), 10, 12, 14, 16, 19 (where a) and b) should be proved for Hausdorff spaces), 30. Exercises 22-29 can also be done, if time so allows.
On pages 78-82 look at 21-24.
Some more exercises:

1. Given two sets A and B we defined $A \sim B$ if there is a one-to-one mapping $f : A \longrightarrow B$ which is onto (i e f is bijective).
   Show that $\sim$ is an equivalence relation on the family of sets.

2. Let $S = \{\{x_n\}_{n=1}^{\infty} : x_n \in \mathbb{N}\}$, i e S is the set of all sequences of integers. Prove that S is uncountable. (Here $\mathbb{N}$ denotes the set of

natural numbers, i e non-negative integers.)
(Hint: Try to find a mapping $f : [0,1] \longrightarrow S$ which is injective.)

3. Let M be a metric space with metric d. We call d an *ultrametric* on M if
$$d(x,z) \leq \max(d(x,y), d(y,z)) \text{ for all } x,\ y,\ z \in M.$$
   a) Let D be the set of all sequences $s = \{s_k\}_{k=0}^{\infty}$ such that s_k is equal to either 0 or 1, for all k. Define
$$d(s,t) = 2^{-n}, \text{ if } s_k = t_k \text{ for } 0 \leq k < n \text{ and } s_n \neq t_n.$$
   Prove that d is an ultrametric on D.
   b) Define a function $\varphi : D \longrightarrow \mathbb{R}$ via
$$\varphi(s) = \frac{2}{3}\sum_{k=0}^{\infty} 3^{-k} s_k.$$
   Prove that $|\varphi(s) - \varphi(t)| \leq d(s,t)$ for all $s,\ t \in D$.

4. The following questions should be answered by "Yes" or "No" and the answers should be motivated by proofs/counterexamples.
   (i) Let $S = \{\{x_n\}_{n=1}^{\infty} : x_n = 2 \text{ or } 3 \text{ for all } n \in \mathbb{N}\}$. Then S is uncountable.
   (ii) Let A and B be two countable, dense subsets of $\mathbb{R}$. Then $A \cap B$ is non-empty.
   (iii) If A_n, $n \in \mathbb{N}$, is a family of countable sets, define A to be the set of all finite arrays $a = (a_{n_1}, a_{n_2}, \ldots, a_{n_k})$ for some $k \in \mathbb{N}$, where $a_{n_i} \in A_{n_i}$, $1 \leq i \leq k$. Then A is countable.
   (iv) Let P denote the set of polynomials (of any degree) with integer coefficients. Then P is countable.

5. Give examples of a collection of closed sets in $\mathbb{R}^2$ whose union is open, and a collection of open sets whose intersection is closed.

6. Give an example of an infinite, countable set (in some topological space) which is
   (i) compact;
   (ii) closed but not compact;
   (iii) bounded but not compact.

7. Prove that the Zariski-topology on $\mathbb{R}^n$ (or $\mathbb{C}^n$) is T_1 but not Hausdorff.

3 Convergent sequences

In this section we study chapter 3 in Rudins book, p. 47–57, and we shall reformulate some of it in terms of topological spaces.

Definition 11. (Convergent sequences) *A sequence* $\{p_n\}_{n\in\mathbb{N}}$ *is said to* converge *if there is a point* $p \in X$*, and for each neighbourhood* V *of* p *there exists* $N \in \mathbb{N}$*, such that* $n \geq N \Rightarrow p_n \in V$*.*

Remark . As is noted in Rudins book the definition of convergence involves not only the sequence, but also the space X. For instance, an approximating sequence of $\sqrt{2}$ of rational number converges to $\sqrt{2}$ in $\mathbb{R}$, but *not* in $\mathbb{Q}$.

The notion of boundedness has no counterpart in general topological spaces, so the text in the first part of page 48 we leave as it is for metric spaces.

Theorem 3.2 is *not* valid as it stands in general topological spaces. Part a) is true (the proof is more or less the same as in Rudins book), part c) has no meaning (since we do not have the concept of boundedness) and parts b) and d) are sometimes false, as is seen from the following example:

Example 11. Let X be an infinite indiscrete space and let $\{p_n\}_{n\in\mathbb{N}}$ be any sequence in X. Given *any* point $p \in X$, the only neighbourhood of p is X itself, and so it contains the *whole* sequence $\{p_n\}_{n\in\mathbb{N}}$. By definition this means that $\{p_n\}_{n\in\mathbb{N}}$ converges to p. Hence, the limit *is not* unique. This disproves part b) of Theorem 3.2.
Part d) is more complicated to disprove – essentially it is not true in topological spaces where a one-point-set $\{p\}$ cannot be written as the intersection of countably many neighbourhoods $\{V_n\}_{n\in\mathbb{N}}$, i e where $\{p\} \neq \cap_{n\in\mathbb{N}} V_n$ for any such sequence of neighbourhoods.

3.1 Subsequences

The definition of subsequences is the same as in Rudins book (Definition 3.5). Theorem 3.6, part a), is *not* true in general topological spaces – it is true however in sequentially compact spaces, but then the result becomes trivial: Let X be a subsequentially compact space and $\{p_n\}_{n\in\mathbb{N}}$ a squence in X. By definition $\{p_n\}_{n\in\mathbb{N}}$ has a limit point in X, so there exists a convergent subsequence. Theorem 3.6 a) actually says that

a compact metric space is sequentially compact

which we proved in the above. Theorem 3.7 we leave as it stands for metric spaces.

4 Continuity

In this section we shall follow chapter 4 in Rudins book and reformulate some of the results for topological spaces. We shall also add a few results that are not given, even for metric spaces, in Rudins book. Chapter 3 in Vretblads kompendium is covered in chapter 4 in Rudins book, except for Sats 3.7 on page 22, saying that a continuous function on a compact interval is integrable – this result comes in chapter 6 in Rudins book.

4.1 Limits of functions

Definition 12. *Let X and Y be topological spaces and $E \subset X$. If f maps E into Y and if p is a limit point of E we say that f* converges to q at p*, if for each neighbourhood V of q, there exists a neighbourhood W of p, such that $f(x) \in V$ whenever $x \in W$, i e if $f(W) \subset V$.*

Theorem 4.2, p. 84, and its corollary *are not* valid in general topological spaces, and without going any deeper into that we just observe that they are true in metric spaces.

As a definition for continuous function between topological spaces we now take the description given in Theorem 4.8, p. 86:

Definition 13. *Let $f : X \longrightarrow Y$ be a function between two topological spaces X and Y. We say that f is* continuous *if $f^{-1}(V)$ is open in X whenever V is open in Y.*

By Theorem 4.8 this definition is equivalent with the one given in Definition 4.5, p. 85, for metric spaces (in particular with the 'old' definition for real valued functions of one variable). The corollary after Theorem 4.8 also holds for topological spaces with exactly the same proof.

