1) Let X : set,

$B_X = \{ R \subseteq X \times X \}$ binary relations on X.

$R, S \in B_X$ Write xRy if $(x,y) \in R$.

$x(R \circ S)y \iff \exists z : xRz \wedge zSy$.

a) Show $(B_X, \circ)$ is a monoid.

$$(R \circ S) \circ T = R \circ (S \circ T)$$

$$x((R \circ S) \circ T)y \iff \exists z_1 : x(R \circ S)z_1 \wedge z_1 Ty$$

$$\iff \exists z_1, z_2 : (xRz_2 \wedge z_2 S z_1) \wedge z_1 Ty$$

$$\Updownarrow$$

$$x(R \circ (S \circ T))y \iff \exists z_1, z_2 : xRz_2 \wedge (z_2 S z_1 \wedge z_1 Ty)$$

So $\underline{(R \circ S) \circ T = R \circ (S \circ T)}$

$I = \{ (x,x) \mid x \in X \}$ i.e $xIy \iff x = y$.

$$x(R \circ I)y \iff \exists z : xRz \wedge zIy$$

$$\iff \exists z : xRz \wedge z = y$$

$$\iff xRy$$

so $\underline{R \circ I = R}$, similarly $\underline{I \circ R = R}$.

So $(B_X, \circ)$ is a monoid.

b) show $B_X \simeq (B_X)^{op}$

$$\varphi : B_X \longrightarrow (B_X)^{op}$$

$$R \longmapsto \tilde{R}$$

$$x \tilde{R} y \Leftrightarrow y R x$$

$$x(\tilde{R} \circ \tilde{S})y \Leftrightarrow \exists z : x \tilde{R} z \wedge z \tilde{S} y$$

$$\Leftrightarrow \exists z : z R x \wedge y S z$$

$$\Leftrightarrow y(S \circ R)x$$

$$\Leftrightarrow x(\widetilde{S \circ R})y$$

$$\varphi(S)\varphi(R) = \varphi(SR)$$

c) Compute $(B_X)^\times$ guess $(B_X)^\times = S_X$

i.e. $R \in (B_X)^\times$ iff $\forall x \exists! y : xRy$
$\forall y \exists! x : xRy$

$R \circ S = I$ means

$$(\exists z : xRz \wedge zSy) \Leftrightarrow x = y$$

$R \circ S = I$ and $S \circ R = I$ means

$$x = y \Leftrightarrow (\exists z : xRz \wedge zSy)$$

$$\Leftrightarrow (\exists z : xSz \wedge zRy)$$

Assume $x = y \iff (\exists z : xRz \wedge zSy)$

$\iff (\exists z : xSz \wedge zRy)$

Let $x \in X$

$x = x \Rightarrow \exists z : xRz \wedge zSx$

If xRz' then $zRx \wedge xSz'$ so $z = z'$

$\{z \mid xRz\}$ has exactly one element

By Symmetry $\{z \mid xSz\}$,

$\{z \mid zRx\}$, all have one

$\{z \mid zSx\}$ element.

Hence R defines a bijection.

so $(Bx)^{*} = Sx$.

(2) Let R: commutative ring
$A, B \in M_{n\times n}(R)$

show $\det(AB) = \det(A)\det(B)$ $(*)$

If R field, use row transformations and elementary matrices.

<u>Recall</u> If $\varphi : R \to S$ ring morphism

$\varphi : M_{n\times n}(R) \longrightarrow M_{n\times n}(S)$ $\varphi((a_{ij})_{ij}) = (\varphi(a_{ij}))_{ij}$

then $\varphi(\det(A)) = \det(\varphi(A))$

If φ is injective and $*$ holds in S then $*$ holds in R.

$$\varphi(\det(AB)) = \det(\varphi(AB)) = \det(\varphi(A)\varphi(B))$$
$$= \det(\varphi(A))\det(\varphi(B)) = \varphi(\det(A)\det(B))$$

φ injective $\Rightarrow$ $\det(AB) = \det(A)\det(B)$

If $*$ holds in R, then $*$ holds in $\operatorname{im}\varphi$

$$\det(\varphi(A)\varphi(B)) = \varphi(\det(AB)) = \varphi(\det(A)\det(B))$$
$$= \det(\varphi(A))\det(\varphi(B)).$$

To prove $*$ for given A, B

it is enough to prove $*$ in

$R' =$ subring generated by $\{a_{ij} \mid i,j\} \cup \{b_{ij} \mid i,j\}$

Let ($2n^2$ variables)

$$\begin{aligned} \varphi : \mathbb{Z}[x_{ij}, y_{ij}] &\longrightarrow R \\ x_{ij} &\longmapsto a_{ij} \\ y_{ij} &\longmapsto b_{ij} \end{aligned}$$

well-def as $\mathbb{Z}[x_{ij}, y_{ij}]$ is free.

