Actions and representations

1 Actions

Recall X sets

$$M_X = \{f : X \to X\} \quad \text{monoid with } \circ$$

$$S_X = (M_X)^\times = \{f : X \to X \mid f \text{ bijective}\} \quad \text{group}$$

$X = \{1, \dots, n\}$ $\quad S_n = S_X$ symmetric group of permutations.

If $\mathcal{C}$: category $\quad X \in \mathcal{C}$

$\mathrm{End}_{\mathcal{C}}(X) = \mathcal{C}(X, X)$: monoid with $\circ$.

$\mathrm{Aut}_{\mathcal{C}}(X) = (\mathrm{End}_{\mathcal{C}}(X))^\times$

Ex 1) $\mathcal{C} = \text{sets} \ni X \quad \mathrm{End}_{\mathcal{C}}(X) = M_X, \quad \mathrm{Aut}_{\mathcal{C}}(X) = S_X$

2) $\mathcal{C} = \mathrm{Vec}(K) \ni V \quad \mathrm{End}_{\mathcal{C}}(V) = \mathrm{End}_K(V)$: K-linear endomorphisms.

$$V = K^n \qquad \mathrm{End}_K(V) \xrightarrow{\sim} M_{n\times n}(K)$$
$$T \longmapsto [T]$$

$\mathrm{Aut}_{\mathcal{C}}(V) = GL(V) = \{f : V \to V \mid f \text{ linear, bijective}\}$

$$V = K^n \qquad GL(V) \simeq GL_n(K)$$
$$T \longmapsto [T].$$

Def Let M : monoid $\quad X$: set. An action on X of M is

$$\mu : M \times X \longrightarrow X, \quad (m, x) \longmapsto mx$$

s.t. $\quad 1x = x$

$\quad (ab)x = a(bx)$

Rmk 1) $\{\mu : M \times X \to X\} \xleftrightarrow{1:1} \{\rho : M \to M_X\}$

$$\mu \longleftrightarrow \rho$$

iff $\mu(m,x) = (\rho(m))(x)$ $\qquad (A^B)^C = A^{BC}$

2) In the above μ is an action iff ρ is a monoid morphism

$1x = x \iff \rho(1) = id_X$

$(ab)x = a(bx) \iff \rho(ab) = \rho(a) \circ \rho(b)$

3) If $M = G$ is a group

$\{\rho : M \to M_X \mid \rho \text{ monoid morphism}\} \xleftrightarrow{1:1} \{\rho' : G \to S_X \mid \rho' \text{ group morphism}\}$

$\rho \longmapsto \rho' \qquad \rho'(g) = \rho(g).$

indeed $g \in G$ inv $\Rightarrow \rho(g)$ inv.

$g_1 g_2 = 1 \Rightarrow \rho(g_1)\rho(g_2) = id_X$

So a group action

$\mu : G \times X \to X \qquad (a,x) \mapsto ax$

$1x = x$

$a(bx) = (ab)x$

has the same information as

$\rho : G \to S_X, \quad \rho(g_1 g_2) = \rho(g_1) \circ \rho(g_2)$

Q Why do we need $1x = x$

Otherwise $x \mapsto ax$ might not be bijective

e.g. μ: constant.

2 Actions in categories

Def Let M: monoid, G group, $X \in \mathcal{C}$ $\quad$ $\mathcal{C}$: category

A $\mathcal{C}$-action of M (or G) on X is

$$\rho : M \longrightarrow \mathrm{End}_{\mathcal{C}}(X) \quad \text{monoid morphism}$$

$$\left(\text{or } \rho : G \longrightarrow \mathrm{Aut}_{\mathcal{C}}(X) \quad \text{group morphism}\right)$$

Ex $\mathcal{C} = Ab$ abelian groups $\quad A \in \mathcal{C}$

$\rho : M \longrightarrow \mathrm{End}_{\mathcal{C}}(A)$ $\qquad$ as actions

$\rho(1_M) = id_A$ $\qquad$ $1x = x$

$\rho(ab) = \rho(a) \circ \rho(b)$ $\qquad$ $(ab)x = a(bx)$

$\rho(a)(x+y) = \rho(a)x + \rho(a)y$ $\qquad$ $a(x+y) = ax + ay$.

Popular choice $\mathcal{C} = \mathrm{Vec}(K)$ $\quad K$ field

or $\mathcal{C} = \mathrm{vec}(K)$ $\quad K = \mathbb{C}$.