Example 12.

a) If X is equipped with the indiscrete topology and $Y = \mathbb{R}$ with the usual topology, then the only continuous functions $f : X \longrightarrow Y$ are the constant ones, since the inverse image of an open set $V \subset Y$ under f has to be either X or $\emptyset$.

b) If X is equipped with the discrete topology and if Y is a topological space, then any function $f : X \longrightarrow Y$ is continuous, since all subsets of X are open.

Sensmoral: *The more open sets in X the easier for a function $f : X \longrightarrow Y$ to be continuous.*

Exercise 1. How does the 'number' of open sets in Y influence the 'difficulty of being continuous' for a function $f : X \longrightarrow Y$?

Definition 14. *Let $\mathcal{U}$ and $\mathcal{V}$ be two topologies on a set X. We say that $\mathcal{U}$ is* weaker *than $\mathcal{V}$ (or that $\mathcal{V}$ is* stronger *than $\mathcal{U}$) if $U \in \mathcal{U} \Rightarrow U \in \mathcal{V}$, i e if any subset of X that is open with respect to $\mathcal{U}$ also is open with respect to $\mathcal{V}$. We write $\mathcal{U} \subset \mathcal{V}$.*

Example 13. a) If $f \in C(X, Y)$ and if we choose a weaker topology on Y and/or a stronger topology on X, then f will still be continuous with respect to these new topologies.

b) Let Y be a topological space, X a set and $f : X \longrightarrow Y$ a function. Then we can define a subbase for a topology on X as follows:
'a subset A of X is open if $A = f^{-1}(V)$ for some open subset of Y.'
Obviously this topology on X makes f continuous and it is also the *weakest* topology on X such that f is continuous, in the sense that if we replace it by any (strictly) weaker topology, then f is no longer continuous.

c) More general we let X and Y be as before and $\mathcal{F}$ a family of functions from X to Y. Then we can define a topology on X by saying that an open subbase for the topology are sets of the form $f^{-1}(V)$, where $f \in \mathcal{F}$ and V is open in Y. This topology on X makes all elements in $\mathcal{F}$ continuous and it is called *the weak topology on X generated by $\mathcal{F}$.*

Next a result for continuous functions which is not presented in Rudin:

Theorem 10. *Let X be a topological space and Y a Hausdorff space. If $f, g : X \longrightarrow Y$ are continuous, then the set*

$$A = \{x \in X : f(x) = g(x)\}$$

is closed in X.

Proof . We shall prove that A^c is open. Let $x \in A^c$. Then $f(x) \neq g(x)$, and since Y is Hausdorff there are neighbourhoods U and V of $f(x)$ and $g(x)$ respectively, such that $U \cap V = \emptyset$. Since f and g are continuous the set $W = f^{-1}(U) \cap g^{-1}(V)$ is a neighbourhood of x and for each point $p \in W$ we obviously have that $f(p) \neq g(p)$, since $U \cap V = \emptyset$. Hence, $W \subset A^c$.

If we let g be a constant function we get the following corollary (note that constant functions are always continuous!):

Corollary 3. *If $f : X \longrightarrow Y$ is continuous, where Y is a Hausdorff space and X is a topological space, then $\{x \in X : f(x) = y\}$ is closed in X for each $y \in Y$. In particular, the zero set of any continuous function $f : X \longrightarrow \mathbb{R}$ is closed.*

4.2 Continuity and compactness

Theorem 4.14 (p. 89) is true for X a compact topological space and Y a topological spaces, with the same proof.
Theorems 4.15 and 4.16 (p. 89) holds (with the same proofs) for X a compact topological space.
Theorem 4.17 (p. 90) holds for X a compact topological space, with the same proof.
Definition 4.18 (p. 90) of uniform continuity and Theorem 4.19 we leave as they are – in general topological spaces they have no counterparts.
Theorem 4.20 (p. 91) and Example 4.21 (p. 93) deals with real-valued functions of one variable.
We go on with a few results that are not presented in Rudins book.
If X and Y are topological spaces, we denote by $C(X, Y)$ the set of all continuous functions from X to Y. In particular, $C(X, \mathbb{R})$ denotes the set of all real-valued continuous functions on X, where $\mathbb{R}$ is equipped with the usual topology. If $X = Y$ we shall write $C(X, X) = C(X)$.

Definition 15. (Separation of points) *We say that* $C(X, Y)$ separates points *on X, if for each pair x and y of different points in X, there exists $f \in C(X, Y)$ such that $f(x) \neq f(y)$.*

Theorem 11. *Let X be a topological space. If $C(X, \mathbb{R})$ separates points on X, then X is Hausdorff.*

Proof . Suppose that $C(X, \mathbb{R})$ separates points on X and let $x \neq y$ be two elements of X. Then there exists $f \in C(X, \mathbb{R})$ such that $f(x) \neq f(y)$, say $f(x) < f(y)$. Then we can find a real number r such that $f(x) < r < f(y)$, so that the sets

$$\{z \in X : f(z) < r\} = f^{-1}(]-\infty, r[) \text{ and } \{z \in X : f(z) > r\} = f^{-1}(]r, \infty[)$$

are disjoint open neighbourhoods of x and y, respectively.

Is the converse true? I e, does $C(X, \mathbb{R})$ separate points on X if X is Hausdorff? We shall prove that compact Hausdorff spaces have this property, by proving a slightly stronger result:

Theorem 12. (Urysohn's lemma) *Let X be a compact Hausdorff space and let A and B be two disjoint, closed subsets of X. Then there exists a continuous, real-valued function f defined on X, all of whose values lie in the closed interval $[0,1]$, such that $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$.*

Remark . Exercise 22, p. 101, in Rudins book leads to a different proof of this theorem for metric spaces.

As an immediate consequence we get:

Corollary 4. *Let X be a compact Hausdorff space and $x \neq y$ be two points in X. Then there exists an $f \in C(X, \mathbb{R})$ such that $0 = f(x) \neq f(y) = 1$, i e $C(X, \mathbb{R})$ separates points on X.*

Combining the last two theorems we get:

Corollary 5. *Let X be a compact topological space. Then $C(X, \mathbb{R})$ separates point on X if and only if X is Hausdorff.*

Before we prove Urysohn's lemma we need two more results:

Lemma 3. *Let X be a compact Hausdorff space and let A and B be two disjoint, closed subsets of X. Then there exists two disjoint open subsets U and V of X, such that $A \subset U$ and $B \subset V$.*

Proof . If A and/or B is empty, then there is nothing to prove (take U and/or $V = \emptyset$), so we assume that they are both non-empty. First we fix $p \in A$ and choose for each $q \in B$ disjoint neighbourhoods $U_{p,q}$ and $V_{p,q}$ of p and q, respectively. Now $\{V_{p,q}\}_{q \in B}$ is an open covering of B and since B is compact, being a closed subset of X, there exists a finite subcovering $\{V_{p,q_1}, \ldots V_{p,q_m}\}$. Then, if we let $V_p = \cup_{i=1}^m V_{p,q_i}$ and $U_p = \cap_{i=1}^m U_{p,q_i}$, we get disjoint neighbourhoods of B and p, respectively. Next we let p vary in A, and conclude that $\{U_p\}_{p \in A}$ is an open covering of A and therefore there is a finite subcovering $\{U_{p_1}, \ldots, U_{p_n}\}$. Finally, we let $U = \cup_{j=1}^n U_{p_j}$ and $V = \cap_{j=1}^n V_{p_j}$ and arrive with two disjoint neighbourhoods of A and B, respectively.

