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Prov i matematik
NV-programmet
Reell analys MN1
2004-05-24

The exam is for 5 hours (2 - 7 pm). The solutions should be accompanied with motivations if nothing else is said.

1. Which of the following statements are true/false? Verify your answers.

a) Let X and Y be metric spaces and $f : X \longrightarrow Y$ a uniformly continuous mapping. If $\{x_n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence in X, then $\{f(x_n)\}_{n=1}^{\infty} \subset Y$ is a Cauchy sequence in Y. (4)

b) Let X be a compact metric space and $\mathcal{F} \subset C(X)$ an equicontinuous family. Suppose that $g \in C(X)$. Then $\mathcal{F} \circ g = \{f \circ g : f \in \mathcal{F}\} \subset C(X)$ is equicontinuous. (4)

c) If A is a countable subset of $\{x \in \mathbb{R} : x \geq 0\}$, then A can be written $A = \{x_n\}_{n=1}^{\infty}$, where $n > m \Rightarrow x_n > x_m$. (4)

d) The set $\{f \in C[0,1] : f(1/2) = 0\}$ is closed in $C[0,1]$ (here $C[0,1]$ has the usual metric $d(f,g) = \max\limits_{0 \leq t \leq 1} |f(t) - g(t)|$). (4)

2. Suppose that α is a monotonically increasing function on $\mathbb{R}$ such that α' is continuous. Define

$$F(x) = \int_0^x f(t)d\alpha(t), \ f \in C(\mathbb{R}).$$

Prove that $F'(x)$ exists for all $x \in \mathbb{R}$ and calculate $F'(x)$. (6)

3. Assume that X is a compact topological space and that $F \subset X$ is closed. Prove that F is compact. (6)

4. Let $\mathcal{L}(\mathbb{R}^n)$ denote the vector space of linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$ with the usual norm

$$\|A\| = \sup_{|x| \leq 1} |Ax|.$$

Define $A^k = A \circ \cdots \circ A$ (k times) for k a non-negative integer, where $A^0 = I$ (= the identity mapping). Let $p(t) = c_0 + c_1 t + \ldots c_m t^m$ be a real-valued polynomial of one real variable, and define

$$p(A) = \sum_{k=0}^{m} c_k A^k = c_0 I + c_1 A + \ldots c_m A^m.$$

Prove that $p : \mathcal{L}(\mathbb{R}^n) \longrightarrow \mathcal{L}(\mathbb{R}^n)$ is continuous. (6)

See also next page!

5. a) Let $\mathcal{F} \subset C[0,1]$ be an equicontinuous family. Prove that $\overline{\mathcal{F}}$ is equicontinuous. (2)

b) Let $\mathcal{F}$ denote the family of all continuous, real valued functions f defined on $[0,1]$ that are differentiable on $]0,1[$ and such that $f(0) = 0$ and $|f'(x)| \leq 1$ for all $x \in]0,1[$. Prove that $\overline{\mathcal{F}}$ is compact in $C[0,1]$.(You may use the result from a), even if you have not solved that problem.) (4)

Good luck!

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1. a) This is true: Let $\varepsilon > 0$ and choose $\delta > 0$ such that $d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon$ (this is possible, since f is uniformly continuous). Since $\{x_n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence, there is an N such that $n, m \geq N \Rightarrow d_X(x_n, x_m) < \delta$. We get

$$n, m \geq N \Rightarrow d_X(x_n, x_m) < \delta \Rightarrow d_Y(f(x_n), f(x_m)) < \varepsilon,$$

which proves that $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence.

b) This is also true: Let $\varepsilon > 0$ and choose $\delta > 0$ such that $d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$ for all $f \in \mathcal{F}$. Then choose $\delta' > 0$ such that $d(x, y) < \delta' \Rightarrow d(g(x), g(y)) < \delta$ (this is possible, since X compact implies g uniformly continuous). We get

$$d(x, y) < \delta' \Rightarrow d(g(x), g(y)) < \delta \Rightarrow d(f(g(x)), f(g(y))) < \varepsilon,$$

for all $f \in \mathcal{F}$.

c) This is false: Let $A = \mathbb{Q}$ and suppose that A can be written as in the text. Let $x = x_n + (x_{n+1} - x_n)/2$. Then $x \in A$ and $x_n < x < x_{n+1}$, so this $x \in A$ cannot be one of the x_n's, which is a contradiction..

