Generators and relations

Recall:

T : type $\quad T \longrightarrow \mathbb{N}$, $\quad \omega \longmapsto n_\omega$ "arity of ω"

A : T-alg $\quad \omega_A : A^{n_\omega} \longrightarrow A$ "operators"

If $n_\omega = 0$ $\quad \omega_A \in A$ constant

A, B : T-alg $\quad \varphi : A \longrightarrow B$ morphism

$$\varphi(\omega_A(x_1 \dots x_{n_\omega})) = \omega_B(\varphi(x_1), \dots, \varphi(x_{n_\omega}))$$

If φ is bijective then φ: isomorphism.
then φ^{-1} is also an isomorphism, $A \simeq B$ "isomorphic"

X : variables

W_X^T : expressions in X using $\omega \in T$

If $f : X \to A$ $\quad \exists!\ \varphi_f : W_X^T \to A$ morphism $\varphi_f(x) = f(x)$

Fix $\mathbb{X}$ countable

If $w_1, w_2 \in W_{\mathbb{X}}^T$ we say $w_1 = w_2$ is satisfied in A
if $\varphi_f(w_1) = \varphi_f(w_2)$ $\quad \forall\ f : \mathbb{X} \to A$

Let $I \subseteq W_{\mathbb{X}}^T \times W_{\mathbb{X}}^T$.

Then $V = V(I)$: T-alg's satisfying $w_1 = w_2$ $\forall (w_1, w_2) \in I$.

1 Quotients

Let A : T-algebra and $\sim$ is an equivalence relation on A.

<u>Def</u> We call $\sim$ a <u>congruence</u> if $\forall \omega \in T$

$$x_i \sim x_i' \quad 1 \le i \le n_\omega \implies \omega_A(x_1 \dots x_{n_\omega}) \sim \omega_A(x_1', \dots x_{n_\omega}')$$

Then $A/\sim = \{\bar{x} \mid x \in A\}$ has a unique T-alg structure st

$$\begin{aligned} \pi : A &\longrightarrow A/\sim \\ x &\longmapsto \bar{x} \end{aligned}$$

is morphism.

namely $\omega_{A/\sim}(\bar{x}_1, \dots, \bar{x}_{n_\omega}) = \overline{\omega_A(x_1, \dots, x_{n_\omega})}$

<u>Ex</u> Let $\varphi : A \to B$. Define

$$x \sim x' :\iff \varphi(x) = \varphi(x')$$

If φ morphism then $\sim$ is a congruence.

$\ulcorner$ If $x_i \sim x_i' \implies \varphi(\omega_A(x_1, \dots x_{n_\omega})) = \omega_B(\varphi(x_1) \dots, \varphi(x_{n_\omega}))$

$$= \omega_B(\varphi(x_1'), \dots, \varphi(x_{n_\omega}'))$$

$$= \varphi(\omega_A(x_1', \dots, x_{n_\omega}'))$$

$\implies \omega_A(x_1, \dots, x_{n_\omega}) \sim \omega_A(x_1', \dots, x_{n_\omega}')$. $\lrcorner$

Thm (Isomorphism) If $\varphi: A \to B$ morphism

then $\tilde{\varphi}: A/\sim \longrightarrow \varphi(A)$, $\bar{x} \longmapsto \varphi(x)$

is an isomorphism, and

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & B \\ A/\sim \arrow[r, "\tilde{\varphi}"] & \varphi(A) \arrow[u, hook] \end{tikzcd}$$

proof $\tilde{\varphi}$ well-defined and bijective by construction.

$$\tilde{\varphi}(\pi(x)) = \tilde{\varphi}(\bar{x}) = \varphi(x). \qquad \square$$

Proposition Let $V = V(I)$ is a variety of T-algebras. Then V is closed under subalgebras, quotients, direct products and directed unions.

proof Subalgebras $B \subseteq A$

If $w_1 = w_2$ is satisfied in A

then —— " —————— in B

direct products $\{A_j\}_{j \in J}$ $A_j \in V$

$$A = \prod_{j \in J} A_j \qquad \omega_A((x_j^1)_{j\in J}, \ldots, (x_j^{n_\omega})_{j \in J})$$
$$= (\omega_{A_j}(x_j^1, \ldots, x_j^{n_\omega}))_{j \in J}.$$

If $w_1 = w_2$ can be checked for each component $j \in J$ separately.

quotients Let $\sim$ is a congruence on A

Let $(w_1, w_2) \in I$.