Lemma 4. *Let X be a compact Hausdorff space, A a closed subset of X and U a neighbourhood of A. Then there exists an open subset U_1 of X such that $A \subset U_1 \subset \overline{U}_1 \subset U$.*

Proof . Let B be the complement of U. Then there exists disjoint open sets U_1 and V_1 such that $A \subset U_1$ and $B \subset V_1$, by the previous lemma. It is clear that U_1 fulfills the requirements.

Proof . (of Urysohn's lemma). Let U and V be disjoint neighbourhoods of A and B, respectively. Let U_1 be the complement of B, so that $A \subset U_1$. By the last lemma we proved, we can find $U_{1/2}$ such that $A \subset U_{1/2} \subset \overline{U}_{1/2} \subset U_1$. Then we can find $U_{1/4}$ and $U_{3/4}$ such that

$$A \subset U_{1/4} \subset \overline{U}_{1/4} \subset U_{1/2} \subset \overline{U}_{1/2} \subset U_{3/4} \subset \overline{U}_{3/4} \subset U_1 = B^c.$$

Inductively, for each integer k, with $0 \leq k \leq 2^n$, we find $U_{k/2^n}$ such that

$$r < s \Rightarrow U_r \subset \overline{U}_r \subset U_s,$$

where r and s both are of the form $k/2^n$ for some integers k and n, $0 \leq k \leq 2^n$. Then we define our function f by

$$\begin{aligned} f(x) = &\quad 0, \text{ if } x \in U_t \text{ for each } t \text{ and} \\ f(x) = &\quad \sup\{t : x \notin U_t\}, \text{ otherwise.} \end{aligned}$$

Obviously the values of f all lie in $[0,1]$, $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$. It remains to prove that f is continuous. Let $a, b \in [0,1]$. Then

$$f(x) < a \text{ if and only if } x \in U_r \text{ for some } r < a,$$

by the definition of f, so that

$$f^{-1}([0,a[) = \cup_{r<a} U_r.$$

Similarly,

$$f(x) > b \text{ if and only if } x \notin \overline{U}_r, \text{ for some } r > b,$$

so that

$$f^{-1}(]b,1]) = \cup_{r>b} (\overline{U}_r)^c.$$

Hence, both $f^{-1}([0,a[)$ and $f^{-1}(]b,1])$ are open subsets of X for every $a, b \in [0,1]$, and since the collection of all sets $[0,a[$ and $]b,1]$, where $a, b \in]0,1[$, form an open subbase for the topology on $[0,1]$ (inherited from $\mathbb{R}$), we see that f is continuous.

4.3 Continuity and connectedness

Here Theorem 4.22 is valid, with the same proof, for X and Y topological spaces.

Page 93, starting with Theorem 4.23, to page 98 deals with real-valued functions of one variable and we leave it as it stands.

The next section contains results not presented in Rudins book.

4.4 Product spaces

Product spaces have been encountered in earlier courses, e g the spaces $\mathbb{R}^n$ for $n > 1$. The elements in these spaces are represented by arrays $(x_1, x_2, \ldots, x_n)$ of real numbers. Furthermore, any set of type

$$A_i = \{(x_1, x_2, \ldots, x_n) : \, |x_i| < r\}, \, 1 \le i \le n$$

where $0 < r \le \infty$, is an open subset of $\mathbb{R}^n$ with the usual topology. One can also prove that *any* open subset of $\mathbb{R}^n$ can be written as a union of finite intersections of such sets (this has been demonstrated during the lectures), i e that these sets form a subbase for the usual topology on $\mathbb{R}^n$.
We also note that if $\pi_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ are the coordinate mappings $\pi_i(x_1, x_2, \ldots x_n) = x_i$, $1 \le i \le n$, then

$$\pi_i^{-1}(V) = \{(x_1, x_2, \ldots, x_n) : x_i \in V\} = \mathbb{R}^{i-1} \times V \times \mathbb{R}^{n-i},$$

is open in $\mathbb{R}^n$ for each open subset V of $\mathbb{R}$, so that each π_i is continuous. Furthermore, the sets A_i, $1 \le i \le n$, above is equal to $\pi_i^{-1}(]-r, r[)$. Hence, the topology on $\mathbb{R}^n$ is identical to the weak topology generated by the coordinate mappings $\{\pi_i\}_{i=1}^n$.
Some more examples of product spaces:

Example 14. a) If $A_1, \ldots, A_n$ is any collection of (non-empty) sets we define their product $A = A_1 \times \cdots \times A_n = \times_{i=1}^n A_i$ as the set of all ordered n-tuples $(a_1, \ldots, a_n)$, such that $a_i \in A_i$, $1 \le i \le n$. If furthermore each A_i is a topological space we can equip A with the weak topology generated by the coordinate mappings $\pi_i : A \longrightarrow A_i$, $\pi_i(a_1, \ldots, a_n) = a_i$.

b) Let X be the set of all sequences of real numbers,

$$(x_1, x_2, \ldots) = (x_i)_{i \in \mathbb{N}}, \, x_i \in \mathbb{R}.$$

This set is a countable product of copies of the real line, and we write $X = \times_{i \in \mathbb{N}} \mathbb{R} = \mathbb{R}^{\mathbb{N}}$. We also have coordinate mappings in this case:

$$\pi_j : \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}, \text{ where } \pi_j((x_i)_{i=1}^{\infty}) = x_j, \, j \in \mathbb{N},$$

and we can define a topology on $\mathbb{R}^{\mathbb{N}}$ making all these mappings continuous (i e the weak topology generated by $\{\pi_i\}_{i=1}^{\infty}$).

c) If we let X be the set of all sequences of 0's and 1's, i e

$$X = \{(x_i)_{i=1}^{\infty} : x_i = 0 \text{ or } 1\},$$

we see that this set is a (countable) product of the set $\{0, 1\}$, i e $X = \{0, 1\}^{\mathbb{N}}$, with the above notation. The coordinate functions here are

$$\pi_j : \mathbb{N} \longrightarrow \{0, 1\},\ \pi_j((x_i)_{i \in \mathbb{N}}) = x_j,\ j \in \mathbb{N}$$

and with the topology $\mathcal{U} = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$ on $\{0, 1\}$, we can define a topology on X as before.

d) Now if A_i are any sets and I is any (index) set, then we can consider 'sequences' (or rather 'arrays') of the form $(a_i)_{i \in I}$, and with the above notation, we write this set of arrays as $\times_{i \in I} A_i$. We have coordinate functions in this case too - $\pi_j : A = \times_{i \in I} A_i \longrightarrow A_j$, $\pi_j((a_i)_{i \in I}) = a_j$, where $j \in I$, and if each A_j is a topological space we can define a topology on A as before.

