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Topological spaces

by

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1 Open sets - why do we need them?

These notes are intended to give the reader, familiar with calculus in one and several variables, some motivation and background to the study of topological spaces, mainly via intuition and examples.

The question 'Why do we need open sets?' will be answered by looking at two familiar concepts from calculus – convergence and continuity. We start by recalling the definition of these two concepts:

Definition 1. (Convergence)
A sequence $\{x_n\}$ of real numbers has a limit as n tends to infinity, if there exists a real number y, and for all $\varepsilon > 0$ there exists a number N, such that if $n \geq N$ then $|x_n - y| < \varepsilon$, or (in a more mathematical language)

$$\exists y \in \mathbb{R}, \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq N \Rightarrow |x_n - y| < \varepsilon.$$

We call y the limit of the sequence and we write $y = \lim_{n\to\infty} x_n$.

Definition 2. (Continuity)
A function $f : A \longrightarrow \mathbb{R}$ is said to be continuous at $a \in A$, if for each $\varepsilon > 0$, there exists a $\delta > 0$, such that if $|x-a| < \delta$ and $x \in A$, then $|f(x)-f(a)| < \varepsilon$, or

$$\forall \varepsilon > 0, \exists \delta > 0 : |x - a| < \delta \text{ and } x \in A \Rightarrow |f(x) - f(a)| < \varepsilon.$$

We then say that f is continuous on A, if it is continuous at each point in A.

1.1 Convergence

Of course one could imagine sequences of elements in any other set X than $\mathbb{R}$, and the question then immediatly arise – how to define convergence of such a sequence? Looking at the definition for sequences of real numbers, we have an intuitive picture, which is formulated rigorously in the definition above, and this picture says that $\{x_n\}$ converges to y if x_n comes 'close' to y as n increases, i e that the distance $|x_n - y|$ can be made arbitrarily small by choosing n large enough. So then the question arise – how to introduce a concept of one point being 'close' to another in an arbitrary set X? Let us choose a set X and formulate the problem for that set:
Let $X = C[0, 1]$ denote the set of all real-valued continuous functions on the interval $[0, 1]$ and let $\{f_n\} \subset X$ be an infinite sequence. How to define that $\{f_n\}$ converges to $g \in X$ as n tends to infinity ?

Actually, there are more than one answer to this question, but we shall find one, which in many senses is the most useful answer. In the definition above we used the fact that we can measure distances between points on the real line. Can we mimic this by finding some way of measuring distances between two continuous functions on $[0, 1]$? Let f, $g \in X$ be two fixed functions and imagine their graphs drawn in a diagram. One way of measuring how much they differ from each other is of course to measure the vertical distance between the graphs at each point, i e to calculate $|f(t) - g(t)|$ for each $t \in [0, 1]$, but of course this number will depend on which t we choose (we could call this the *pointwise distance* between f and g). If we do not want this we can instead look at the maximum of all such distances, as t varies between 0 and 1:

$$\max_{t\in[0,1]} |f(t) - g(t)| = d(f, g),$$

where we write $d(f, g)$ for this maximum. Saying that $d(f, g) < \varepsilon$ now means geometrically that the graph of f (or g) lies in a strip of width 2ε centered (like a worm) around the graph of g (or f, respectively). Furthermore, this distance function shares some properties with the distance function on $\mathbb{R}$. For both of them we have (we let the letter d stand for both of them, and the letter X can be either $\mathbb{R}$ or $C[0, 1]$):

M1) $d(p, q) = d(q, p)$, $\forall p, q \in X$ (reflexivity)

M2) $d(p, q) \le d(p, r) + d(r, q)$, $\forall p, q, r \in X$ (triangle inequality)

M3) $d(p, q) \ge 0$ with equality if and only if $p = q$.

In the case $X = \mathbb{R}$ this is well-known, and in the case $X = C[0,1]$ properties M1) and M3) can be checked by inspection and M2) is neither hard to convience oneself of (do that!).

Using this distance function on $X = C[0,1]$, the frase '$\{f_n\}$ converges to $g \in X$' then means that the graph of f_n (at each point simultanously) can be made arbitrarily close to the graph of the function g by choosing n large enough.