Then $\operatorname{im} \varphi = R'$

But $\mathbb{Z}[x_{ij}, y_{ij}] \xhookrightarrow{\iota} \mathbb{Q}(x_{ij}, y_{ij}) =: K$

K field so $*$ holds in K

$\Rightarrow$ $*$ holds in $\mathbb{Z}[x_{ij}, y_{ij}]$ $\Rightarrow$ $*$ holds in R'.

Alt. Choose any set X st

$\varphi\ \mathbb{Z}[X] \longrightarrow R$ is surjective

$x \longmapsto r_x$

e.g. $X = R$.

and $\mathbb{Q}(X)$.

③ Let $F : \mathcal{C} \to \mathcal{D}$ a covariant functor

- F is called full (faithful) if $\forall X, X' \in \mathcal{C}$ $\mathcal{C}(X, X') \longrightarrow \mathcal{D}(FX, FX')$ is surjective (injective)
- F is called dense if $\forall Y \in \mathcal{D}\ \exists X \in \mathcal{C}$: $FX \simeq Y$.

Show F is an equivalence iff F is full, faithful and dense.

Assume F : equivalence

i.e $\exists G : \mathcal{D} \to \mathcal{C}$ st

$\phi : F \circ G \simeq id_{\mathcal{D}}$ $\psi : G \circ F \simeq id_{\mathcal{C}}$

Any $Y \in \mathcal{D}$ $\phi_Y : F(G(Y)) \simeq Y$ so F dense.

$$\mathcal{C}(X, X') \xrightarrow{F} \mathcal{D}(F(X), F(X')) \xrightarrow{G} \mathcal{C}(G(F(X)), G(F(X'))) \xrightarrow{\Psi} \mathcal{C}(X, X')$$

$h \longmapsto \psi_{X'} h \psi_X^{-1}$

$\Psi \circ G \circ F = id_{\mathcal{C}(X,X')}$ $\Rightarrow$ F : faithful by symmetry G : faithful and Ψ bijective

Let $h : F(X) \to F(X')$

Then $\Psi G F \Psi G(h) = \Psi G(h)$ $\quad$ ΨG injective so $F \Psi G(h) = h$ so F full.

Assume F full, faithful dense

Need to construct $G : \mathcal{D} \to \mathcal{C}$

Assume some strong choice principle. e.g. $\mathcal{D}$ small.

For each $Y \in \mathcal{D}$ choose $X_Y \in \mathcal{C}$ and

$$\varphi_Y : Y \xrightarrow{\sim} F(X_Y) \quad \text{as } F \text{ is dense}$$

Set $G(Y) = X_Y$ so $\varphi_Y : Y \xrightarrow{\sim} F(G(Y))$

For any $g : Y \to Y'$ need to define $G(g)$

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\varphi_Y"'] & Y' \arrow[d, "\varphi_{Y'}"] \\ F(G(Y)) \arrow[r, "F(G(g))"] & F(G(Y')) \end{tikzcd}$$

Need

Define $G(g) = f$ where $F(f) = \varphi_{Y'} g \varphi_Y^{-1}$

$\exists !$ such f as F full, faithful

<u>Claim</u> G functor $Y = Y'$ $g = 1_Y$ $f = 1_{G(Y)}$ work

so $G(1_Y) = 1_{G(Y)}$.

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\varphi_Y"'] & Y' \arrow[r, "g'"] \arrow[d, "\varphi_{Y'}"] & Y'' \arrow[d, "\varphi_{Y''}"] \\ FG(Y) \arrow[r, "F(f)"'] & FG(Y') \arrow[r, "F(f')"'] & FG(Y'') \end{tikzcd}$$

$$G(Y) \xrightarrow{f} G(Y') \xrightarrow{f'} G(Y''$$

$$\begin{tikzcd} Y \arrow[r, "g' g"] \arrow[d, "\varphi_Y"'] & Y'' \arrow[d, "\varphi_{Y''}"] \\ FG(Y) \arrow[r, "F(f' f)"] & FG(Y'') \end{tikzcd}$$

$$G(Y) \xrightarrow{f' f} G(Y'')$$

$\varphi : id_{\mathcal{D}} \longrightarrow F \circ G$ is natural by construction

$\varphi_{F(\cdot)} : F \longrightarrow F \circ G \circ F$ isomorphism

F full faithful $\quad \varphi_{F(X)} = F(\tilde{\varphi}_X)$

$\tilde{\varphi} : id \longrightarrow G \circ F$ natural iso.

so F is an equivalence