Before we continue we note that a function f from a set I to a set A can be thought of as 'the array of all its values', namely $(f(i))_{i \in I}$, i e it can be identified with an element in the product space $\times_{i \in I} A = A^I$. Hence, the product $\times_{i \in I} A_i$ can be identified with the set of all functions f defined on I, such that $f(i) \in A_i$ for each $i \in I$.

Example 15. a) $A = A_1 \times \cdots \times A_n = \times_{i=1}^{n} A_i$ can be identified with the set of all functions a defined on $\{1, \ldots, n\}$ such that $a(i) \in A_i$, $1 \leq i \leq n$. In particular, $\mathbb{R}^n$ can be identified with the set of all functions $a : \{1, \ldots, n\} \longrightarrow \mathbb{R}$.

b) The set of all sequences of real numbers, $\mathbb{R}^{\mathbb{N}}$, can be identified with the set of all functions $x : \mathbb{N} \longrightarrow \mathbb{R}$.

c) The set of all sequences of 0's and 1's, $\{0, 1\}^{\mathbb{N}}$, can be identified with the set of all functions $f : \mathbb{N} \longrightarrow \{0, 1\}$.

Now we give the formal definition of a product of topological spaces and product topology:

Definition 16. (Product spaces)

a) Let $\{X_i\}_{i \in I}$ *be a family of sets. The* product set $X = \times_{i \in I} X_i$ *is the set of all arrays* $(x_i)_{i \in I}$ *such that* $x_i \in X_i$ *for each* i *(or, equivalently, the set of all functions* x *defined on* I*, such that* $x(i) \in X_i$, $i \in I$*). We call* X_i the i^{th} coordinate set.

b) *The* coordinate functions $\pi_j : X \longrightarrow X_j$*,* $j \in I$*, are the mappings* $\pi_j((x_i)_{i\in I}) = x_j$*.*

c) *If each* X_i *is a topological space, then the* product topology *on* $X = \times_{i\in I} X_i$ *is the weak topology generated by the coordinate mappings.*

Example 16. We repeat once more what we did in the beginning of this section. Let $I = \{1, \ldots, n\}$ and $X_i = \mathbb{R}$ with the usual topology, for each i, and let

$$X = \times_{i\in\{1,\ldots,n\}} X_i$$

according to the above definition. Then the open subbase for the topology on X as defined in c) above, is the one generated by the coordinate functions $\pi_i(x) = \pi_i((x(1), \ldots, x(n))) = x(i)$, $1 \le i \le n$, where $x = (x(1), \ldots, x(n))$. One proves rather easily that this topology is the same as the usual topology on $\mathbb{R}^n$ (this has been done during the lectures).

Remark . By viewing the product space $X = \times_{i\in I} X_i$ as the set of all functions x defined on I, such that $x(i) \in X_i$ for each $i \in I$, it is easier to draw 'pictures' of the product space - if we draw two 'coordinate axes' in the plane (as usual) and let the horisontal 'axes' represent I, then each vertical 'line' (through say i_0 on the horisontal 'axes') in this diagram may be thought of as the set X_{i_0}, and an element in the product space can be represented by the graph of a function f in this diagram. (Note here that 'axes' and 'line' can be e g discrete sets: if $X_i = \{0, 1\}$, for each i, and $I = \mathbb{N}$, then the horisontal 'axes' consists of $\mathbb{N}$ and the vertical of $\{0, 1\}$.) An open subset U in the subbase for the topology is achieved by fixing such a function f, finitely many values on i, say $i_1, \ldots, i_k \in I$ and open neighbourhoods $V_{i_j} \subset X_{i_j}$ of $f(i_j)$ for $j = 1, \ldots, k$, and then define

$$U = \{g : g(i_j) \in V_{i_j}, j = 1, \ldots, k\},$$

(or, equivalently, $U = \{(x_i)_{i\in I} : x_{i_j} \in V_{i_j}, j = 1, \ldots, k\}$, if we prefer viewing the product as a set of arrays).

We give an important result for product spaces without proof (the proof is rather long and it uses Zorn's lemma – in fact it is equivalent with Zorn's lemma and thus equivalent with the Axiom of Choice).

Theorem 13. (Tychonoff's theorem) *The product of any non-empty class of compact topological spaces is compact.*

Tychonoff's theorem can be used to prove the Heine–Borel theorem, which is proved in Rudins book (Theorem 2.41, p. 40) with slightly different methods:

Corollary 6. (The Heine–Borel theorem) *A subset of* $\mathbb{R}^n$ *is compact if and only if it is closed and bounded.*

Proof . Assume first that F is a closed and bounded subset of $\mathbb{R}^n$. Then it is contained in a product of n closed and bounded intervals $\{I_j\}_{j=1}^n$, and therefore it is a closed subset of the compact Hausdorff space $\times_{j=1}^n I_j$, and thus F is compact.
Conversely, if F is compact in $\mathbb{R}^n$, then it is closed. It also has to be bounded, since otherwise one could find a sequence going out to infinity, and not having any limit points.

Next we recall the definition of a norm on a vectorspace:

Definition 17. (Norm and normed spaces) *A function* $\|\cdot\| : X \longrightarrow \mathbb{R}$ *on a vectorspace* X *is called a* norm*, if*

I1) $\|tx\| = |t| \cdot \|x\|$ *for all* $x \in X$ *and all* $t \in \mathbb{R}$ *(or* $t \in \mathbb{C}$*, if* X *is a complex vectorspace),*

I2) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$ *(the triangle inequality),*

I3) $\|x\| \geq 0$ *for all* $x \in X$*, with equality if and only if* $x = 0$*.*

The vectorspace X *equipped with a norm is called a* normed space

Example 17. a) The usual length of a vector, $\|(x_1, \ldots, x_n)\| = (\sum_{i=1}^n |x_i|)^{1/2}$, is a norm on $\mathbb{R}^n$. Items *I1* and *I3* are trivial to prove and item *I2* is proved using the Cauchy–Schwarz inequality.

b) If we define $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we also get a norm on $\mathbb{R}^n$.

c) One can prove that, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $1 \leq p < \infty$,

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}$$

defines a norm on $\mathbb{R}^n$. For $p = 2$ this is the same as example a). Only item *I2* is non-trivial to prove. For these norms item *I2* is known as *Minkowski's inequality.*

d) If we let $C[a, b]$ denote the set of real (or complex) valued functions defined on the compact interval $[a, b] \subset \mathbb{R}$ and $1 \leq p < \infty$, then

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{1/p}, \ f \in C[a, b]$$

defines a norm on $C[a,b]$. This is also the case for

$$\|f\|_\infty = \max_{a\le t\le b} |f(t)|.$$

The perhaps most interesting cases here are $p = 2$ and ∞.