We now make a more general definition from these observations:

Definition 3. (Metric spaces)
Let X be a set and assume that there exists a mapping $d : X \times X \longrightarrow \mathbb{R}$ satisfying M1)–M3) above. We then call d a metric on X and (together with this metric, or 'rule to measure distances') X is called a metric space. If we want to emphasize the metric in question we write (X, d) instead of just X.

There are many other examples of metric spaces, and we list a few of them:

(I) $X = \mathbb{R}^n$ with various metrics (we list three of them):

a) $d(p,q) = \left(\sum_{k=1}^{n} |p_k - q_k|^2\right)^{1/2}$ (the ususal euclidean metric – to prove item M2) in the definition of a metric one uses Cauchy–Schwarz' inequality),

b) $d(p,q) = \sum_{k=1}^{n} |p_k - q_k|$,

c) $d(p,q) = \max_{1 \le k \le n} |p_k - q_k|$,

where $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$.

(II) $X = \{\{x_k\}_{k=1}^{\infty} : x_k \in \mathbb{R}, \forall k \text{ and } \sum |x_k| \text{ converges}\}$, i e the set of all infinite sequences of real numbers such that the sum of the numbers in the sequence converges absolutely. As a metric on this set we can choose

$$d(p,q) = \sum_{k=1}^{\infty} |x_k - y_k|, \text{ where } p = \{x_k\}_{k=1}^{\infty} \text{ and } q = \{y_k\}_{k=1}^{\infty}.$$

It is straightforward to verify that this is a metric.

(III) We modify (II) a bit:
$X = \{\{x_k\}_{k=1}^{\infty} : x_k \in \mathbb{R}, \forall k \text{ and } \sum |x_k|^2 \text{ converges}\}$. As a metric we choose

$$d(p,q) = \left(\sum_{k=1}^{\infty} |x_k - y_k|^2\right)^{1/2}, \text{ where } p = \{x_k\} \text{ and } q = \{y_k\}.$$

To verify that this is a metric – use Cauchy–Schwartz' inequlity for item M2) in the definition of a metric.

(IV) We modify (II) again:
$X = \{\{x_k\}_{k=1}^{\infty} : x_k \in \mathbb{R}, \forall k \text{ and } \lim_{k\to\infty} x_k = 0\}$. As metric we choose in this case

$$d(p,q) = \max_{k\in\mathbb{N}} |x_k - y_k|$$

and it is immediate to verify that this is a metric.

Exercise 1. Draw the sets $\{p \in \mathbb{R}^2 : d(p,0) < 1\}$ for the various metrics d in (I) above.

Once we have a metric space, we can make the following definition for convergence:

Definition 4. (Convergence in metric spaces)
Let X be a metric space with metric d and let $\{p_n\}$ be an infinite sequence in X. We say that $\{p_n\}$ converges if

$$\exists q \in X, \forall \varepsilon > 0, \exists N \in \mathbb{N} : n \geq N \Rightarrow d(p_n, q) < \varepsilon.$$

We name q the limit of the sequence and denote this by $\lim_{n\to\infty} p_n = q$.

Now, can we pull this generalization even further? Answering this we come to the question 'why do we need open sets?'.
First we make some observations for metric spaces. Let X be a metric space with metric d and let $p \in X$. Denote by $I_p(r)$ the set $\{q \in X : d(p,q) < r\}$ (i e the set of all q in X at distance less than r to p; in the case $X = \mathbb{R}^n$ with the usual euclidean metric this set is nothing but the open ball of radius r centered at p). We call $I_p(r)$ *the open ball of radius r centered at p.* If we now choose a point $x \in I_p(r)$ and let $\delta > 0$ be less than $r - d(x,p)$, we get that $I_x(\delta) \subset I_p(r)$, since if we pick a point $y \in I_x(\delta)$, then

$$d(y,p) \leq d(y,x) + d(x,p) < \delta + d(x,p) < r - d(x,p) + d(x,p) = r.$$

Hence, if $x \in I_p(r)$, then there is an open ball centered at x which is contained in $I_p(r)$. To put it more intuitively – any point in $I_p(r)$ is surrounded by nothing but points in $I_p(r)$. This observation leads us to the following definition:

Definition 5. (Open subsets of metric spaces)
Let X be a metric space with metric d. Let U be a subset of X and let $p \in U$. We say that p is an interior point of U, if there is an open ball $I_p(\delta)$ centered at p of radius $\delta > 0$ which is contained in U. The set U is called open, if all points of U are interior points.
If $p \in X$ and if U is an open subset of X containing p, then we call U a neighbourhood of p.