Terminology Let G group. A representation of G is (ρ, V), $V \in \mathrm{Vec}(K)$

and $\rho : G \longrightarrow GL(V)$ group morphism.

i.e. ρ is a linear action of G on V

Ex $G = \langle s, r \mid s^2 = 1,\ r^n = 1,\ srs = r^{-1} \rangle$

$$\rho : G \longrightarrow GL(\mathbb{R}^2) \qquad \text{representation.}$$

$$s \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$r \longmapsto \begin{bmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{bmatrix}$$

Def Let M: monoid, $X, Y \in \mathcal{C}$ and

$\rho_1 : M \to \mathrm{End}_{\mathcal{C}}(X)$, $\rho_2 : M \to \mathrm{End}_{\mathcal{C}}(Y)$.

A morphism $\varphi : \rho_1 \to \rho_2$ is $\varphi \in \mathcal{C}(X, Y)$

st $\forall m \in M$

$$\begin{tikzcd} X \arrow[r, "\rho_1(m)"] \arrow[d, "\varphi"'] & X \arrow[d, "\varphi"] \\ Y \arrow[r, "\rho_2(m)"] & Y \end{tikzcd}$$

commute

$$\rho_2(m) \circ \varphi = \varphi \circ \rho_1(m)$$

Ex Let $V, W \in \mathrm{Vec}(K)$ and $\varphi : V \to W$ linear

$(K, \cdot)$ acts on V and W in Ab

as
$$1v = v$$
$$(a(bv)) = (ab)v$$
$$a(v+w) = av + aw$$

$\varphi \in Ab(V, W)$ as $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2)$

φ is a morphism of actions as

$$\varphi(av) = a\varphi(v)$$

Def Let G: group. $\mathrm{Rep}_K G$: category of rep's

objects: $\rho : G \to GL(V)$

morphisms: $\varphi : \rho_1 \to \rho_2$ morphism of actions in $\mathrm{Vec}(K)$

$\mathrm{rep}_K G \subseteq \mathrm{Rep}_K G$ full subcategory of (ρ, V) where $\dim V < \infty$.

3 Complex rep's of finite groups

Let G group $|G| < \infty$

<u>Aim</u> understand $\text{rep}_{\mathbb{C}} G$

<u>Def</u> Let (ρ, V) be a representation

1) A subrepresentation is a subspace $V' \subseteq V$ st $\rho(g)(V') = V'$ $\forall g \in G$

(ρ', V') is a representation $\rho'(g)v = \rho(g)v$

inclusion $V' \hookrightarrow V$ is a morphism.

2) (ρ, V) is called irreducible if $V \neq \{0\}$ and $\{0\}$, V are the only subrep's of (ρ, V).

<u>Ex</u> If $\dim V = 1$ then (ρ, V) is irreducible.

<u>Proposition</u> If G abelian, then (ρ, V) irreducible iff $\dim V = 1$

<u>proof</u> "$\Leftarrow$" clear

"$\Rightarrow$" Let $g \in G$ $\rho(g) = T : V \to V$ Let $\lambda \in \mathbb{C}$ eigenvalue of T so $E_\lambda := \ker(T - \lambda I) \subseteq V$

(ρ, V) irr. $\dim V \geq 1$

i.e $v \in E_\lambda \Leftrightarrow \rho(g)v = \lambda v$

Now let $v \in E_\lambda$

$\rho(g)\rho(h)v = \rho(gh)v = \rho(hg)v = \rho(h)\rho(g)v = \rho(h)\lambda v = \lambda\rho(h)v$

$\rho(h)v \in E_\lambda$ so $V = E_\lambda$ $g \in G$ any

$\rho(g)v = \lambda_g v$ for some $\lambda_g \in \mathbb{C}$ $\forall v \in V$.

Then any $V' \subseteq V$ is a subrepresentation.

(ρ, V) irreducible $\dim = V$. $\square$.

<u>Lemma</u> Let $(\rho, V) \in \mathrm{rep}_{\mathbb{C}} G$. Then there is an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ s.t.

$$\langle \rho(g) v, \rho(g) w \rangle = \langle v, w \rangle \quad \forall g \in G$$

i.e. $\rho(g) \in U(V)$.

<u>proof</u> Since $\dim V < \infty$ $V \simeq \mathbb{C}^n$ for some n

so V admits an inner product $(\cdot, \cdot)$

set $\langle v, w \rangle = \sum_{g \in G} (\rho(g) v, \rho(g) w)$ inner product

Now $\langle \rho(h) v, \rho(h) w \rangle = \sum_{g \in G} \langle \rho(g) \rho(h) v, \rho(g) \rho(h) w \rangle$

$= \sum_{g \in G} \langle \rho(gh) v, \rho(gh) w \rangle = \sum_{gh \in G} \langle \rho(gh) v, \rho(gh) w \rangle$

$\left(\begin{array}{l} G \to G \\ g \mapsto gh \end{array} \text{ bijective} \right)$ $= \langle v, w \rangle$ $\square$.

