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Prov i matematik
MN-programmet
Reell analys MN1, deltentamen
2000-10-07

*The exam is for 5 hours (9 am – 2 pm). Nothing, except for pen/pencil and rubber, is allowed. The solutions should be accompanied with motivations.*

**1.** Let $X = \mathbb{R}^2$ with the usual topology. (6)

a) Give an example of closed subsets $F_n$ of $X$, $n \in \mathbb{N}$, whose union $\cup_{n\in\mathbb{N}} F_n$ is not closed.

b) Give an example of open subsets $U_n$, $n \in \mathbb{N}$, of $X$ whose intersection $\cap_{n\in\mathbb{N}} U_n$ is not open.

c) Find an open subcover of the set $E = \{(x,y) \in \mathbb{R}^2 : y = 0,\ 0 < x \leq 1\}$ that does not have a finite subcover.

Try to illustrate your solutions in diagrams.

**2.** Let $\{x_n\}_{n=1}^{\infty}$ be the sequence

$$\begin{cases} x_1 = & \sqrt{2} \\ x_{n+1} = & \sqrt{x_n + 2},\ n \geq 1 \end{cases}$$

of real numbers. (9)

a) Show that $\{x_n\}_{n=1}^{\infty}$ is increasing and bounded from above, and hence convergent.

b) Show that if $x = \lim_{n\to\infty} x_n$, then $x$ is a root of the equation $x^2 - x - 2 = 0$.

c) Calculate $\lim_{n\to\infty} x_n$.

**3.** Which of the following statements are true/false? Give proofs/counterexamples to verify your answers. (9)

a) If $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and $\limsup_{n\to\infty} x_n = b$, then there exists $N$ such that $x_n \leq b$ for all $n \geq N$.

b) If $A$ and $B$ are non-empty subsets of $\mathbb{R}$, then $\sup(A+B) = \sup A + \sup B$, where $A + B = \{x + y : x \in A,\ y \in B\}$.

c) If $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are bounded sequences in $\mathbb{R}$, then $\limsup_{n\to\infty}(x_n+y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$.

**4.** Let $\{p_n\}_{n=1}^{\infty}$ be a convergent sequence in a topological space $X$, with limit point $p \in X$. Prove that $K = \{p_n\}_{n=1}^{\infty} \cup \{p\}$ is compact. (3)

**5.** Let $X = \mathbb{R}$ with the topology $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathbb{R} \cup \{\pm\infty\}}$, where $U_\alpha = ]\alpha, \infty[$ if $-\infty \le \alpha < +\infty$, and $U_\alpha = \emptyset$ if $\alpha = +\infty$. Let furthermore $E = [0, 1]$. (9)

a) Is $E$ closed in $X$?

b) Determine $\overline{(E^c)}$.

c) Is $E$ compact in $X$?

**6.** Let $\{V_\alpha\}_{\alpha \in A}$ be a family of non-empty open subsets of $\mathbb{R}$ (with the usual topology). Suppose that $V_\alpha \cap V_\beta = \emptyset$ for all $\alpha, \beta \in A$, such that $\alpha \neq \beta$. Show that $A$ is at most countable. (4)

## Svar till tentamen i Reell analys MN1, deltentamen 2000-10-07

**1.** a) $F_n = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 - 1/n\} \implies \cup_{n \in \mathbb{N}} F_n = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.

b) $U_n = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1/n\} \implies \cap_{n \in \mathbb{N}} U_n = \{(0,0)\}$.

c) We have that $E \subset \cup_{n \in \mathbb{N}} V_n$, where $V_n = \{(x,y) \in \mathbb{R}^2 : 1/n < x < 1 + 1/n\}$, $n \in \mathbb{N}$, has no finite subcover.

**2.** The easiest way is to pretend as if we know that the sequence is convergent, with limit $x$ say. Then

$$x = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} \sqrt{x_n + 2} = \sqrt{x + 2}.$$

Hence, $x$ (if it exists) must satisfy the equation $x = \sqrt{x+2}$, which implies the equation $x^2 = x + 2 \Leftrightarrow x^2 - x - 2 = 0 \Leftrightarrow x = 2$ or $x = -1$. (This is also the solution to b)).

a) From what we just did it is indicated that we should be able to prove that $x_n \leq 2$ for all $n$. Obviously $x_1 = \sqrt{2} \leq 2$, so suppose that $x_n \leq 2$ for some $n \geq 1$. Then

$$x_{n+1} = \sqrt{x_n + 2} \leq \sqrt{2 + 2} = 2,$$

so, by induction, $x_n \leq 2$ for all $n$.