(Note that all points in an open ball are interior points, from the discussion preceeding the definition.) We can now refrase our definition of convergence as follows:

Definition 6. (Convergence in terms of open sets)
Let X be a metric space with metric d and let $\{x_n\}$ be an infinite sequence in X. We say that $\{x_n\}$ is convergent if

$$\exists y \in X, \forall U\ (U\, a\ neighbourhood\ of\ y), \exists N \in \mathbb{N} : n \geq N \Rightarrow x_n \in U.$$

So, the thing really needed to define convergence is a concept of open sets. Let us investigate what additional properties these sets have. Let now $\mathcal{U}$ denote the collection of all open subsets of a metric space X. Then we have ($\emptyset$ denotes the empty set):

T1) $X \in \mathcal{U}$ and $\emptyset \in U$, since each point in X is obviously the center of an open ball contained in X, and since $\emptyset$ contains no points, 'each' point in $\emptyset$ is also the center of an open ball contained in X.

T2) If $U_1, \ldots, U_k$ are elements of $\mathcal{U}$, and if $p \in \cap_{j=1}^k U_j$, then p is the center of open balls $I_p(r_j)$ contained in U_j, $1 \leq j \leq k$. Let r denote the smallest of these numbers r_j, $1 \leq j \leq k$. Then obviously $I_p(r) = \cap_{j=1}^k I_p(r_j) \subset \cap_{j=1}^k U_j$, so $\cap_{j=1}^k U_j$ is also an open set, i e an element of $\mathcal{U}$.

T3) If $\{U_\alpha\}_{\alpha \in A}$ is an arbitrary collection of elements in $\mathcal{U}$ (as many as you please), and if $p \in \cup_{\alpha \in A} U_\alpha$, then $p \in U_{\alpha_o}$ for some $\alpha_o \in A$, and therefore $I_p(\delta) \subset U_{\alpha_o}$ for some $\delta > 0$. We get that $I_p(\delta) \subset U_{\alpha_o} \subset \cup_{\alpha \in A} U_\alpha$, so $\cup_{\alpha \in A} U_\alpha$ is also an open set, i e an element of $\mathcal{U}$.

Properties T2) and T3) says that $\mathcal{U}$ is closed under finite intersections and arbitrary unions. Property T1) guarantees that $\mathcal{U}$ is non-empty.

1.2 Continuity

As for convergence, the definition of continuity leeds to the problem of defining what a continuous function (or mapping) $f : A \longrightarrow B$, where A and B are two sets, should be. The definition given above, in the case $A \subset \mathbb{R}$ and $B = \mathbb{R}$, intuitively says that f is continuous on A if given $a \in A$, the value of f at x comes 'close' to $f(a)$ (in B) if we choose x 'close enough' to a (in A), i e we need a concept of closeness in both A and B if we want to generalize the given definition - just as in the case of convergence. Of course, one can use the definition of convergence to define continuity (we write it down in the case of real-valued functions of one real variable):
Let $A \subset \mathbb{R}$. A function $f : A \longrightarrow \mathbb{R}$ is continuous on A if