<u>Proposition</u> Let $(\rho, V) \in \mathrm{rep}_{\mathbb{C}} G$ and $V' \subsetneq V$ subrepresentation.

Then $\exists V'' \subseteq V$ subrep. st $V' \oplus V'' = V$

<u>proof</u> choose $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ as in Lemma

set $V'' = (V')^{\perp} = \{ v \in V \mid \langle v, v' \rangle = 0 \ \forall v' \in V' \}$.

Then $V = V' \oplus V''$ and $V'' \subseteq V$ is a subrep.

Let $v'' \in V''$ $\forall v' \in V'$ $\langle \rho(g) v'', v' \rangle = \langle v'', \rho(g)^{-1} v' \rangle = \langle v'', \underbrace{\rho(g^{-1}) v'}_{\in V'} \rangle = 0$ $\square$

$\Rightarrow \rho(g) v'' \in V''$.

Rmk This proposition works even if $|G| = \infty$ if we can find $\langle \cdot, \cdot \rangle$.

Theorem (Maschke) Let $(\rho, V) \in \mathrm{rep}_{\mathbb{C}} G$.

Then $V = \bigoplus_{i=1}^{n} V_i$ for some (ρ_i, V_i) irreducible

proof Induction $\dim V$

If $\dim V = 0$ then $n = 0$

If $\dim V > 0$ Let $0 \neq V_1 \subseteq V$ subrep of $\dim V_1$ minimal

$\dim V_1 < \infty \Rightarrow V_1$ irreducible

[Indeed $0 \neq U \subseteq V_1$ the $\dim U = \dim V_1$ (as $< \infty$)

so $U = V_1$]

Now $V = V_1 \oplus V''$ $V'' \subseteq V$ subrep.

$\dim V'' = \dim V - \dim V_1 < \dim V$

By induction hypothesis $V'' = \bigoplus_{i=2}^{n} V_i$ (ρ_i, V_i) irr.

and $V = \bigoplus_{i=1}^{n} V_i$ $\square$

Ex Let $G = S_3$ symmetric group on $\{1,2,3\}$

$V = \mathbb{C}^3 \qquad \rho(\sigma) e_i = e_{\sigma(i)}$

$$\rho(\tau)(\rho(\sigma) e_i) = \rho(\tau)(e_{\sigma(i)}) = e_{\tau\sigma(i)}$$

$$= e_{(\tau\sigma)(i)} = \rho(\tau\sigma) e_i.$$

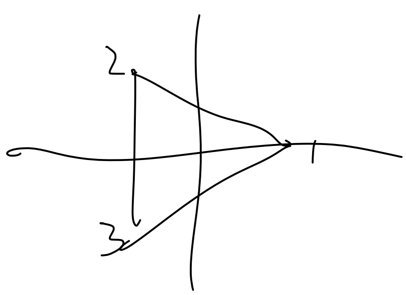
$\rho(\sigma)(1,1,1) = (1,1,1)$

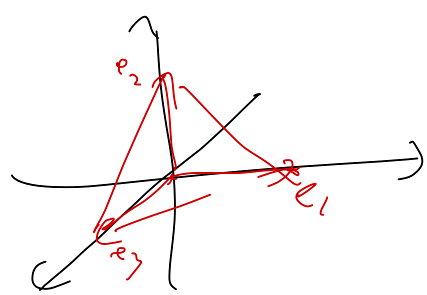
$V_1 = \{(x,x,x) \mid x \in \mathbb{C}\} \subseteq V$ irr subrep. $\dim V_1 = 1$

$V_2 = \{(x,y,z) \mid x+y+z = 0\}$ subrep

and $V = V_1 \oplus V_2 \qquad \dim V_1 = 1 \qquad \dim V_2 = 2.$

$S_3 = \langle s, r \mid s^2 = 1,\ r^3 = 1,\ srs = r^{-1} \rangle \qquad \mathbb{R}^3 \subseteq \mathbb{C}^3$





$\rho(\sigma)\rho(\tau) \neq \rho(\tau)\rho(\sigma)$ on V_2

so V_2 is irreducible.

Fact For S_3 there are 3 irreducible reps up to iso V_1, V_2, sign rep.