Let $X \xrightarrow{f} A/\sim$

Choose $X \xrightarrow{f'} A$ st $\pi \circ f' = f$

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[dr, "f"'] & A \arrow[d, "\pi"] \\ & A/\sim \end{tikzcd}$$

$$\begin{tikzcd} W_X^T \arrow[r, "\varphi_{f'}"] \arrow[dr, "\varphi_f"'] & A \arrow[d, "\pi"] \arrow[dl, phantom, "\circlearrowleft"] \\ & A/\sim \end{tikzcd}$$

commutes by uniqueness of φ_f.

$\varphi_{f'}(w_1) = \varphi_{f'}(w_2)$ as $A \in V$ (apply π)

$\Rightarrow \varphi_f(w_1) = \varphi_f(w_2) \Rightarrow A/\sim \in V$.

direct union $\{A_j\}_{j\in J}$ is totally ordered by $\subseteq$

$B = \bigcup_{j\in J} A_j$, $A_j \in V$.

$w_1 = w_2$ involves only finitely many variables.

So it can be checked in B by considering some A_j. $\square$

Rmk Let R, S domains $R \times S$ is not a domain.

as $(1,0)(0,1) = (0,0)$.

Ex Let G: group $N \subseteq G$ subgroup.

$gN := \{gh \mid h \in N\}$ $Ng = \{hg \mid h \in N\}$.

N: normal $gN = Ng$ $\forall g \in G$.

define $g \sim h$ by $gN = hN$

$(\Leftrightarrow Ng = Nh \Leftrightarrow gh^{-1} \in N \Leftrightarrow hg^{-1} \in N)$

Then $\sim$ is a congruence

$\omega = \varepsilon$ constant automatic.

$\omega = \mu$ Assume $g_1 \sim h_1$ $g_2 \sim h_2$

$g_1 g_2 N = g_1 N g_2 = h_1 N g_2 = h_1 N h_2 = h_1 h_2 N$

so $\mu(g_1, g_2) \sim \mu(h_1, h_2)$.

$\omega = \iota$ $g \sim h$ $g^{-1}N = Ng^{-1} = h^{-1}hNg^{-1} = h^{-1}N \underbrace{hg^{-1}}_{\in N}$
$= h^{-1}N$.

so $\iota(g) \sim \iota(h)$.

Write $G/\sim = G/N$

Let $\varphi: G \to H$ $\ker\varphi := \varphi^{-1}(1_H) \subseteq G$
is a normal subgroup.

$$\varphi(g) = \varphi(h) \Leftrightarrow \underset{\displaystyle \varphi(gh^{-1})}{\underset{\shortparallel}{\varphi(g)\varphi(h)^{-1}}} = 1_H$$

$\Leftrightarrow gh^{-1} \in \ker\varphi$

2) Let R ring. subgroup $I \subseteq R$ wrt $+$.
is called ideal if $\forall a \in R, \forall x \in I$ $ax \in I$, $xa \in I$.
then $x \sim y :\Leftrightarrow x - y \in I$ is a congruence

$R/\sim = R/I$

3) For semigroups congruences are not given by subsemigroups.

2 Free algebras

Let $V = V(I)$ variety of T-algebras.

Def Let $F \in V$ and X set with $X \xrightarrow{f} F$.

We say F is free on X in V if

$\forall B \in V$ with $g: X \to B$ $\exists!\ \varphi : F \to B$ morphism

st $\varphi(f(x)) = g(x)$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, bend right, "g"'] & F \arrow[d, "\exists!\varphi"] \\ & B \end{tikzcd}$$

Rmk 1) If A, A' free on X we have

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, bend right, "f'"'] & A \arrow[d, shift right, "\varphi"'] \\ & A' \arrow[u, shift right, "\varphi'"'] \end{tikzcd}$$

By uniqueness $\varphi \circ \varphi' = id_{A'}$

$\varphi' \circ \varphi = id_A$

So $A \simeq A'$

2) If $I = \emptyset$ ie $V =$ all T-algebras.

then W_X^T is free on X.

<u>Construction</u> Let $\sim$ be the smallest congruence on W_X^T st $w_1 = w_2$ is satisfied in $W_X^T/\sim$ for all $(w_1, w_2) \in I$.

Set $F = W_X^T/\sim$ and
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & F \\ W_X^T \arrow[ur, two heads, "\pi"'] & \end{tikzcd}$$

Note $F \in V$.