$$x \in A \text{ and } x \to a \Rightarrow f(x) \to f(a).$$

Guided from the discussion on convergence above, we investigate if the concept of open sets can be used to define continuity, and study the case of metric spaces afterwards. First we look at the case of a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$. From the definition it follows that, given a small open interval centered at $f(a) \in \mathbb{R}$, say $I_\varepsilon(f(a)) = (f(a)-\varepsilon, f(a)+\varepsilon)$, there exists a small open interval $I_\delta(a) = (a - \delta, a + \delta)$ centered at a, such that f maps $I_\delta(a)$ into $I_\varepsilon(f(a))$.
Let now $V \subset \mathbb{R}$ (V is thought to be in the target space of f, not in the domain of definition) be an open subset. Let furthermore $b \in f^{-1}(V) = \{x \in \mathbb{R} : f(x) \in V\}$ (=the inverse image, or preimage, of the set V under the function f). Since $b \in \mathbb{R}$ and f is continuous on $\mathbb{R}$ we have that for each (small) open interval J centered at $f(b)$, there exists a (small) open interval I centered at b, such that $f(I) \subset J$. We assumed that V is open and that $f(b) \in V$, so if we now choose $J \subset V$, we get that $I \subset f^{-1}(J) \subset f^{-1}(V)$ (since $f(I) \subset J \subset V$). Hence, each point $b \in f^{-1}(V)$ has an open neighbourhood contained in $f^{-1}(V)$, which proves that $f^{-1}(V)$ is open. I e, *if f is continuous, then the preimage of any open set is an open set.*
Conversely, suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ has the property that the preimage of any open set is open. Let $\varepsilon > 0$ and $a \in \mathbb{R}$ be fixed and let J be the open interval centered at $f(a)$ of length 2ε. Then $f^{-1}(J)$ is open by assumption and $a \in f^{-1}(J)$, so there exists a $\delta > 0$ such that the open interval I centered at a of length 2δ is contained in $f^{-1}(J)$. Obviously $f(I) \subset J$, i e if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$. Since $a \in \mathbb{R}$ was arbitrarily chosen, this proves that f is continuous on A.

Remark . In the discussion above we have used the following fact for preimages of sets under a function $f : A \longrightarrow B$:

If $I \subset f^{-1}(J)$, then $f(I) \subset J$. This is so, since if $I \subset f^{-1}(J)$ and $x \in I$, then there exists an element $y \in J$ such that $f(x) = y$ and, since f is a function, this y is unique.

Next we look at the case when $f : A \longrightarrow \mathbb{R}$, $A \subset \mathbb{R}$. If we agree to say that a subset $W \subset A$ is open (in A), if $W = V \cap A$ for some open subset V of $\mathbb{R}$, then we get that $\mathcal{U}_A = \{V \cap A : V \in \mathcal{U}\}$ satisfies properties T1)–T3) in the above (verify this!). By just repeating step by step what we did above, we see that f is continuous if and only if $f^{-1}(V)$ is open in A for each open subset V of $\mathbb{R}$.

Definition 7. (Continuity in terms of open sets) *If A and B are two sets and $\mathcal{U}_A$ and $\mathcal{U}_B$ are collections of subsets of A and B satisfying items T1)–T2) above, then we say that a function $f : A \longrightarrow B$ is continuous if $f^{-1}(V) \in \mathcal{U}_A$ for all $V \in \mathcal{U}_B$.*

So, as soon as we have a function $f : A \longrightarrow B$ and collections of open sets in A and B (satisfying properties T1)–T3) above), we can give meaning to the concept of continuity of f: f is continuous if $f^{-1}(V)$ is open in A for each open subset V of B.

What does this mean in terms of metrics on metric spaces? Let $f : X \longrightarrow Y$ be a function between two metric space X and Y. If we look at the discussion we had above, proving that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous if and only if $f^{-1}(V)$ is open in $\mathbb{R}$ for each open subset V of $\mathbb{R}$, and if we replace the metric $d(x, y) = |x - y|$ on $\mathbb{R}$ with the metrics on X and Y, denote them by d_X and d_Y respectively, we get, by repeating the discussion step by step, that f is continuous (with the new definition involving open sets) if and only if

$$\forall p \in X, \forall \varepsilon > 0, \exists \delta > 0 : d_X(p, x) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon.$$

Example . Let us look at a familiar mapping, namely integration. We let $X = C[0, 1]$ equipped with the metric introduced above, $d_X(f, g) = \max_{0 \le t \le 1} |f(t) - g(t)|$ and we let $Y = \mathbb{R}$ with the usual metric. Obviously $\int_0^1 f(s)ds \in \mathbb{R}$ for each $f \in X$, so

$$\int_0^1 : X \longrightarrow Y.$$

Is this mapping continuous (with respect to the given metrics)? Let $f \in X$ and $\varepsilon > 0$ be given. We have to prove that there exists $\delta > 0$ such that

$$d_X(f, g) = \max_{0 \le t \le 1} |f(t) - g(t)| < \delta \Rightarrow$$

$$\Rightarrow d_Y\left(\int_0^1 f(s)ds, \int_0^1 g(s)ds\right) = \left|\int_0^1 f(s)ds - \int_0^1 g(s)ds\right| < \varepsilon.$$