Definition 18. (Equivalent norms) *We say that two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on a normed space X are* equivalent *if there exists positive constants c and C such that $c\|x\|_a \le \|x\|_b \le C\|x\|_a$ for all $x \in X$.*

Remark . a) If two norms are equivalent, then they induce the same topology on X. To see this we let U be an open subset with respect to the topology induced by $\|\cdot\|_a$ and $x \in U$. Then there exists $\delta > 0$ such that the ball

$$N_a(x,\delta) = \{y \in X : \|x-y\|_a < \delta\} \subset U.$$

Now, $c\|x\|_a \le C\|x\|_b$, so $\|x-y\|_b < \delta/c \Rightarrow \|x-y\|_a < \delta$, i e $N_b(x,\delta/c) \subset U$, so U is open in the topology induced by $\|\cdot\|_b$.
The converse is obtained similarly, using the inequality $\|x\|_a \le C\|x\|_b$.

b) If we let $\|\cdot\|_a \sim \|\cdot\|_b$ denote that the two norms on X are equivalent, then $\sim$ is an equivalence relation on the set of norms on X, i e

(i) it is reflexive: $\|\cdot\|_a \sim \|\cdot\|_b$ for all norms (which is obvious),

(ii) it is symmetric: $\|\cdot\|_a \sim \|\cdot\|_b \Rightarrow \||\cdot\|_b \sim \|\cdot\|_a$ for all norms, since $c\|x\|_a \le \|x\|_b \le C\|x\|_a$ obviously implies that $\frac{1}{C}\|x\|_b \le \|x\|_a \le \frac{1}{c}\|x\|_b$,

(iii) it is transitive: $\|\cdot\|_a \sim \|\cdot\|_{b'}$ and $\|\cdot\|_{b'} \sim \|\cdot\|_b \Rightarrow \|\cdot\|_a \sim \|\cdot\|_b$, since $c\|x\|_a \le \|x\|_{b'} \le C\|x\|_a$ and $c'\|x\|_{b'} \le \|x\|_b \le C'\|x\|_{b'}$ implies that $cc'\|x\|_a \le \|x\|_b \le CC'\|x\|_a$.

The next result follows from Tychonoff's theorem:

Corollary 7. *All norms on $\mathbb{R}^n$ are equivalent, i e if $\|x\|_a$ and $\|x\|_b$ are two different norms on $\mathbb{R}^n$, then there are positive constants C and c (depending on a, b and n, but not on x) such that*

$$c\|x\|_a \le \|x\|_b \le C\|x\|_a$$

for all $x \in \mathbb{R}^n$.

Proof . Since equivalence between norms is an equivalence relation, it is enough to prove that any norm on $\mathbb{R}^n$ is equivalent to the max-norm $\|x\|_\infty = \max_{1\le i\le n}|x_i|$. Let $e_1,\dots,e_n$ be the unit vectors in $\mathbb{R}^n$ and write $x = x_1e_1 + \cdots + x_ne_n$. Then

$$\|x\|_a = \|x_1e_1 + \cdots + x_ne_n\|_a \le \text{ (triangle ineq.)}$$

$$\le |x_1|\|e_1\|_a + \cdots + |x_n|\|e_n\|_a \le (|x_1| + \dots |x_n|) \max_{1\le i\le n}\|e_i\|_a \le$$

$$\le C\|x\|_\infty$$

where $C = n\max_{1\le i\le n}\|e_i\|_a$. This proves one inequality, and also that $\|x\|_a$ is continuous as a function of x with respect to the norm $\|\cdot\|_\infty$, since

$$|\|x\|_a - \|y\|_a| \le \|x-y\|_a \le C\|x-y\|_\infty.$$

To prove the other inequality we let S denote the 'unit circle' centered at the origin for the max-norm, i e S is the boundary of the box

$$\{x\in\mathbb{R}^n : \max_{1\le i\le n}|x_i| \le 1\} = \times_{1\le i\le n}[-1,1]$$

Then S is closed and bounded, so it is compact by the Heine–Borel theorem, and therefore the a–norm obtains a minimum on S, say at x_0. I e $\|x\|_a \ge \|x_0\|_a$ for all $x\in S$. Now, if x is any element in $\mathbb{R}^n\setminus\{0\}$, then $x/\|x\|_\infty \in S$. We have that

$$\left\|\frac{x}{\|x\|_\infty}\right\|_a \ge \|x_0\|_a,$$

and therefore $\|x_0\|_a\|x\|_\infty \le \|x\|_a$, so by choosing $c = \|x_0\|_a$ we get the other inequality.

Remark . If we let X_i, $i = 1,\dots n$, be metric spaces with metric d_i on X_i, then we can define a metric d on the product space $X = \times_{i=1}^n X_i$ in many different ways (for $z\in X$ we write $z = (z_1,\dots,z_n), z_i \in X_i$):

(i) $d_\infty(x,y) = \max_{1\le i\le n} d_i(x_i,y_i)$,

(ii) $d_p(x,y) = (d_1(x_1,y_1)^p + \cdots + d_n(x_n,y_n)^p)^{1/p} = \left(\sum_{i=1}^n d_i(x_i,y_i)^p\right)^{1/p}$, $1\le p<\infty$.

Note the analogy with the different norms on $\mathbb{R}^n$. One can prove that these metrics on X are all equivalent, i e that there exists positive constants C and c depending on n,p and q, such that

$$cd_q(x,y) \le d_p(x,y) \le Cd_q(x,y)$$

for each $p, q \in [1, \infty]$. Therefore all these metrics define the same topology on the product space X and, since the topology induced by d_∞ obviously is identical with the product topology, they are all equivalent to the product topology. One also easily verifies that equivalence between metrics on a metric space X is an equivalence relation (in the same way as we did for norms).

We prove the statement made in the remark above:

Corollary 8. *The different metrics in the remark above are all equivalent.*

Proof . We see that $d_p(x, y) = \|(d_1(x_1, y_1), \ldots, d_n(x_n, y_n))\|_p$, $1 \leq p \leq \infty$, where we regard $(d_1(x_1, y_1), \ldots, d_n(x_n, y_n))$ as a vector in $\mathbb{R}^n$ and $\|\cdot\|_p$ is the corresponding norm on $\mathbb{R}^n$, i e $d_p(x, y)$ is the composition of the two mappings

$$X \times X \ni ((x_1, y_1), \ldots, (x_n, y_n)) \mapsto (d_1(x_1, y_1), \ldots, d_n(x_n, y_n)) \in \mathbb{R}^n$$

and

$$\mathbb{R}^n \ni (t_1, \ldots, t_n) \mapsto \|(t_1, \ldots, t_n)\|_p \in \mathbb{R}.$$

The result now follows from what we did for norms on $\mathbb{R}^n$.