Next we prove that the sequence is increasing. First we see that $x_2 = \sqrt{2 + \sqrt{2}} > \sqrt{2} = x_1$. Suppose now that $x_{n+1} \geq x_n$ for some $n \geq 1$. Then

$$x_{n+2} - x_{n+1} = \sqrt{x_{n+1} + 2} - \sqrt{x_n + 2} \geq \sqrt{x_n + 2} - \sqrt{x_n + 2} = 0,$$

so that $x_{n+2} \geq x_{n+1}$ and by induction we get that the sequence increases. Hence, it is convergent.

b) See above for the solution.

c) Since the sequence is increasing and $x_1 = \sqrt{2}$, the limit must be equal to 2 – the only possible limits are -1 and 2, according to what we did at the beginning.

**3.** a) False.
Counterexample: with $x_n = 1/n \in \mathbb{R}$, $n \in \mathbb{N}$, we have that $\limsup_{n \to \infty} x_n = \lim_{n \to \infty} x_n = 0$, but $x_n > 0$ for *all* $n$.

b) True.
Proof: Let $\alpha, \beta \in \mathbb{R}$ be upper bounds for $A$ and $B$, respectively, i e $a \leq \alpha$ and $b \leq \beta$ for all $a \in A$ and $b \in B$. Then obviously $c \leq \alpha + \beta$ for all $c = a + b \in A + B$. This proves that $\sup(A + B) \leq \sup A + \sup B$. Conversely, let $\gamma$ be an upper bound for $A + B$, i e $a + b \leq \gamma$ for all $a \in A$ and $b \in B$, which is equivalent with $a \leq \gamma - b$ for all $a \in A$ and $b \in B$. Hence, $\gamma - b$ is an upper bound for $A$ for each $b \in B$, so if $\beta_0$ is the smallest upper bound of $B$ we get $a \leq \gamma - \beta_0$ for all $a \in A$. Let $\alpha_0$ be the smallest upper bound of $A$. We have $\alpha_0 \leq \gamma - \beta_0$, so that $\alpha_0 + \beta_0 \leq \gamma$, which proves the other inequality.

**4.** Let $K \subset \cup_{\alpha \in A} V_\alpha$ be an open cover of $K$. Then there is an $\alpha = \alpha_0 \in A$ such that $p \in V_{\alpha_0}$. Since $p_n \to p$, $n \to \infty$, and $V_{\alpha_0}$ is a neighbourhood of $p$, the definition of convergence implies that there exists $N$ such that $n \geq N \implies p_n \in V_{\alpha_0}$. For $1 \leq n \leq N - 1$ we can choose $\alpha_1, \ldots, \alpha_{N-1} \in A$ such that $p_i \in V_{\alpha_i}$, $1 \leq i \leq N - 1$. Then $K \subset \cup_{i=0}^{N-1} V_{\alpha_i}$ is a finite subcover, which proves that $K$ is compact.

**5.** a) $E$ is not closed in $X$, since $E^c = ]-\infty, 0[ \cup ]1, \infty[$ is not an open subset of $X$.

b) The closed subsets of $X$ are all of the form $F = U_\alpha^c = \mathbb{R} \setminus ]\alpha, \infty[ = ]-\infty, \alpha]$, and the smallest such set that contains $E^c = ]-\infty, 0[ \cup ]1, \infty[$ is $X = \mathbb{R}$ itself. Hence, $\overline{(E^c)} = X = ]-\infty, \infty[$.

c) Yes, $E$ is compact, since if $E \subset \cup_{\beta \in A} V_\beta$ is an open cover of $E$, then at least one of the $V_\beta$'s must contain $0 \in X$, i e must be of the form $V_\beta = ]\beta, \infty[$, for some $\beta < 0$, and then obviously $E \subset V_\beta$, which is a finite subcover.

**6.** To each $\alpha \in A$ we can associate a rational number $p$, such that $p \in V_\alpha$, since $V_\alpha \cap \mathbb{Q} \neq \emptyset$ for each $\alpha \in A$. (Actually $V_\alpha \cap \mathbb{Q}$ is infinite for each $\alpha \in A$, but we only choose *one* element in the intersection, for each $\alpha \in A$.) This provides us with a bijection from $A$ to a subset of $\mathbb{Q}$, and since $\mathbb{Q}$ is countable, we get that $A$ is at most countable (it could of course be finite too).