But this follows from the following inequalities (with $\delta = \varepsilon$):

$$\left|\int_0^1 f(s)ds - \int_0^1 g(s)ds\right| \leq \int_0^1 |f(s) - g(s)|ds \leq \max_{0\leq t\leq 1} |f(t) - g(t)|,$$

where the first inequality is just the triangle inequality for integrals and the second one is obvious. Hence, the answer is yes – integration in this case is a continuous mapping.

Remark . The space $C[0,1]$ in the above example can be replaced by $C[a,b]$ for any closed and bounded interval $[a,b]$ on the real line and $\int_0^1$ by $\int_a^b$, and the conclusion holds.

2 Topological spaces.

We begin by giving the definition of a topological space, which is motivated in the previous section:

Definition 8. (Topological space)
Let X be a set and $\mathcal{U}$ a collection of subsets of X satisfying the following conditions:

T1) $X, \emptyset \in \mathcal{U}$,

T2) if $U_1, \ldots, U_k \in \mathcal{U}$, *then* $\cap_{j=1}^k U_j \in \mathcal{U}$,

T3) if $U_\alpha \in \mathcal{U}$, *for* $\alpha \in A$, *then* $\cup_{\alpha\in A} U_\alpha \in \mathcal{U}$.

We say that $U \subset X$ is open if $U \in \mathcal{U}$. The collection $\mathcal{U}$ is called a topology on X, and X equipped with the topology $\mathcal{U}$ is called a topological space (sometimes denoted $(X,\mathcal{U})$, if we want to emphasize the topology in question).

The motivation for this definition and some of its purposes has hopefully been clarified in the preceeding section, and now we turn to some more examples.

2.1 Different topologies.

If we look at the definition of topological spaces, there are two topologies (on any set) that are easy to see:

Example . (The indiscrete topology.) Let X be a set and let $\mathcal{U} = \{X, \emptyset\}$. Then obviously T1)–T3) are satisfied. The set X equipped with this topology is called *an indiscrete space.*

Example . (The discrete topology.) Let X be a set and let $\mathcal{U}$ consist of *all* subsets of X. Then $X, \emptyset \in \mathcal{U}$ and since any intersection and any union of subsets of X are subsets of X, T2) and T3) are also satisfied. The set X equipped with the discrete topology is called *a discrete space.*

These two examples are the two extremes for topologies on a set X. The indiscrete topology only satisfies the minimum requirement for a topology and in the discrete topology *all* subsets are considered as open. Inbetween them there is a whole spectrum of topologies, and we shall pay some attention to these and see some differences between them, but first we define convergence and continuity:

Definition 9. (Convergence in topological spaces)
Let $(X,\mathcal{U})$ be a topological space and let $\{x_n\}$ be an infinite sequence in X. We say that $\{x_n\}$ is convergent, if

$$\exists x \in X,\, \forall U \in \mathcal{U}\,(x \in U),\, \exists N \in \mathbb{N} :\, n \geq N \Rightarrow x_n \in U.$$

Definition 10. (Continuity in topological spaces)
Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two topological spaces and let $f : X \longrightarrow Y$ be a function. We say that f is continuous, if $f^{-1}(V) \in \mathcal{U}$ for all $V \in \mathcal{V}$, i e if $f^{-1}(V)$ is open in X for all open subsets V of Y.

