Algebras and Modules

1 R-modules

Let $\mathcal{C} = Ab$: abelian groups, $A, B \in \mathcal{C}$

$\mathcal{C}(A,B) \in Ab$ by

$(f+g)(a) = f(a) + g(a)$

Moreover $End_{\mathcal{C}}(A)$ is a ring under $\circ$ and $+$

as $(f \circ (g+h))(a) = f(g(a) + h(a)) = f(g(a)) + f(h(a))$ and

$((f+g) \circ h)(a) = (f+g)(h(a)) = f(h(a)) + f(g(a))$

A ring morphism $R \to End_{\mathcal{C}}(A)$ corresponds

to a $\mathcal{C}$-action $\mu: R \times A \to A$, $(r,a) \mapsto ra$

such that $(r+s)a = ra + sa$

$\mathcal{C}$-action $\begin{cases} 1a = a \\ (rs)a = r(sa) \\ r(a+b) = ra + rb \end{cases}$

We call $(A = (A, \mu, +))$ an <u>R-module</u>

If A, B are R-modules, then a morphism of $\mathcal{C}$-actions $\varphi: A \to B$ is called an R-module morphism.

$\rightsquigarrow$ Category $R\text{-}Mod$.

<u>Ex</u> $R = K$: field $\quad R\text{-}Mod = Vec\,K$.

<u>Rmk</u> 1) We can consider R-Mod as a variety of T-algebras by extending Ab with infinitely many operations $\omega^r : a \mapsto ra$ $\quad r \in R$ and infinitely many identities

$$1a = a$$
$$(rs)a = r(sa) \qquad r, s \in R$$
$$r(a+b) = ra + rb \qquad r \in R$$
$$(r+s)a = ra + sa \qquad r, s \in R$$

$\Rightarrow$ We have submodules, quotients (by submodules), products, free R-modules etc.

X : set $\quad R^{(X)} = \left\{ \sum_{x \in X} r_x x \;\middle|\; \begin{array}{l} r_x \in R,\ r_x = 0 \text{ for all} \\ \text{but finitely many } x \in X \end{array} \right\}$

is free on X :

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & M \\ R^X \arrow[ur, "\varphi_f"'] & \end{tikzcd} \quad \varphi_f : \sum_{x \in X} r_x x \mapsto \sum_{x \in X} r_x f(x)$$

2) R-Mod is the category of left R-modules.

We may also consider

Mod-R : category of right R-modules:

$$A \times R \longrightarrow A, \quad (a, r) \mapsto ar$$

$$a1 = a$$
$$a(rs) = (ar)s$$
$$(a+b)r = ar + br$$
$$a(r+s) = ar + as$$

Note R^{op}-Mod = Mod-R.

Ex Let $A, B \in Ab$ $R = \mathrm{End}_{Ab}(A)$ $S = \mathrm{End}_{Ab}(B)$

$M = \mathrm{Hom}_{Ab}(A, B)$

$$\begin{aligned} M \times R &\longrightarrow M \\ (f, r) &\longmapsto f \circ r \end{aligned} \qquad M \in \mathrm{Mod}\text{-}R$$

$$\begin{aligned} S \times M &\longrightarrow M \\ (s, f) &\longmapsto s \circ f \end{aligned} \qquad M \in S\text{-}\mathrm{Mod}$$

Note $s \circ (f \circ r) = (s \circ f) \circ r$

Def If $M \in \mathrm{Mod}\text{-}R$, $M \in S\text{-}\mathrm{Mod}$ (same abelian group)

we call M an S-R-bimodule if

$$(sm)r = s(mr)$$

$S\text{-}\mathrm{Mod}\text{-}R$: category of S-R-bimodules.

Rmk 1) If R : commutative $R^{op} = R$ so

we may identify $R\text{-}\mathrm{Mod} = \mathrm{Mod}\ R$.

If $M \in R\text{-}\mathrm{Mod}$ we may consider

$M \in R\text{-}\mathrm{Mod}\text{-}R$ by $smr = srm$.

Warning For $N \in R\text{-}\mathrm{Mod}\text{-}R$ we may have $rn \neq nr$

so $R\text{-}\mathrm{Mod} \neq R\text{-}\mathrm{Mod}\text{-}R$.

2) Note $\mathbb{Z}$ is the free ring on $\emptyset$

indeed

$$\begin{tikzcd} \emptyset \arrow[r, "\exists!"] \arrow[d, "\exists!"'] & R \\ \mathbb{Z} \arrow[ur, "\exists!\,\varphi"'] & \end{tikzcd}
\qquad
\varphi(n) = \begin{cases} \overbrace{1_R + \cdots + 1_R}^{n \text{ times}} & n > 0 \\ 0 & 0 \\ -(\underbrace{1_R + \cdots + 1_R}_{-n \text{ times}}) & n < 0 \end{cases}$$

So any $A \in Ab$ admits a unique $\mathbb{Z}$-module structure. We may identify

$$Ab = \mathbb{Z}\text{-Mod}$$

② K-algebras

Let K: commutative ring (e.g. K field)

Def 1) A K-algebra is $A \in K\text{-Mod}$

with $\mu: A \times A \longrightarrow A$, $(a,b) \longmapsto ab$

which is K-bilinear, i.e.

$$(a+b)c = ac + bc$$
$$a(b+c) = ab + ac$$
$$(\lambda a)b = a(\lambda b) = \lambda(ab) \qquad \lambda \in K$$

2) A is called an associative K-algebra (with unit)

if $(A, \mu, +)$ is a ring

Rmk 1) If A: associative K-alg

$$K \longrightarrow A, \quad \lambda \longmapsto \lambda 1_A$$

is a central ring morphism i.e. $(\lambda 1_A)a = a(\lambda 1_A)$

In fact any ring R with a central ring morphism

$$\varphi: K \longrightarrow R \qquad (\varphi(\lambda)a = a\varphi(\lambda))$$

defines an associative K-algebra.