We end this section with a discussion on linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$. Let F be such a mapping and let $e_1, e_2, \ldots, e_n$ be the unit vectors in $\mathbb{R}^n$. Then $F(x) = x_1 F(e_1) + \ldots x_n F(e_n)$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Using the triangle inequality we get

$$\|F(x)\|_\infty \leq |x_1|\|F(e_1)\|_\infty + \cdots + |x_n|\|F(e_n)\|_\infty \leq$$
$$\leq \|x\|_\infty| \max_{1 \leq i \leq n} \|F(e_i)\|_\infty = C\|x\|_\infty,$$

where $C = \max_{1 \leq i \leq n} \|F(e_i)\|_\infty$. Hence, since F is linear,

$$\|F(x) - F(y)\|_\infty = \|F(x - y)\|_\infty \leq C\|x - y\|_\infty$$

(where the norm in the right-hand-side is the max-norm in $\mathbb{R}^m$ and in the left-hand-side the max-norm in $\mathbb{R}^n$). Since all norms on $\mathbb{R}^n$ are equivalent, the same inequality holds (with a different constant C) for any other norms on $\mathbb{R}^n$ and $\mathbb{R}^m$. This proves that:

Corollary 9. *Linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$ are continuous (with respect to any choice of norms).*

Remark . It is seen from the proof that the image space $\mathbb{R}^m$ can be replaced by any normed linear space and the conclusion in the corollary holds. I e linear mappings $F : \mathbb{R}^n \longrightarrow Y$, where Y is a normed linear space, is continuous.

4.5 Exercises

On page 98–102 in Rudins book, look at: 2 (also for topological spaces), 3 (for topological spaces), 4, 5 (use the result of exercise 29, chapter 2), 6, 17, 18, 20, 21.

Some more exercises:

1. Prove that there is no continuous surjection $f : [0,1] \longrightarrow]0,1[$. Give also an example of a continuous surjection from a closed set to an open set.

2. Which of the following statements are true/false? (Proof or counterexample.)

   a) $\{(x,y) \in \mathbb{R}^2 : \sin(xy) < 1\}$ is open.

   b) $\{(x,y) \in \mathbb{R}^2 : \sin(xy) \leq 1\}$ is closed.

   c) $\{(u,v) \in \mathbb{R}^2 : \exists (x,y) \in \mathbb{R}^2,\, x^2 + y^2 \leq 1,\, u = x+y,\, v = \sin(xy)\}$ is closed.

3. Give an example of a continuous function $f : [0,\infty[\longrightarrow \mathbb{R}$ whose range is the open interval $]0,1[$.

4. Let $X = \mathbb{R}^2$ with the usual Euclidean metric and let M be the set of non-empty compact subsets of X. For each $K \in M$ we define a function φ_K via

   $$\varphi_K(p) = \inf\{d(p,q) : q \in K\} - d(p,0),\ p \in X.$$

   a) Show that φ_K is bounded on X for each $K \in M$.

   b) Define a function Δ on $M \times M$ via

   $$\Delta(K,L) = \sup_{p \in X} |\varphi_K(p) - \varphi_L(p)|,\ K, L \in M.$$

   Show that Δ is a metric on M.

   c) Let $M_{\mathbb{Q}}$ be the set of all finite subset of rational points in X. Show that $M_{\mathbb{Q}}$ is countable and dense in M (with the metric Δ).

5. Let X and Y be topological spaces and $f : X \longrightarrow Y$ a continuous mapping. Which of the following statements are true/false? If the answer is *true* a proof should be given, if the answer is *false* a counterexample:

a) If $E \subset X$ is open, then $f(E)$ is an open subset of Y.

b) If $F \subset X$ is closed, then $f(F)$ is a closed subset of Y.

c) If $K \subset X$ is compact, then $f(K)$ is a compact subset of Y.

d) If $K \subset Y$ is compact, then $f^{-1}(K)$ is a compact subset of X.

6. Let X be a topological space, Y a Hausdorff space and $f : X \to Y$ a continuous mapping. We say that f is *proper* if $f^{-1}(K)$ is compact in X, for each compact $K \subset Y$.

   a) Prove that if X is compact, then f is proper.

   b) Prove that if $X = Y = \mathbb{R}$, then f is proper if and only if

   $$\lim_{n\to\infty} |x_n| = \infty \;\Rightarrow\; \lim_{n\to\infty} |f(x_n)| = \infty.$$

   c) Give an example of a continuous bijection which is not proper.

7. Describe the open subsets of $\mathbb{R}^{\mathbb{N}}$ in the product topology.

8. Draw the 'unit circles' $\{(x_1, x_2) \in \mathbb{R}^2 : \|(x_1, x_2)\|_p = 1\}$, $1 \leq p \leq \infty$, for some values of p (in particluar for $p = 1, 2$ and ∞).

5 Families of functions

In chapter 7 Rudin only considers real (or complex) valued functions. We shall study functions with values in a metric space whenever possible.

5.1 Introduction

In many problems in mathematics and its applications one often encounters the problem of interchanging limits. For instance, if $f_n(x) \to f(x)$, $n \to \infty$, for $x \in I$ is it then true that

$$\lim_{n\to\infty} \int_I f_n(x)dx = \int_I f(x)dx \text{ and/or } \lim_{n\to\infty} f_n'(x) = f'(x)?$$

If we let

$$f_n(x) = \begin{cases} 0 & , |x| \geq 1/n \\ n(1-n|x|) & , |x| < 1/n \end{cases}$$

(draw the graph!) then we obviously have that $\int_{-1}^{1} f_n(x)dx = 1$ for all n, but if $f(x) = \lim_{n\to\infty} f_n(x)$, then $f(x) = 0$ for $x \neq 0$, so

$$\lim_{n\to\infty} \int_{-1}^{1} f_n(x)dx \neq \int_{-1}^{1} f(x)dx = 0.$$

Other examples of this type are e g Examples 7.5 and 7.6 on page 146 in Rudins book. Note that both integration and differentiation are defined by limits (integration as limits of Riemann sums).
The concept that allows us to interchange limits, as presented above, is *uniform convergence*. Items 7.7 – 7.10 on pages 147 – 148 in Rudins book deals with arbitrary sequences of functions (not necessarily continuous functions), whereas items 7.11 – 7.15 deals with uniformly convergent sequences of continuous functions. The important results here are that

a) If $f_n \to f$ uniformly, and if each f_n is continuous, then f is continuous.

b) If Y is complete, then the space $C_b(X,Y)$ (of bounded continuous functions), with the metric

$$d(f,g) = \sup_{x\in X} d_Y(f(x), g(x)),$$

is complete (note that if X is compact, e g if $X = [0,1]$, then $C_b(X,Y) = C(X,Y)$).

Remark . Completeness of Y guarantees that $\{f_n(x)\}_{n=1}^{\infty}$ is convergent in Y, whenever $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence.

Only real- (complex-) valued functions of one real variable are considered. We interpret the notation $\mathcal{R}(\alpha)$ as the set of (Riemann-) integrable functions and $d\alpha = dx$. Theorem 7.16 answers the problem raised at the beginning about interchanging limits of sequences of functions and integration. For the Riemann integral the proof of this theorem goes as follows:
Let $f_n \in C[a,b]$ and suppose that $f_n \to f$ uniformly on $[a,b]$. Let $\varepsilon > 0$ be given and choose N so that

$$n \geq N \Rightarrow \|f_n - f\| = \max_{a \leq x \leq b} |f_n(x) - f(x)| < \frac{\varepsilon}{b-a}.$$

Then

$$\left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| \leq \int_a^b |f_n(x) - f(x)|dx \leq \varepsilon$$

This proves that $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx$.