Remark . With the indiscrete topology on X any infinite sequence $\{x_n\}$ is convergent (to any point in X), since if $x \in X$ then the only open subset of X containing x is X itself, and of course $x_n \in X$ (for all n). In particular, the limit is not unique.
With the discrete topology on X the only convergent sequences are those for which $x_n = x_N$ if $n \geq$ some N, since if $x \in X$, then $U = \{x\}$ is an open neighbourhood of X, and $x_n \in U$ for n large enough then means that $x_n = x$.
Sensmoral: *The more open sets, the harder to be convergent.*

Remark . If X is equipped with the indiscrete topology and $Y = \mathbb{R}$ with the usual topology, then the only continuous functions $f : X \longrightarrow \mathbb{R}$ are the constant ones, since the inverse image of an open set $I \subset \mathbb{R}$ under f has to be either X or $\emptyset$.
If X is equipped with the discrete topology and $Y = \mathbb{R}$ with the usual topology, then any function $f : X \longrightarrow \mathbb{R}$ is continuous, since all subsets of

X are open.
Sensmoral: *The more open sets in X the easier for a function $f : X \longrightarrow \mathbb{R}$ to be continuous.*

Exercise 2. How does the 'number' of open sets in Y influence the 'difficulty of being continuous' for a function $f : X \longrightarrow Y$?

In order to see the properties of the topologies lying between the indiscrete and the discrete topologies we once again turn our attention to the real line (with the usual topology), and a bit more generally to metric spaces. If we pick two different points x and y on the real line we can find two open intervals I_x and I_y containing x and y respectively, such that $I_x \cap I_y = \emptyset$, i e any two pairs of different points on the real line are contained in disjoint open sets. The same is true for metric spaces, but if e g X is a set consisting of more than one point and if X is equipped with the indiscrete topology, then this is no longer true. Or, if $X = \mathbb{R}$ and if we define 'open set' to mean that the set is equal to an interval (a, ∞), $a \in \mathbb{R} \cup \{\pm\infty\}$ (which turns X into a topological space – verify this!), then it is neither true. What is true in the second case is that for any two points x and y on the real line, there exists *one* open set I containing one of the points (the bigger one), but not the other. We introduce the following terminology:

Definition 11. (Separation of points; Hausdorff spaces)
Let $(X,\mathcal{U})$ be a topological space. We say that $\mathcal{U}$ separates points on X, if for each pair of different points x and y in X, there exists $U, V \in \mathcal{U}$, such that $U \cap V = \emptyset$ and $x \in U$, $y \in V$. If $\mathcal{U}$ separates points in X, then we call $(X,\mathcal{U})$ a Hausdorff space(or T_2-space).
(If $\mathcal{U}$ is such that for any pair of different points x and y in X, there exists one open set $U \in \mathcal{U}$ such that one of x and y belongs to U, but the other does not, then we say that X is a T_1-space.)

We shall almost only study Hausdorff spaces in what is coming.

2.2 Closed sets

On the real line we have a concept of closed intervals (or sets) – an interval is closed, if it contains its endpoints (where the endpoints might be $-\infty$ and/or $+\infty$, in which case they are not included). The complement of a closed (bounded) interval is an open set (if the interval is $[a, b]$ then the complement is $(-\infty, a) \cup (b, +\infty)$). More generally:

Definition 12. (Closed sets)
A subset F of a topological space $(X,\mathcal{U})$ is called closed, if the complement $\neg F$ of F is an open set.

One important thing for Hausdorff spaces the fact that

a one-point-set of a Hausdorff space is closed.

I e if $x \in X$ and if X is Hausdorff, then $\{x\}$ is closed in X. The proof of this goes as follows:
Given any other point $y \in X$, there exists an open set U_y, such that $y \in U_y$ but $x \notin U_y$. The union of all such sets, as y varies over $X \setminus \{x\}$ is then an open subset of X whose complement is equal to $\{x\}$. (Note: Here we only used that X is T_1!)

From the definition of closed sets and the definition of topology, it follows that if X is a topological space then

C1) X and $\emptyset$ are closed subset of X,

C2) if $F_1, \ldots, F_k$ are closed subsets of X, then $\cup_{j=1}^{k} F_j$ is a closed subset of X and

C3) if $\{F_\alpha\}_{\alpha \in A}$ is an arbitrary collection of closed subsets of X, then $\cap_{\alpha \in A} F_\alpha$ is a closed subset of X.

