Categories and functors

1 Categories

Def A category $\mathcal{C}$ consists of

- A class of objects $\mathrm{ob}\,\mathcal{C}$
- For all $X, Y \in \mathrm{ob}\,\mathcal{C}$ a set
  $\mathcal{C}(X, Y)$ $(= \mathrm{Hom}_{\mathcal{C}}(X, Y))$ of morphisms from X to Y.
- Composition maps
  $\mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(X, Z)$, $(g, f) \longmapsto g \circ f$

such that all morphism sets are disjoint

- $\exists\, 1_X \in \mathcal{C}(X, X)$ st $1_X \circ f = f$ $\quad g \circ 1_X = g$
- $h \circ (g \circ f) = (h \circ g) \circ f$.

Notation $f \in \mathcal{C}(X, Y)$ write $X \xrightarrow{f} Y$

$f \in \mathcal{C}(X, Y)$ is an isomorphism if
$\exists\, g \in \mathcal{C}(Y, X)$ st $f \circ g = 1_Y$ $\quad g \circ f = 1_X$. $\quad X \simeq Y$.

Ex 1) Sets $\quad$ objects : sets
morphisms : functions
$1_X : X \to X$, $x \longmapsto x$
$\circ$: function composition.

2) Let T : type
T-alg $\quad$ objects : T-algebras
morphisms : morphisms of T-algebras.

<u>Def</u> Let $\mathcal{C}$: category. We call $\mathcal{D}$ a <u>subcategory</u> of $\mathcal{C}$ if $\mathrm{ob}\,\mathcal{D} \subseteq \mathrm{ob}\,\mathcal{C}$, $\mathcal{D}(X,Y) \subseteq \mathcal{C}(X,Y)$ $\forall X, Y \in \mathrm{ob}\,\mathcal{D}$, and identities and composition are induced.

$\mathcal{D}$ is called <u>full</u> if $\mathcal{D}(X,Y) = \mathcal{C}(X,Y)$.

3) Let T: type and $\mathcal{C}$ is a class of T-alg's (e.g. $\mathcal{C} = V(I)$). Then we regard

$$\mathcal{C} \subseteq T\text{-alg} \text{ as a full subcategory}$$

so Semigroups, Monoids, Groups, Rings, ComRings, Domains, Fields,

4) Let K: field. $\mathrm{Vec}(K)$: vectorspaces / K with linear maps.

$\mathrm{vec}(K) \subseteq \mathrm{Vec}(K)$ full subcategory of finite dimensional vector spaces.

5) Let M: monoid. Define $\mathcal{C}_M$: category

$\mathrm{ob}\,\mathcal{C}_M = \{*\}$

$\mathcal{C}_M(*,*) = M$.

composition : multiplication.

6) Top : topological spaces with continuous maps.

$\mathrm{Haus} \subseteq \mathrm{Top}$ full subcategory of Hausdorff top. spaces.

7) Top_0 : Objects (X, x_0) $X \in \mathrm{Top}$, $x_0 \in X$

morphisms $(X, x_0) \xrightarrow{f} (Y, y_0)$, $f: X \to Y$ continuous, $f(x_0) = y_0$.

2 Functors

Def Let $\mathcal{C}, \mathcal{D}$ categories

A <u>covariant</u> (contravariant) functor $F : \mathcal{C} \to \mathcal{D}$ consists of

- For each $X \in ob\,\mathcal{C}$ an object $F(X) \in ob\,\mathcal{D}$.
- For every $X \xrightarrow{f} Y$ in $\mathcal{C}$ a morphism $F(X) \xrightarrow{F(f)} F(Y)$ in $\mathcal{D}$
  $F(X) \xleftarrow{F(f)} F(Y)$

s.t. a) $F(1_X) = 1_{F(X)}$

b) $F(g \circ f) = F(g) \circ F(f)$.
$F(g \circ f) = F(f) \circ F(g)$

<u>Ex</u> 1) Let $\mathcal{C}$: category and $X \in ob\,\mathcal{C}$

covariant

$$\begin{aligned} \mathcal{C}(X,-) : \mathcal{C} &\longrightarrow Sets \\ Y &\longmapsto \mathcal{C}(X,Y) \\ (Y \xrightarrow{f} Z) &\longmapsto \left(\mathcal{C}(X,Y) \xrightarrow{f\circ -} \mathcal{C}(X,Z) \right) \\ g &\longmapsto f \circ g \end{aligned}$$

contravariant

$$\begin{aligned} \mathcal{C}(-,X) : \mathcal{C} &\longrightarrow Sets \\ Y &\longmapsto \mathcal{C}(Y,X) \\ (Y \xrightarrow{f} Z) &\longmapsto \left(\mathcal{C}(Y,X) \xleftarrow{-\circ f} \mathcal{C}(Z,X) \right) \\ g \circ f &\longmapsfrom g \end{aligned}$$

<u>Alt</u> $\mathcal{C}(-_1,-_2) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow Sets$ covariant

$ob\,\mathcal{C}^{op} = ob\,\mathcal{C}$ $\quad \mathcal{C}^{op}(X,Y) = \mathcal{C}(Y,X)$ $\quad g \circ_{op} f = f \circ g$.

