(1) $A = M_{2\times 2}(\mathbb{C})$. Describe all left A-modules

A : basis $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$e^2 = e \quad f^2 = f \quad e + f = 1 \qquad xy = e \qquad fye = y$

$ef = 0 \quad fe = 0 \qquad\qquad yx = f \qquad exf = x$

$M \in A\text{-mod} \qquad M = eM \oplus fM$

$$eM \underset{x\_}{\overset{y\_}{\rightleftarrows}} fM$$ mutually inverse

pick basis $b_1, \dots, b_r$ of eM

$\Rightarrow \quad yb_1 \dots yb_r$ basis of fM

$$e \circlearrowright b_i \underset{x\_}{\overset{y\_}{\rightleftarrows}} yb_i \circlearrowleft f \qquad \text{span}\{b_i, yb_i\} \simeq \mathbb{C}^2$$

$M \simeq (\mathbb{C}^2)^{\oplus r} \qquad \Rightarrow \quad \nexists$ $\mathbb{C}$-alg morphism $M_{2\times 2}(\mathbb{C}) \to M_{3\times 3}(\mathbb{C})$

(2) $A = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ c = 0 \right\} \subseteq M_{2\times 2}(\mathbb{C})$ subalg.

$A \simeq \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}.$

Describe all A-modules

$$\begin{matrix} \mathbb{C}^2 = \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \end{bmatrix} \\ \cup \qquad \cup \\ \mathbb{C}\times\{0\} = \begin{bmatrix} \mathbb{C} \\ 0 \end{bmatrix} \end{matrix}$$

are A-modules

and $\mathbb{C}^2 / \mathbb{C}\times\{0\} =: \begin{bmatrix} 0 \\ \mathbb{C} \end{bmatrix}$

In general $M \in A\text{-mod}$ $\quad M \simeq \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \end{bmatrix}^{\oplus a} \oplus \begin{bmatrix} \mathbb{C} \\ 0 \end{bmatrix}^{\oplus b} \oplus \begin{bmatrix} 0 \\ \mathbb{C} \end{bmatrix}^{\oplus c}$

A has basis e, f, x $\quad$ same relations

$$M = eM \oplus fM$$

$$e \circlearrowright eM \xleftarrow[x\cdot]{} fM \circlearrowleft f \qquad \text{choose bases}$$

$$\rightsquigarrow \quad \mathbb{C}^m \xleftarrow{X} \mathbb{C}^n \qquad \text{change of basis}$$

$$\begin{tikzcd} \mathbb{C}^m \arrow[d, "T_1"'] & \mathbb{C}^n \arrow[l, "X"'] \arrow[d, "T_2"] \arrow[dl, phantom, "\circlearrowright"] \\ \mathbb{C}^m & \mathbb{C}^n \arrow[l, "X'"] \end{tikzcd}$$

i.e $X' = T_1 X T_2^{-1}$

$M = \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \end{bmatrix} \rightsquigarrow X' = [1]$

$M = \begin{bmatrix} \mathbb{C} \\ 0 \end{bmatrix} \rightsquigarrow X' : 1\times 0$

$M = \begin{bmatrix} 0 \\ \mathbb{C} \end{bmatrix} \rightsquigarrow X' : 0\times 1$

WLOG $X' = \begin{array}{c|cc} & a & c \\ \hline a & \begin{matrix} 1 & \\ & \ddots \\ & & 1 \end{matrix} & 0 \\ b & 0 & 0 \end{array}$

Let G fin. group

Recall If $U \subseteq \mathbb{C}G$ submodule then

$\exists V \subseteq \mathbb{C}G$ submodule st $\mathbb{C}G = U \oplus V$

(3) $G = \langle g \mid g^n = 1 \rangle$

$\mathbb{C}G = U_0 \oplus \cdots \oplus U_{n-1}$ $\quad U_i = \mathbb{C}$ with $gu = \zeta^i u$

$$\mathbb{C}G = \mathbb{C}[T]/(T^n - 1) \simeq \prod_{i=0}^{n-1} \underbrace{\mathbb{C}[T]/(T-\zeta^i)}_{\simeq U_i}$$

$\zeta^n = 1$ primitive

$g \longleftarrow T$

as $T^n - 1 = \prod_{i=0}^{n-1} (T - \zeta^i)$

(4) Let $K = \mathbb{Z}/(p)$ $\quad G$ as above $n = p$

p : prime.

