1 Algebraic structures

Idea: set with operations satisfying certain axioms

Examples Let M : set with an operation

- $\mu : M \times M \longrightarrow M$, $(a,b) \longmapsto ab$

  $(ab)c = a(bc)$ associativity

  $\mu(\mu(a,b),c) = \mu(a,\mu(bc))$

Then M is called a semigroup

- $e \in M$, st $ea = a = ae$ (Then e is unique)

  Neutral element

  M is called a monoid (If we have both μ, e)

- $\forall a \in M \ \exists a^{-1} \in M$: $a^{-1}a = e = aa^{-1}$

  Inverse

  M is called a group if we have all three.

- $\forall a,b \in M$ $ab = ba$ commutative

  If all of the above M is called an abelian group

Often write $\mu(a,b) = a+b$ $e = 0$

Examples • $\mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$ with $+$ semigroup

• $\mathbb{N} = \{0, 1, 2, \dots\}$ with $+$ monoid
with $\cdot$ monoid.

• $\mathbb{Z}$ with $+$ abelian group.

• $GL_n(\mathbb{C}) = \{ M : n \times n\text{-matrices over } \mathbb{C} \mid \det M \neq 0 \}$.
with mult.

non-abelian group.

• X set $M_X = \{ f : X \to X \mid f : \text{map} \}$

with composition is a monoid

• X set $S_X = \{ f \in M_X \mid f \text{ bijective} \}$.

with composition is a group.

Ex • Let R set with

$\mu : R \times R \to R$, $(a,b) \longmapsto ab$

$\alpha : R \times R \to R$ $(a,b) \longmapsto a+b$.

Then R is a ring if

(R, α) : abelian group.

(R, μ) : monoid and

$a(b+c) = ab + ac$

$(a+b)c = ac + bc$ } distributivity.

Ex $M_n(\mathbb{C})$ with matrix mult and addition.

R commutative if $ab = ba$ Ex $\mathbb{Z}$

R domain if in addition $1 \neq 0$ and $ab = 0 \Rightarrow a = 0$ or $b = 0$

Ex $\mathbb{Z}$

R field if $1 \neq 0$ and $\forall a \neq 0 \ \exists a^{-1} : aa^{-1} = 1$ Ex $\mathbb{Q}$

2 Universal algebra

Def • A *type* is a set T (think operations) with $T \longrightarrow \mathbb{N}$, $\omega \mapsto n_\omega$ (think arity)
$n_\omega = 2$ binary operation

• A *T-algebra* is a set A with maps $\omega_A : A^{n_\omega} \longrightarrow A$ for each $\omega \in T$

Ex $T = \{\mu\}$, $n_\mu = 2$

$$\mu_A : A^2 \longrightarrow A$$

For instance a semigroup is T-alg A st μ_A is associative.

Rmk If $n_\omega = 0$ then $A^0 = \{*\}$ singleton

$$\omega : A^0 \longrightarrow A$$

We identify ω with $\omega(*) \in A$ constant.

Ex $T = \{\mu, \varepsilon\}$ $n_\mu = 2$ $n_\varepsilon = 0$

$$\mu_A : A^2 \to A \qquad \varepsilon_A \in A$$

e.g. a monoid $\mu_A(a,b) = ab$, $\varepsilon_A = e$.
is a T alg, where μ_A associative
ε_A neutral element.

Ex $T = \{\mu, \varepsilon, \iota\}$ $n_\mu = 2$, $n_\varepsilon = 0$, $n_\iota = 1$
e.g. group with $\mu(a,b) = ab$ $\varepsilon = e$, $\iota(a) = a^{-1}$.
inverse $\forall a$ $\mu(a, \iota(a)) = \varepsilon = \mu(\iota(a), a)$.

Ex $T = \{\alpha, \varepsilon^0, \iota, \mu, \varepsilon^1\}$ e.g. R: ring
n: 2 0 1 2 0
$\alpha(a,b) = a+b$, $\varepsilon^0 = 0$ $\iota(a) = -a$ $\mu(a,b) = ab$ $\varepsilon^1 = 1$

3 Subalgebras and morphisms

Def Let A be a T-algebra.
A subset $B \subseteq A$ st $\forall \omega \in T$
$$\omega_A(B^{n_\omega}) \subseteq B$$
is called a subalgebra and is itself a T-algebra by restriction
$$\omega_B = \omega_A|_{B^{n_\omega}}$$

Ex Let $B_i \subseteq A$ subalgebras $\forall i \in I$

i) Then $\bigcap_{i \in I} B_i \subseteq A$ is a subalg.

ii) If $(B_i)_{i \in I}$ is totally orderd by $\subseteq$
then $\bigcup_{i \in I} B_i \subseteq A$ is a subalg.