Remark . Note that this says that the mapping $\int_a^b : C[a,b] \longrightarrow \mathbb{R}$ is continuous.

The results of Theorems 7.17 and 7.18 (p. 153 – 154) are important.

5.2 Equicontinuity

Recall that a function $f : E \longrightarrow \mathbb{R}$ is *continuous*, if

$$\forall \varepsilon > 0,\ \underline{\forall x \in E},\ \underline{\exists \delta > 0},\ \forall y \in E :\ d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon,$$

If we change places between the two quantifiers underlined above we get

$$\forall \varepsilon > 0,\ \underline{\exists \delta > 0},\ \underline{\forall x \in E}, \forall y \in E :\ d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon,$$

which is the definition for f being *uniformly continuous.* In the first definition we (may) have to change δ if we choose a different ε and/or x, i e δ depends on both ε and x, whereas in the second definition the same δ suffices for all x once ε is chosen, i e δ depends only on ε. It is clear also that the choice of δ in both cases depends on the function f in question. I e in the first case $\varepsilon = \varepsilon(\delta, x, f)$ and in the second case $\varepsilon = \varepsilon(\delta, f)$. This becomes even clearer if we in both definitions above write

$$\forall f \in C(E,\mathbb{R}),\ \forall \varepsilon > 0,\ \underline{\forall x \in E},\ \underline{\exists \delta > 0},\ \forall y \in E :$$

$$d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon,$$

and

$$\forall f \in C(E,\mathbb{R}),\ f \text{ unif.cont},\ \forall \varepsilon > 0,\ \underline{\exists \delta > 0},\ \underline{\forall x \in E}, \forall y \in E:$$

$$d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon,$$

Next we make a definition so that ε becomes independent of f (within some set of functions):

Definition 19. (Equicontinuity) *A family $\mathcal{F}$ of functions from E to Y, where $E \subset X$ and X and Y are metric spaces, is said to be* equicontinuous *(on E) if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall f \in \mathcal{F},\ \forall x, y \in E:\ d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon.$$

Remark . Note that the position of the quantifiers implies that δ only depends on ε (and $\mathcal{F}$). Theorems 7.23 – 7.25 can be formulated for functions with values in a complete metric space Y. (In Theorem 7.24 completeness of Y is not needed.)

Definition 20. (Pointwise compact) *We say that $\mathcal{F} \subset C(E,Y)$ is* pointwise compact*, if $\{f(x) :\ f \in \mathcal{F}\}$ is compact in Y for each fixed $x \in E$.*

The following result is not explicitly mentioned in Rudins book:

Theorem 14. (Arzela–Ascolis theorem) *Let X and Y be metric spaces and suppose that $K \subset X$ is compact. Then $\mathcal{F} \subset C(K,Y)$ is compact in $C(K,Y)$ if and only if $\mathcal{F}$ is closed, equicontinuous and pointwise compact.*

Lemma 5. *Let $K \subset X$ be compact, X a metric space. Then, for any $\delta > 0$, there is a finite set $C_\delta = \{y_1, \dots, y_k\} \subset X$ such that $K \subset \cup_{i=1}^k N(y_i, \delta)$.*

Proof . The family of all balls $\{N(y,\delta)\}_{y \in K}$ covers K, so there is a finite subcover, say $\{N(y_i,\delta)\}_{i=1}^k$.

Proof . (Proof of Arzela–Ascolis theorem.)
$\Leftarrow$): Suppose that $\mathcal{F}$ is closed, equicontinuous and pointwise compact. It is enough to prove that $\mathcal{F}$ is sequentially compact, since $C(K,Y)$ is a metric space. I e that given a sequence $\{f_j\}_{j=1}^\infty$ in $\mathcal{F}$, there exists a convergent subsequence.
Choose a sequence $\{f_j\}_{j=1}^\infty$ in $\mathcal{F}$ and let $C = \cup_{n=1}^\infty C_{1/n}$, where each $C_{1/n}$ satisfies the conclusion of the lemma. Each $C_{1/n}$ is finite, so C is countable, say equal to $\{x_k\}_{k=1}^\infty$. Since $\mathcal{F}$ is pointwise compact, there is a subsequence of $\{f_n(x_1)\}_{n=1}^\infty$ which is convergent, say $\{f_{1n}(x_1)\}_{n=1}^\infty$. Similarly

$\{f_{1n}(x_2)\}_{n=1}^{\infty}$ contains a convergent subsequence, $\{f_{2n}(x_2)\}_{n=1}^{\infty}$, etc. We get sequences $\{f_{kn}\}_{n=1}^{\infty}$, which is are subsequences of $\{f_{jn}\}_{n=1}^{\infty}$ for $j < k$, such that $\{f_{kn}(x_j)\}_{n=1}^{\infty}$ converges for $j \leq k$. Now look at the diagonal sequence $g_n = f_{nn}$. We get that $\{g_n(x_j)\}_{n=1}^{\infty}$ converges for each j, since it is a subsequence of the convergent sequence $\{f_{kn}(x_j)\}_{n=1}^{\infty}$, $j \leq k$.
Next let $\varepsilon > 0$ and $\delta > 0$ be as in the definition of equicontinuity, i e such that $\forall x, y \in K : d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$, if $f \in \mathcal{F}$. and let $\cup_{i=1}^{k} N(y_i, \delta)$ be a cover of K, as in the lemma. We can assume that $y_i = x_i$ for $1 \leq i \leq k$. Since the sequence $\{g_n(y_i)\}_{n=1}^{\infty}$ converge for each i, $1 \leq i \leq k$, there exists N such that

$$n, m \geq N \Rightarrow d_Y(g_n(y_i), g_m(y_i)) < \varepsilon, \; 1 \leq i \leq k.$$

Furthermore, if $x \in K$ then there is an i, $1 \leq i \leq k$, such that $d_X(x, y_i) < \delta$. Hence, the equicontinuity implies that $d_Y(g_n(x), g_n(y_i)) < \varepsilon$ for all n. Therefore

$$d_Y(g_n(x), g_m(x)) \leq$$
$$\leq d_Y(g_n(x), g_n(y_j)) + d_Y(g_n(y_j), g_m(y_j))) + d_Y(g_m(y_j), g_m(x)) < 3\varepsilon,$$

so $\sup_{x \in K} d_Y(g_n(x), g_m(x)) \leq 3\varepsilon$. The uniform convergence of the sequence $\{g_n\}_{n=1}^{\infty}$ follows from the Cauchy criterion. The limit will belong to $\mathcal{F}$, since $\mathcal{F}$ is closed by assumption. Hence, $\mathcal{F}$ is sequentially compact.
$\Rightarrow$): Suppose that $\mathcal{F}$ is compact. Then obviously $\mathcal{F}$ is closed.
Furthermore, since the mapping $f \mapsto f(x)$ is a continuous mapping from $C(K,Y)$ to Y for each fixed $x \in K$, the image of $\mathcal{F}$ under this mapping is compact in Y, so $\mathcal{F}$ is pointwise compact.
Let $\varepsilon > 0$ be given. Since $\mathcal{F}$ is compact, and since the sets $N(f, \varepsilon) = \{g \in C(K,Y) : d(f,g) < \varepsilon\}$, $f \in \mathcal{F}$, covers $\mathcal{F}$, there is a finite subcover, say $N(f_i, \varepsilon)$, $1 \leq i \leq k$. I e if $g \in \mathcal{F}$, then there exists i, $1 \leq i \leq k$, such that $d(g, f_i) < \varepsilon$. Now, each f_i is uniformly continuous on K, since K is compact, so there is a $\delta_i > 0$ such that

$$d_X(x,y) < 2\delta_i \Rightarrow d_Y(f_i(x), f_i(y)) < \varepsilon, \; x, y \in K.$$