<u>Claim</u> F is free on X in V

Let $g : X \to B \in V$ morphism.
$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & B \\ W_X^T \arrow[ur, "\varphi_g"'] & \end{tikzcd}$$

φ_g defines a congruence on W_X^T which satisfies the condition as $B \in V$

So since $\sim$ is smallest we get
$$\begin{tikzcd} & X \arrow[dl] \arrow[dr, "g"] & \\ W_X^T \arrow[rr, "\varphi_g"] \arrow[rr, phantom, bend left, "\circlearrowleft"] \arrow[dr, two heads, "\pi"'] & & B \\ & F = W_X^T/\sim \arrow[ur, "\varphi"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

$\square$.

Ex 1) V = monoids $X = \{a, b, c\}$.

F = strings in X

$= \{1, a, b, c, aa, ab, ac, \dots, aabca$

Now let M a monoid

$f : X \to M$, $f(a) = A$
$f(b) = B$
$f(c) = C$

then $\varphi : F \to M$, $abcaab \mapsto ABCAAB$.

e.g. $M = (\mathbb{N}, +)$, $A = 2$, $B = 0$, $C = 1$

$\varphi(abcaab) = 2 + 0 + 1 + 2 + 2 + 1 = 8$.

2) $X = \{x, y\}$ V = rings.

$F = \mathbb{Z}\langle x, y\rangle$ noncommutative polynomials in x, y

e.g. $1 + 2x + y$ $\quad 4 + 3x^2 - y^2$

$xyx + yx^2 + x^2y$

3) $X = \{x, y\}$ V: commutative rings.

$F = \mathbb{Z}[x, y]$ polynomials in x, y

$f : X \to \mathbb{C}$ $f(x) = 1$ $f(y) = i$

$\varphi : F \to \mathbb{C}$, $1 + x + y \mapsto 1 + 1 + i = 2 + i$.

4) $X = \emptyset$ V = rings $F = ?$

3 Relations

Let V : variety

X : set

F = free on X in V. $\quad X \xrightarrow{f} F$.

$R \subseteq F \times F$ relations $(r_1, r_2) \in R$.

Let $\sim$ be the smallest congruence on F

st $r_1 \sim r_2 \quad \forall (r_1, r_2) \in R$.

The algebra in V given by generators X and relations R is

$$\langle X \mid R \rangle := F/\sim$$

Ex V = Groups $\quad X = \{r, s\}$.

$F = \{1, r, s, \cancel{r^{-1}}, \cancel{s^{-1}}, r^2, rs, \cancel{sr}, s^2, r^2s, \ldots \quad r^{-1}sr^2srs^{-1}\}$

Let $R = \{(s^2, 1), (r^3, 1), (srs, r^{-1})\}$.

$$G = \langle X \mid R \rangle = \langle r, s \mid s^2 = 1, r^3 = 1, srs = r^{-1} \rangle.$$

dihedral group

$s^2 = 1 \implies s = s^{-1}$

$r^3 = 1 \implies r^2 = r^{-1}$

$srs = r^{-1} = r^2 \implies sr = r^2 s$

$G = \{1, r, r^2, s, sr, sr^2\}$

Define $\varphi : G \longrightarrow GL_2(\mathbb{R})$

$$s \longmapsto S$$

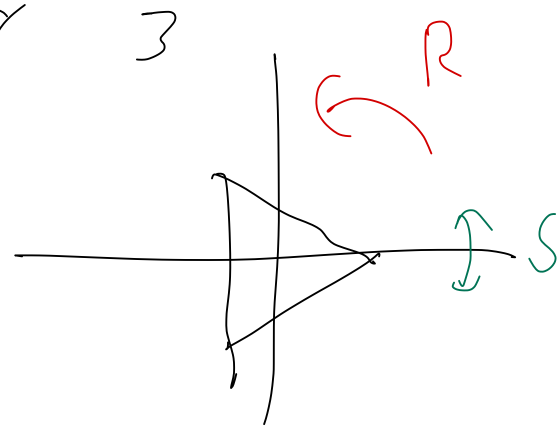
$$r \longmapsto R$$

such that $S^2 = I$, $R^3 = I$

$$SRS = R$$

e.g. R: rotation by $\frac{2\pi}{3}$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



then I, R, R^2

S SR SR^2

are distinct so $|G| = 6$