Def Fix a type T
Let A, B T-algebras. A map $\varphi : A \to B$ is called a morphism of T-algebras
if $\varphi(\omega_A(x_1, \dots, x_{n_\omega})) = \omega_B(\varphi(x_1), \dots, \varphi(x_{n_\omega}))$
$\forall \omega \in T \quad \forall x_i \in A$

Ex $T = \{\mu, \varepsilon\}$ $\quad n_\mu = 2 \quad n_\varepsilon = 0$ as before

M, N monoids. $\quad \varphi : M \to N$

$\varphi(\mu_M(a,b)) = \varphi(ab) \qquad \mu_N(\varphi(a), \varphi(b)) = \varphi(a)\varphi(b)$

$\varphi(\varepsilon_M) = \varphi(1_M) \qquad \varepsilon_N = 1_N$

Ex $\quad T = \{\mu, \iota, \varepsilon\} \quad n_\mu = 2 \quad n_\iota = 1 \quad n_\varepsilon = 0$

G, H groups are T-algebras.

i) $\varphi(ab) = \varphi(a)\varphi(b)$

ii) $\varphi(a^{-1}) = \varphi(a)^{-1}$

iii) $\varphi(1_G) = 1_H$

In this case i) $\Rightarrow$ ii), iii) follows from axioms.

$$\varphi(1_G 1_G) \overset{i)}{=} \varphi(1_G)\varphi(1_G) \quad \text{multiply by } \varphi(1_G)^{-1}$$

$$\overset{\shortparallel}{\varphi(1_G)} \quad \Rightarrow \quad 1_H = \varphi(1_G).$$

similarly i) $\Rightarrow$ ii).

Ex
- If A T-algebra and $B \subseteq A$ subalgebra then inclusion $B \to A$ is a morphism.
- If $\varphi : A \to B$ is a morphism, then $\varphi(A) \subseteq B$ is a subalgebra.

$$\omega_B(\varphi(a_1), \dots, \varphi(a_{n_\omega})) = \varphi(\underbrace{\omega_A(a_1, \dots, a_{n_\omega})}_{\in A}) \in \varphi(A).$$

4 Expressions

Let T : type and X set (think variables)

The set W_X^T of expressions (or words)

is the smallest set s.t.

- $X \subseteq W_X^T$
- If $\omega \in T$ and $w_1, \dots, w_{n_\omega} \in W_X^T$

  then $(\omega, w_1, \dots, w_{n_\omega}) \in W_X^T$

  think $\omega(w_1, \dots, w_{n_\omega})$

Note each $\omega \in T$ can be considered as an operation $\left(W_X^T\right)^{n_\omega} \longrightarrow W_X^T$, $(w_1, \dots, w_{n_\omega}) \longmapsto (\omega, w_1, \dots, w_{n_\omega})$

So W_X^T is a T-algebra.

Ex $T = \{\mu, \varepsilon\}$ $n_\mu = 2$ $n_\varepsilon = 0$ $X = \{a, b\}$.

write $\varepsilon = e$ $(\mu, w_1, w_2) = w_1 w_2$

Then $W_X^T = \{a, b, e,$

$aa, ab, ae, ba, bb, be, ea, eb, ee$

$a(aa), a(ab), \dots$

$b(aa), b(ab), \dots$

$e(aa), e(ab), \dots$

$(aa)a, (aa)b, \dots$

$\vdots$

$(b((a(ab))a))e$ etc.

Proposition Let A : T-algebra and $f : X \longrightarrow A$ map.

Then there is a unique morphism

$$\begin{tikzcd} \varphi_f : W_X^T \arrow[r] & A \\ \cup\!\!\!\!\!\!\ \quad X \arrow[ur, "f"'] & \end{tikzcd}$$

st $\varphi(x) = f(x)$ $\forall x \in X$

proof $\varphi_f(x) = f(x)$ $\forall x \in X$ given

$\varphi_f(\omega, w_1, \dots, w_{n_\omega}) = \omega_A(\varphi_f(w_1, \dots, w_{n_\omega}))$

only choice to make φ_f a morphism

But notice this defines φ_f recursively.

Def Fix a countable infinite set $X = \{a, b, c, \dots\}$

- Let $w_1, w_2 \in W_X^T$ We say $w_1 = w_2$ is satisfied in a T-algebra A if

$$\varphi_f(w_1) = \varphi_f(w_2) \quad \text{for all } f : X \to A.$$

Ex $T = \{\mu, \varepsilon\}$ as before $\mu(a,b) = ab$ $\varepsilon = e$

i) $w_1 = (ab)c$ $\qquad w_2 = a(bc)$

Then $w_1 = w_2$ is satisfied in A iff μ associative

ii) $w_1 = a$ $\qquad w_2 = ae$ $\qquad w_3 = ea$

$\left\{\begin{array}{l} w_1 = w_2 \\ w_1 = w_3 \end{array}\right.$ are satisfied in A iff ε neutral

Def Let $J \subseteq W_X^T \times W_X^T$. The class of T-algebras satisfying $w_1 = w_2$ $\forall (w_1, w_2) \in J$ is called a variety (of type T)

Ex
- Semigroups, monoids, groups, rings abelian group, commutative rings. are varieties
- Domains, fields are not varieties

Intuition: $\left.\begin{array}{l} ab = 0 \Rightarrow a = 0 \text{ or } b = 0 \\ \forall a \neq 0 \ \exists a^{-1} : aa^{-1} = e \end{array}\right\}$ can't be expressed as identities

proof silly reason No field with 1 element.
serious reason next time.