d) This is true: The mapping $T_p(f) = f(p)$ is continuous from $C[0, 1]$ to $\mathbb{R}$, since

$$|T_p(f) - T_p(g)| = |f(p) - g(p)| \leq \max_{0 \leq t \leq 1} |f(t) - g(t)| = d(f, g).$$

The set $\{f \in C[0, 1] : f(1/2) = 0\}$ is equal to $T_p^{-1}(\{1/2\})$ and therefore it is closed.
Alternatively: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\{f \in C[0, 1] : f(1/2) = 0\}$, such that $f_n \to f \in C[0, 1]$. Then

$$|f(1/2)| = |f(1/2) - f_n(1/2)| \leq \max_{0 \leq t \leq 1} |f(t) - f_n(t)| \to 0, \ n \to \infty$$

so $f(1/2)$ must be equal to 0.

2. By a theorem

$$F(x) = \int_0^x f(t) d\alpha(t) = \int_0^x f(t)\alpha'(t) dt$$

since α' is continuous, and thereby Riemann integrable on each compact interval. We get

$$\frac{F(x + h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)\alpha'(t) dt = f(\xi)\alpha'(\xi)$$

for some ξ between x and $x + h$. Let $h \to 0$. We get $F'(x) = f(x)\alpha'(x)$.

3. Let $\{V_i\}_{i \in I}$ be an open cover of F. Since F is closed $\{V_i\}_{i \in I} \cup (X \setminus F)$ is an open cover of X. X compact implies that there is a finite subcover, which must be of the form $\{V_{i_j}\}_{j=1}^{n} \cup (X \setminus F)$, and therefore there is a finite subcover $\{V_{i_j}\}_{j=1}^{n}$ of F.

4. It is obvoius that $p(A) \in \mathcal{L}(\mathbb{R}^n)$ for each $A \in \mathcal{L}(\mathbb{R}^n)$. Futhermore, if we choose a basis in $\mathbb{R}^n$, each element in $\mathcal{L}(\mathbb{R}^n)$ can be identified with an $n \times n$ matrix, and the coefficients of the matrix $p(A)$ are polynomial expressions of the coefficients of A. Hence, the coefficients of $p(A)$ depends continuously of the coefficients of A. By a result in Rudins book, this shows that p is continuous.
Alternatively formulated: polynomials are continuous and so are linear mappings. The mapping $p(A)$ is the composition of a polynomial and a linear mapping and compositions of continuous mappings are continuous.

5. a) Suppose that $f \in \overline{\mathcal{F}}$. Choose $g \in \mathcal{F}$ such that $d(f,g) < \varepsilon/3$ ($\varepsilon > 0$ arbitrary). Then choose $\delta > 0$ so that $|x-y| < \delta \Rightarrow |g(x)-g(y)| < \varepsilon/3$. We get

$$|f(x)-f(y)| \leq |f(x)-g(x)| + |g(x)-g(y)| + |g(y)-f(y)| \leq$$

$$\leq 2d(f,g) + |g(x)-g(y)| < \varepsilon.$$

This proves that $\overline{\mathcal{F}}$ is equicontinuous.

b) If $f \in \mathcal{F}$, then $|f(x)| = |f(x)-f(0)| = |f'(\xi)| \cdot |x-0| \leq |x|$ for som $\xi \in]0,x[$, by the mean value theorem. Hence, for fixed x we get $\{f(x) : f \in \mathcal{F}\} \subset [-x,x]$. In fact, $\{f(x) : f \in \mathcal{F}\} = [-x,x]$, since the functions $f(x) = \alpha x$, $|\alpha| \leq 1$, are in $\mathcal{F}$. This proves that $\mathcal{F}$ is pointwise compact. Furthermore, $\mathcal{F}$ is equicontinuous, since $f \in \mathcal{F}$ implies

$$|f(x)-f(y)| = |f'(\xi)| \cdot |x-y|,$$

for some ξ between x and y. So given $\varepsilon > 0$, there exists δ (in this case we can choose $\delta = \varepsilon$) such that

$$|x-y| < \delta \Rightarrow |f(x)-f(y)| = |f'(\xi)| \cdot |x-y| \leq |x-y| < \varepsilon$$

for all $x, y \in [0,1]$ and all $f \in \mathcal{F}$. It follows from a) that $\overline{\mathcal{F}}$ is equicontinuous. Hence, $\overline{\mathcal{F}}$ is closed, pointwise compact and equicontinuous, so by the Arzela–Ascoli theorem it is compact.