Here items C2)and C3) follows from the equalities

$$\neg(E \cap F) = \neg E \cup \neg F \text{ and } \neg(E \cup F) = \neg E \cap \neg F.$$

C1) follows from the fact that $\neg X = \emptyset$ and $\neg \emptyset = X$.

When we look at metric spaces there is also another (equivalent) description of closed sets. Suppose first that F is a closed subset of a metric space X and let $\{x_n\}$ be an infinite sequence of points, all belonging to F. Suppose furthermore that this sequence converges to a point $x \in X$. If $x \notin F$, then $x \in \neg F$, which is an open set, so if $x_n \to x$ we must have that $x_n \in \neg F$ if n is large enough, which contradicts the assumption that $x_n \in F$ for all n. Hence, the limit must belong to F.
Conversely, suppose that whenever $\{x_n\}$ is an infinite sequence of points in F, such that $\{x_n\}$ converges to some point $x \in X$, then the limit too belongs to F. Let $y \in \neg F$ be an arbitrary, fixed point. We want to prove that there is an open neighbourhood of y that do not meet F (i e to prove that y is an interior point of $\neg F$). Suppose that this is not the case. Then the sequence

$$U_n = \{z \in X : d(z, y) < 1/n\}, \ n \in \mathbb{N}$$

of open neighbourhoods of y all contains points from F, so for each $n \in \mathbb{N}$ we can choose a point $x_n \in U_n \cap F$. This sequence obviously belongs to F and converges to y, which contradicts our assumption about F. Hence, $\neg F$ must be open.

Remark . (1) What we need in the second half of our discussion above is that for each point $y \in X$, there is a decreasing sequence of open neighbourhoods $\{U_n\}$ such that for any given neighbourhood U of y, we have $U_n \subset U$ if n is large enough and $\cap_n U_n = \{y\}$. This requirement on the topology on X is actually almost equivalent with saying that X is a metric space.
(2) If we leave the universe of metric spaces, there are examples of topological Hausdorff spaces where the two criterions for closed sets are not the same, but we leave them be here.

In order to prove that a subset of a metric space is closed it therefore suffices to prove that limit points of convergent sequences contained in the sets are also members of the set. Let us formulate this:

Definition 13. (Limit points)
Let F be a subset of a metric space X. A point $y \in X$ is called limit point of F, if there exists an infinite sequence $\{x_n\}$ of points in F such that $x_n \to y$ as $n \to \infty$.

Remark . Note that we do not require that the sequence $\{x_n\}$ consists of distinct points. Inparticular, for each $x \in X$, x is a limit point of $F = \{x\}$: take all the x_n's equal to x in the above definition.

Definition 14. (Closed sets – an alternative definition)
Let F be a subset of a metric space X. We say that F is closed, if it contains all its limit points.

How does closed sets behave under continuous functions $f : X \longrightarrow Y$, where X and Y are topological spaces? Suppose that $F \subset Y$ is closed and that f is continuous. Then $f^{-1}(\neg F)$ is open and, since $X = f^{-1}(F) \cup f^{-1}(\neg F)$ and $\emptyset = f^{-1}(F) \cap f^{-1}(\neg F)$, we get that $\neg f^{-1}(F) = f^{-1}(\neg F)$, so $f^{-1}(F)$ is closed. Hence, *inverse images of closed sets under continuous functions are closed.*
The converse is also true: if $f : X \longrightarrow Y$ has the property that $f^{-1}(F)$ is closed in X whenever F is closed in Y, then f is continuous (verify this!).

Definition 15. *(Closure) Let F be a subset of a topological space X. The closure of F, denoted $\overline{F}$, is definied as the smallest closed subset of X that contains F.*

Remark . Since arbitrary intersections of closed sets are closed sets, it follows that the closure of F is equal to the intersection of all closed sets that contains F.

Remark . If $F = \{x_n\}$ is an infinite sequence of points in a metric space X, and if this sequence is convergent to $x \in X$, then $\overline{F} = \{x_n\} \cup \{x\}$. If $\{x_n\}$ is not convergent, then $\overline{F}$ is equal to $\{x_n\}$ together with all the limits of convergent subsequences of $\{x_n\}$, e g if $F = \{\cos \sqrt{n}\}$ then $\overline{F} = [0, 1]$ (verify this!).