2) $\mathcal{C} = Vec(K)$

dual space
contravariant

$D : Vec(K) \longrightarrow Vec(K)$ $\quad (= Vec(K)(V, K))$

$V \longmapsto Hom_K(V, K)$

$D(f) = - \circ f$

3) Let M : monoid $\quad M^\times = \{ m \in M \mid m \text{ invertible} \}$.

$M^\times$ is a group.

If $\varphi : M \to N$ monoid morphism

then $\varphi(M^\times) \subseteq N^\times$ (check)

and $\varphi|_{M^\times} : M^\times \longrightarrow N^\times$ is a group morphism.

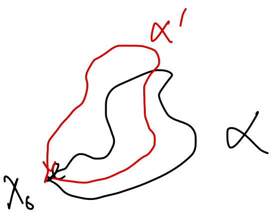
So $(\ )^\times : \text{Monoids} \longrightarrow \text{Groups}$ covariant functor.

similarly $(\ )^\times : \text{Rings} \to \text{Groups}$

$R \longmapsto (R, \cdot)^\times$

5) Let $(X, x_0) \in \text{Top}_0$

$$\pi_1(X, x_0) = \frac{\{ \alpha : [0,1] \to X \text{ continuous} \mid \alpha(0) = \alpha(1) = x_0 \}}{\text{homotopy}}$$



$\pi_1(X, x_0)$ group $\quad \overline{\alpha}\,\overline{\alpha'} = \overline{\alpha''}$

$$\alpha'' = \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ \alpha'(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

$(X, x_0) \xrightarrow{f} (Y, y_0) \quad \rightsquigarrow \quad f_* : \pi(X, x_0) \longrightarrow \pi(Y, y_0)$

$\overline{\alpha} \longmapsto \overline{f \circ \alpha}$

In fact $\pi_1 : \text{Top}_0 \longrightarrow \text{Groups}$.

6) Let R commutative ring

$$GL_n(R) = \{A : n\times n\text{-matrices} / R \mid A \text{ invertible}\}$$
$$= (M_{n\times n}(R))^\times$$

$f : R \to S$ ring morphism

$$\tilde{f} : M_{n\times n}(R) \to M_{n\times n}(S),\ (a_{ij}) \mapsto (f(a_{ij}))$$

ring morphism

$$\tilde{f}^\times : M_{n\times n}(R)^\times \to M_{n\times n}(S)^\times$$

$$GL_n(f) : GL_n(R) \longrightarrow GL_n(S) \quad \text{group morphism.}$$

$$GL_n : \text{ComRings} \to \text{Groups.}$$

3 Natural transformations

Let $\mathcal{C}, \mathcal{D}$: categories

$F, G : \mathcal{C} \to \mathcal{D}$ functors.

A natural transformation $\phi : F \to G$ consists of morphisms $\phi_X : F(X) \to G(X)$ in $\mathcal{D}$ for each $X \in \mathcal{C}$.

st $\forall f \in \mathcal{C}(X, Y)$

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"'] & F(Y) \arrow[d, "\phi_Y"] \\
G(X) \arrow[r, "G(f)"] & G(Y)
\end{tikzcd}$$

commutes

i.e. $G(f) \circ \phi_X = \phi_Y \circ F(f)$

If ϕ_X isomorphism $G(f) = \phi_Y \circ F(f) \circ \phi_X^{-1}$. Then we call ϕ a natural isomorphism and $F \simeq G$.

$\underline{\text{Ex}}$ 1) Recall $GL_n : \text{ComRings} \to \text{Groups}$

$(-)^\times : \text{Com Rings} \to \text{Groups}$

$\det : GL_n \to (-)^\times$

$\det_R : GL_n(R) \to R^\times$

$A = (a_{ij}) \longmapsto \det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} \in R^\times$

show later.

group morphism as $\det_R(AB) = \det_R(A)\det(B)$.