Let $\delta = \min\{\delta_i : 1 \leq i \leq k\}$ and let $N(x_j, \delta)$, $1 \leq j \leq m$, be a cover of K, according to the lemma. Then, for $f \in \mathcal{F}$, we get

$$d_Y(f(x), f(y)) \leq$$
$$\leq d_Y(f(x), g_i(x)) + d_Y(g_i(x), g_i(x_j)) + d_Y(g_i(x_j), g_i(y)) + d_Y(g_i(y), f(y)).$$

Now, to start with we choose i so that the first and the last term in the right-hand-side is $< \varepsilon$. Then we choose x and y in K so that $d_X(x,y) < \delta/2$ and j

so that $d_X(x, x_j) < \delta$, which implies that $d_X(y, x_j) \leq d_X(y, x) + d_X(x, x_j) < 3\delta/2 < 2\delta$, and hence the two middle terms are also $< \varepsilon$. This proves that $\mathcal{F}$ is equicontinuous.

Remark . The assumption that assumption that $\mathcal{F}$ is pointwise compact in Arzela–Ascolis theorem implies that $\{f_n(x)\}_{n=1}^{\infty}$ is convergent in Y whenever $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence and $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$, even in the case when Y is not complete.

5.3 Weierstrass' theorem

Theorem 7.26 is Weierstrass contribution to the Stone–Weierstrass' theorem. The theorem shows that the set of all (complex-valued) polynomials is dense in $C[a, b]$, where $[a, b] \subset \mathbb{R}$ is a closed and bounded interval. Since each polynomial can be uniformly approximated by polynomials with rational coefficients, and since the set of all polynomials with rational coefficients is countable, this proves that $C[a, b]$ is separable.

5.4 Exercises

Rudin p. 165–171: 1, 4, 7, 8, 11, 15, 16, 17, 20, 24.

1. a) If $p \in X$ is fixed, then

$$U = \{f \in C(X) : f(p) = 0\}$$

is a closed subset of $C(X)$.

b) If $X = [a, b] \subset \mathbb{R}$, then the mapping $T : C[a, b] \longrightarrow \mathbb{R}$ defined by

$$T(f) = \int_a^b f(x)dx$$

is continuous.

c) If $X = [a, b] \subset \mathbb{R}$, then

$$V = \{f \in C[a, b] : \int_a^b f(x)dx > 0\}$$

is open in $C[a, b]$. (4)

(You may use the result of b), even if you have not proved it.)

2. Suppose that $f \in C[0,2]$ and let $f_n(x) = f(x + 1/n)$ for $x \in [0,1]$ and $n \in \mathbb{Z}_+$. Calculate the pointwise limit of the sequence $\{f_n\}_{n=1}^{\infty}$ for each $x \in [0,1]$. Is the convergence uniform on $[0,1]$? Is $\{f_n\}_{n=1}^{\infty}$ equicontinuous?

6. Let $X = C[0,1]$ with metric

$$d(f,g) = \int_0^1 |f(t) - g(t)|dt$$

and let $\mathcal{P}$ denote the set of all real-valued polynomials with rational coefficients. Prove that $\mathcal{P}$ is a dense subset of X.
(*Hint:* You can use Weierstrass' theorem, but note the metric here!)

3. a) Let K be a compact metric space and let $C(K)$ denote the metric space of all realvalued, continuous functions defined on K, with metric

$$d(f,g) = \sup_{x \in K} |f(x) - g(x)|.$$

Give the definition of an equicontinuous subset of $C(K)$.

b) Let $\mathcal{F} = \{x^n\}_{n=1}^{\infty}$. Is $\mathcal{F}$ an equicontinuous subset of $C[0,1]$?

c) Let $\mathcal{F}$ be as in b) above. Is $\mathcal{F}$ an equicontinuous subset of $C[0, 1-\delta]$, $\delta > 0$?

4. Let $\{f_n\}_{n=1}^{\infty}$ be an equicontinuous sequence of real-valued functions defined on a compact metric space X, such that $f_n(x) \to f(x)$ pointwise as $n \to \infty$ for all $x \in X$. Prove that $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded on X.

5. Let $X = L(\mathbb{R}^n, \mathbb{R}^n)$ be the set of all linear operators on $\mathbb{R}^n$. On X we define a norm

$$\|B\| = \max_{|x|=1} |Bx|, \; B \in X,$$

where $|x|$ is the usual norm on $\mathbb{R}^n$. Show that $\mathcal{A} = \{B \in X : \|B\| \leq M\}$, where $M \in \mathbb{R}$, is an equicontinuous subset of X.

6. A function $f : [a,b] \to \mathbb{R}$ is said to satisfy a Lipschitz condition of order $\alpha > 0$ if there is a real number M such that

$$|f(x) - f(y)| \leq M|x - y|^{\alpha}, \; \forall x, y \in [a,b].$$

We denote this by $f \in \text{Lip}^{\alpha}[a,b]$. Obviously $\text{Lip}^{\alpha}[a,b] \subset C[a,b]$, for all $\alpha > 0$.

a) Prove that if $\alpha > 1$ and $f \in \mathrm{Lip}^{\alpha}[a, b]$, then f is constant.

b) Let M be a fixed real number and prove that the set

$$\{f \in \mathrm{Lip}^{\alpha}[a, b] : |f(x) - f(y)| \leq M|x - y|^{\alpha}, \forall x, y \in [a, b]\}$$

, $\alpha > 0$, is equicontinuous.

7. Let X and Y be metric spaces and let $K \subset X$ be compact. Suppose that $\mathcal{F} \subset C(K, Y)$ is compact.

   a) Prove that the mapping $H : C(K, Y) \times K \longrightarrow Y$ defined by $H(f, x) = f(x)$ is continuous.

   b) Use uniform continuity of H restricted to $\mathcal{F} \times K$ to prove that $\mathcal{F}$ is equicontinuous.

   (This is an alternative proof of the last part of Arzela–Ascolis theorem.)

References:

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Simmons, George F: *Topology and modern analysis*, McGraw-Hill Co, Inc.
Royden, H. L: *Real analysis*, The Macmillan Co.

The last two references exists in newer editions and all three of them should be available in the Beurling library (anyone interested could take an almost arbitrary book from the section on topology in the library and read more). These references also serve as references to the part on Basic topology that we did before.