2) If R: associative K-algebra, then

$M \in R\text{-Mod} \rightsquigarrow M \in K\text{-Mod}$ by

$\lambda m = \varphi(\lambda)m$ $\quad \varphi$ as above.

3) (associative) K-algebras form a variety

Ex 1) R : ring, then $R \in \mathbb{Z}$-Mod

in fact R : $\mathbb{Z}$-algebra

2) $K[x]$ polynomial alg over K

3) $K[x_i \mid i \in I]$ polynomial alg over K in variables $\{x_i \mid i \in I\}$

4) $V \in K$-Mod $\quad \mathrm{End}_K(V)$ is a K-alg, via

$$\begin{aligned} K &\longrightarrow \mathrm{End}_K(V) \\ \lambda &\longmapsto (v \mapsto \lambda v). \end{aligned}$$

5) Let G : group and K field

$\rightsquigarrow$ KG : group algebra

KG free K-module on G with

$$\Big(\sum_{g \in G} \lambda_g g\Big)\Big(\sum_{g' \in G} \lambda_{g'} g'\Big) = \sum_{g, g'} \lambda_g \lambda_{g'} g g' = \sum_{h \in G} \Big(\sum_{gg' = h} \lambda_g \lambda_{g'}\Big) h$$

In fact KG has the unique K-algebra multiplication such that $G \longrightarrow KG$ is a monoid morphism.

If $M \in KG\text{-Mod}$ we have

$$\begin{tikzcd}
KG \arrow[rr, "\varphi"] & & \mathrm{End}_K(M) \\
G \arrow[u, hook] \arrow[urr, "\varphi|_G"] \arrow[rr, "\underline{\tilde{\varphi}|_G}"'] & & GL_K(M) = \mathrm{End}_K(M)^\times \arrow[u, hook]
\end{tikzcd}$$

representation of G

If $\rho : G \to GL_K(V)$ representation of G, then

$$\begin{tikzcd}
G \arrow[r, "\rho"] \arrow[d, hook] \arrow[dr, bend left] & GL_K(V) \arrow[d, hook] \\
\underline{KG} \arrow[r, "\exists!\, \varphi"'] & \mathrm{End}_K(V)
\end{tikzcd}
\qquad \varphi(g) = \rho(g)$$

free K-mod on G $\qquad$: K-linear

$\varphi(1_G) = \rho(1_G) = 1_V \qquad \varphi(\lambda_g g$

$$\varphi\left(\left(\sum_{g \in G} \lambda_g g\right)\left(\sum_{g' \in G} \lambda_{g'} g'\right)\right) = \sum_{g, g'} \lambda_g \lambda_{g'} \varphi(g)\varphi(g')$$

$$= \varphi\left(\sum_{g \in G} \lambda_g g\right) \varphi\left(\sum_{g' \in G} \lambda_{g'} g'\right)$$ so φ ring morphism.

Hence $V \in KG\text{-Mod}$.

This gives $KG\text{-Mod} \longrightarrow \mathrm{Rep}_K G$ equivalence

Now assume $|G| < \infty$ and $K = \mathbb{C}$. Let V irr. rep.

and $v \in V \setminus \{0\}$ $\qquad$ $KG \xrightarrow{\varphi} V$ surjective as V irr.

$$1 \longmapsto v$$

free on $\{1\}$

Choose $V' \in KG$ subrep st $KG = \ker \varphi \oplus V'$ then $V' \xrightarrow{\varphi_{V'}} V$ iso

Every irreducible rep. appears as a direct summand of KG.

6) Let M : smooth manifold

$C^\infty(M) = \{ f : M \to \mathbb{R} \mid f = \text{smooth} \}$

$\mathbb{R}$-algebra : $(f+g)(p) = f(p) + g(p)$

$(fg)(p) = f(p)g(p)$

$\mathbb{R} \longrightarrow C^\infty(M)$

$\lambda \longmapsto (p \mapsto \lambda)$

$\Theta(M)$ = vector fields on M i.e.

$X(p) = X_p \in T_p M \qquad p \in M$

varying smoothly in p

$\Theta(M) \in \mathrm{Vec}\,\mathbb{R} \qquad (X+Y)(p) = X_p + Y_p$

$(\lambda X)(p) = \lambda X_p$

In fact $\Theta(M) \in C^\infty(M)\text{-Mod}$ via

$C^\infty(M) \times \Theta(M) \longrightarrow \Theta(M), \ (f, X) \longmapsto fX$

$(fX)(p) = f(p) X_p \in T_p(M)$