Can we decompose KG is the same way?

$$KG = K[T]/(T^p - 1) = K[T]/((T-1)^p) \simeq K[X]/(X^p) = A$$

$T^p - 1 = (T-1)^p$ $\quad T - 1 \longleftarrow X$

$T \longmapsto X + 1$

$X : A \longrightarrow A,\ a \longmapsto Xa$ in basis $(1, \ldots, X^{p-1})$

we set X : matrix $\begin{bmatrix} 0 & 0 & & 0 \\ 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}$

But if $A = U \oplus V$ submodules $\quad$ matrix $\left[\begin{array}{c|c} X_1 & \\ \hline & X_2 \end{array}\right]$

choose bases

$\Rightarrow U = 0$ or $V = 0$.

Note $0 \subseteq (x^{p-1}) \subseteq \cdots (x^2) \subseteq (x) \subseteq A$ submodules indecomposable

5 Let K any field st $\mathrm{char}(K) = p > 0$ with $p \mid |G|$

Show $\exists U \leq KG$ st $\nexists V \leq KG : U \oplus V = KG$.

$C_p \subseteq G$ $\quad KC_p \leq KG$ closed under action of C_p but not G

Consider $x = \sum_{h \in G} h \in KG$

Note $gx = \sum_{h \in G} gh = x$

$xg = x$

$U = KGx \cong Kx$ Note $(KGx = xKG = U)$

Assume $KG = U \oplus V \ni u+v$ $\quad u \in U,\ v \in V$

$U \ni x(u+v) = xu + xv = xv \in V$

$xu = 0$ as $x^2 = |G|x = 0$

$\Rightarrow x(u+v) \in U \cap V = \{0\}.$

$\Rightarrow xKG = 0$ $\lightning$ as $x \in KG$ $\lightning$.

Note $|G| \neq 0$ we get $\left(\frac{1}{|G|}x\right)^2 = \frac{1}{|G|}x = e$

$\Rightarrow KG = eKG \oplus (1-e)KG$

(7) Recall $\mathcal{H} = \{ f : \mathbb{C} \longrightarrow \mathbb{C} \mid f \text{ holomorphic} \}$.

$\mathcal{H}$: domain but does not admit (finite) factorizations e.g. $f(z) = \sin z$.

So $\mathcal{H}$: not Noetherian.

Find $I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \dots \mathcal{H}$

$$\sin z = \prod_{k=0}^{n-1} (z - \pi k) \, f_n(z) \qquad f_0(z) = \sin z$$

$$(f_0) \subseteq (f_1) \subseteq (f_2) \subseteq \dots \mathcal{H}$$

$f_k \notin (f_{k-1})$ because $f_k(\pi(k-1)) \neq 0$

Take any discrete $Z \subseteq \mathbb{C}$

$$I_Z = \{ f \in \mathcal{H} \mid f(z) = 0 \ \forall z \in Z \}$$

Others $I = \bigcup I_{\{n, n+1, \dots\}} \subsetneq \mathcal{H}$

(8) Let p : prime number

$$\mathbb{Z}(p^\infty) = \{ z \in \mathbb{C}^\times \mid z^{p^n} = 1 \text{ for some } n \geq 1 \} \subseteq \mathbb{C}^\times \quad \text{Prüfer group.}$$

Let $R = S^{-1}\mathbb{Z}$ $\quad S = \{ p^n \mid n \geq 0 \}$

$$G = (R, +)$$

Show $G/\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}(p^\infty)$

$$G = \{ \tfrac{a}{p^n} \mid a \in \mathbb{Z}, n \geq 0 \}$$

Idea $\quad \frac{a}{p^n} \longmapsto e^{\frac{2\pi i a}{p^n}}$

$$\varphi : (\mathbb{Q}, +) \longrightarrow \mathbb{C}^\times \qquad \ker \varphi = \mathbb{Z}$$
$$x \longmapsto e^{2\pi i x}$$

$$\begin{tikzcd}
\mathbb{Z} \arrow[rr, hook] \arrow[dr] & & \mathbb{Q} \\
& S^{-1}\mathbb{Z} \arrow[ur, "\psi"] &
\end{tikzcd}
\qquad p^{-n}a \mapsto \frac{a}{p^n}$$

$$\operatorname{im} \varphi\psi = \{ e^{\frac{2\pi i a}{p^n}} \mid a \in \mathbb{Z}, n \geq 0 \} = \mathbb{Z}(p^\infty)$$

Now $\quad G \xrightarrow{\varphi\psi} \mathbb{Z}(p^\infty)$ surj.

ker is $\mathbb{Z}$ so $G/\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}(p^\infty)$.