$\varphi : R \to S$ ring morphism

$$\begin{tikzcd} GL_n(R) \arrow[r, "GL_n(\varphi)"] \arrow[d, "\det_R"'] & GL_n(S) \arrow[d, "\det_S"] \\ R^\times \arrow[r, "\varphi|_{R^\times}"] & S^\times \end{tikzcd}$$

$$\begin{tikzcd} A = (a_{ij}) \arrow[r, mapsto] \arrow[d, mapsto] & B = (\varphi(a_{ij})) \arrow[d, mapsto] \\ \sum_\sigma \operatorname{sign}(\sigma) \prod_i a_{i\sigma(i)} \arrow[r, mapsto, "\varphi"] & \sum_\sigma \operatorname{sign}(\sigma) \prod_i \varphi(a_{i\sigma(i)}) \end{tikzcd}$$

2) Let $D : \text{Vec}(K) \to \text{Vec}(K)$

$\varphi_V \; V \longrightarrow D(D(V))$ linear

$v \longmapsto (\underset{D(V)}{f} \longmapsto \underset{K}{f(v)}) \in D(D(V))$

If $T : V \to W$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\varphi_V"'] \arrow[dr, phantom, "G"] & W \arrow[d, "\varphi_W"] \\ D(D(V)) \arrow[r, "D(D(T))"] & D(D(W)) \end{tikzcd}$$

$$\begin{tikzcd} v \arrow[r, mapsto] \arrow[d, mapsto] & T(v) \arrow[d, mapsto] \\ \varphi_V(v) \arrow[r, mapsto] & D(D(T))(\varphi_V(v)) = \varphi_V(v) \circ D(T) : f \mapsto f(T(v)) \end{tikzcd}$$

$\varphi_W(T(v)) : f \longmapsto f(T(v))$

4 Equivalences and adjoints

Def A functor $F : \mathcal{C} \to \mathcal{D}$ is called an **equivalence** if $\exists\, G : \mathcal{D} \to \mathcal{C}$ st

$$F \circ G \simeq id_{\mathcal{D}}$$

$$G \circ F \simeq id_{\mathcal{C}}$$

Ex $D : Vec(K) \to Vec(K)$

$\varphi_V : V \longrightarrow D(D(V))$ is injective

$$\varphi_V(v) = 0 \Rightarrow \underset{\substack{\| \\ f(v)}}{\varphi_V(v)(f)} = 0 \quad \forall f \Rightarrow v = 0.$$

But if $\dim V < \infty$ then $\underset{\substack{\| \\ \dim D(D(V))}}{\dim D(V)} = \dim V.$

so φ_V is bijective.

So $D : vec(K) \to vec(K)$ is an equivalence

as $\varphi : id_{vec(K)} \xrightarrow{\sim} D \circ D$ natural isomorphism.

Def Let $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$ functors

we call (F, G) an adjoint pair if

$$\mathcal{C}(FY, X) \xrightarrow{\sim} \mathcal{D}(Y, GX) \quad \text{natural in } X, Y$$

i.e. $\mathcal{C}(F-_1, -_2) \simeq \mathcal{D}(-_1, G-_2)$ as

functors $\mathcal{D}^{op} \times \mathcal{C} \longrightarrow$ sets.

Ex 1) Let V : variety of T-algebras

$V \subseteq$ T-alg full subcat.

$$F : \text{Sets} \longrightarrow V$$
$$X \longmapsto F(X) \quad \text{free on } X \text{ in } V \quad X \xrightarrow{f_X} F(X)$$

$X \xrightarrow{g} Y$ map

$$\begin{tikzcd} X \arrow[r, "f_X"] \arrow[d, "g"'] & F(X) \arrow[d, dashed, "\exists!\varphi"'] \\ Y \arrow[r, "f_Y"] & F(Y) \end{tikzcd} \quad =: F(g) \qquad \text{uniqueness} \Rightarrow F(g \circ h) = F(g) \circ F(h)$$

$G : V \longrightarrow$ Sets, forgetful functor

$A \longmapsto A$ underlying set

$\varphi \longmapsto \varphi$ underlying function.

$\mathrm{Hom}_V(F(Y), A) = \{ \varphi : F(Y) \to A \text{ morphism} \}$.

$\mathrm{Hom}_{sets}(Y, G(A)) = \{ g : Y \to A \mid g \text{ function} \}$.

$$\begin{tikzcd} Y \arrow[r, "f_Y"] \arrow[dr, "\forall g"'] & F(Y) \arrow[d, dashed, "\exists!\varphi"] \\ & A \end{tikzcd}$$

so $\Phi_{Y,A} : \mathrm{Hom}_V(F(Y), A) \longrightarrow \mathrm{Hom}_{sets}(Y, G(A))$
$$\varphi \longmapsto \varphi \circ f_Y$$

is bijective i.e. (F, G) adjoint pair.