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Tentamen i matematik
Reell Analys MN1
1999-08-13

Skrivtid: 9-14.
Tillåtna hjälpmedel: Manuella skrivdon.
För betygen Godkänd/Väl godkänd krävs (minst) 18 respektive 28 poäng.

1. Let X denote the metric space $C[0,1]$ with the usual sup-norm. Which of the following sets are closed? (4)

$$F_1 = \{f \in X : f(p) = 0\} \text{ ,where } p \in [0,1] \text{ is fixed.}$$

$$F_2 = \{p \in [0,1] : g(p) = 1\} \text{ ,where } g \in X \text{ is fixed.}$$

2. Let X and Y be two metric spaces and $f : X \longrightarrow Y$ a continuous mapping. Prove that $f(\overline{E}) \subset \overline{f(E)}$ for each subset E of X. Give also an example which shows that the inclusion may be stict. (5)

3. Which of the following statements are true? (Give proof or counterexample.)

   a) Let X be a metric space and $K \subset Y \subset X$. If K is compact relatively Y, then K is compact relatively X. (3)

   b) Let K, Y and X be as in a). If K is compact relatively X, then K is compact relatively Y. (3)

4. Let X be a set equipped with the metric

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

   Which subsets of X are open? Closed? Compact? (6)

5. Suppose that $\{f_n\}_{n=1}^{\infty}$ is an equicontinuous sequence on a compact, metric space K and that $\{f_n\}_{n=1}^{\infty}$ converges pointwise on K. Prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on K. (6)

SEE NEXT PAGE!

6. Let $f : [0,1] \longrightarrow \mathbb{R}$ be continuous and suppose that (6)

$$\int_0^1 f(t)g(t)dt = 0$$

for each *real analytic* function g on $[0,1]$ (i e each function g that can be expanded in a convergent power series around every point in $[0,1]$). Prove that $f(x) = 0$ for all $x \in [0,1]$.

7. Let X be a metric space with metric d and define for $A \subset X$

$$\rho_A(y) = \inf_{x \in A} d(x,y),\ x \in X$$

Let now E and F be two closed, disjoint subsets of X and define

$$f(x) = \frac{\rho_E(x)}{\rho_E(x) + \rho_F(x)},\ x \in X.$$

a) Prove that f is a continuous function from X to $[0,1]$. (3)

b) Prove that $f(x) = 0$ if and only if $x \in E$ and that $f(x) = 1$ if and only if $x \in B$. (2)

c) Let $U = f^{-1}([0, \frac{1}{2}])$ and $V = f^{-1}((\frac{1}{2}, 1])$. Prove that U and V are disjoint and open subsets of X. (2)