

# Localizations

Ex  $\mathbb{Q} = \left\{ \frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\} \right\}$   
 $\phi: \mathbb{Z} \longrightarrow \mathbb{Q}, a \longmapsto \frac{a}{1}$   
 $\phi(a)$  is inv.  $\Leftrightarrow a \in \mathbb{Z} \setminus \{0\}$ .

Let  $R$ : ring  $S \subseteq R$  subset

Q When can we construct a ring  $S^{-1}R$  of fractions  $s^{-1}r$ ,  $s \in S$ ,  $r \in R$  with a morphism  $R \longrightarrow S^{-1}R$ ,  $r \longmapsto r^{-1}r$   
 $s \longmapsto$  inv. elements.

Def We call  $S \subseteq R$  a

1) multiplicative set if  $(S, \cdot) \subseteq (R, \cdot)$  is a submonoid

2) regular if

$\forall a \in R \quad sa \neq 0 \Rightarrow a \neq 0$  (i.e.  $L_s: R \rightarrow R$  injective  $\forall s \in S$ )  
 $\forall a \in R \quad aS \neq 0 \Rightarrow a \neq 0$  (i.e.  $R_s: R \rightarrow R$  injective  $\forall s \in S$ )

Ex a)  $R = \mathbb{Z}$ ,  $S = \mathbb{Z} \setminus \{0\}$  is a regular mult. set

b)  $R = \mathbb{Z}$ ,  $p$ : prime number  $S = \{m \in \mathbb{Z} \mid p \nmid m\}$   $\dashv$

Motivation 1)  $r^{-1} = 1$   $s^{-1}t^{-1}$  should be  $(ts)^{-1}$ .

2)  $sa = 0$  should imply  $s^{-1}sa = a = 0$

# Commutative rings

Assume  $R$  commutative,  $S \subseteq R$  mult. set.

Define  $S^{-1}R = S \times R / \sim$  where

$$(s, a) \sim (t, b) \Leftrightarrow \exists u \in S : uta = usb$$

Think  
 $\frac{a}{us} = \frac{a}{ut}$

$\sim$  is an equivalence

Reflexive: use  $1 \in S$

Symmetric: trivial

Transitive: use  $SS \subseteq S$  (exercise)

Prop 1) If  $S$  regular, then  $(s, a) \sim (t, b) \Leftrightarrow ta = sb$

2)  $(s, a) \sim (ts, ta) \quad \forall t \in S$  as  $1(ts)a = 1s(ta)$

Write  $[s, a] = s^{-1}a$  and define

$$(s^{-1}a)(t^{-1}b) = (ts)^{-1}ab$$

$$s^{-1}a + t^{-1}b = (ts)^{-1}(ta + sb)$$

Exercise  $S^{-1}R$  is a well-defined ring with

$$0 = 1^{-1}0, \quad 1 = 1^{-1}1$$

Moreover  $\forall s, t \in S$

$s^{-1}t$  is inv. with inverse  $t^{-1}s$

Note  $(s, a) \sim (1, 0) \Leftrightarrow \exists u \in S : ua = 0$

So  $s^{-1}a = 0 \Leftrightarrow sa = 0$

Proposition  $\tau: R \rightarrow S^{-1}R, a \mapsto \tau^{-1}a$  is a ring morphism, which is injective iff  $S$  is regular.

Proof  $\tau(a+b) = \tau^{-1}a + \tau^{-1}b = \tau^{-1}(a+b) = \tau^{-1}a + \tau^{-1}b$ .

$$\tau(1) = \tau^{-1}1 = 1.$$

$$\tau(ab) = \tau^{-1}(ab) = \tau^{-1}a \tau^{-1}b = \tau^{-1}a \tau^{-1}b.$$

$$\tau^{-1}a = \ominus \quad s a \ominus 0$$

$$\text{so } \ker \tau = \{a \in R \mid s a \ominus 0\}$$

so  $\ker \tau = \{0\}$  iff  $S$  regular.

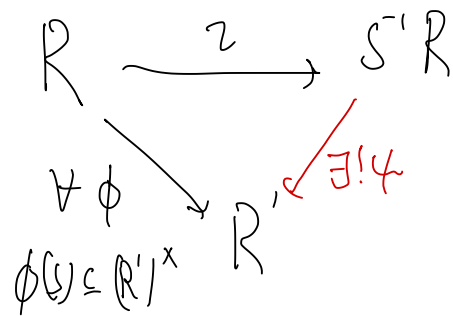
Thm Let  $R$  commutative ring,  $S \subseteq R$  multiplicative

and  $\tau: R \rightarrow S^{-1}R, a \mapsto \tau^{-1}a$ . Then

$\tau^{-1}(s)$  is inv.  $\forall s \in S$  and if

$\phi: R \rightarrow R'$  st  $\phi(s)$  is inv.  $\forall s \in S$ , then

there is a unique  $\psi: S^{-1}R \rightarrow R'$  st  $\psi \circ \tau = \phi$



In fact  $\psi(\tau^{-1}a) = \phi(s)^{-1} \phi(a)$ .

$$R \xrightarrow{\quad \tau \quad} S^{-1}R$$

$$\forall \phi \searrow \begin{matrix} R \\ \phi(s) \in R' \setminus \{0\} \end{matrix} \quad R'$$

$$\psi(s^{-1}a) = \phi(s)^{-1} \phi(a)$$

proof

$\psi$  : well-defined

$$\tau(ua) = usb$$

$$s, t, u \in S \quad a, b \in R$$

$$\Rightarrow \phi(u) \phi(t) \phi(a) = \phi(u) \phi(s) \phi(b)$$

$$\Rightarrow \phi(s)^{-1} \phi(a) = \phi(t)^{-1} \phi(b)$$

$$(\psi \circ \tau)(a) = \psi(\tau^{-1}a) = \phi(1)^{-1} \phi(a) = \phi(a) \quad \text{ok}$$

$$\psi(s^{-1}a) \psi(t^{-1}b) = \phi(s)^{-1} \phi(a) \phi(t)^{-1} \phi(b) = \phi(s)^{-1} \phi(t)^{-1} \phi(a) \phi(b) \\ = \phi(st)^{-1} \phi(ab) = \psi(st^{-1}ab)$$

$R$  comm.

$$\psi(s^{-1}a + t^{-1}b) = \psi(s^{-1}a) + \psi(t^{-1}b) \quad \text{similar.}$$

$$\psi \text{ unique} \quad \psi(a) = \psi(s s^{-1}a) = \underbrace{\psi(s)}_{=\phi(s)} \psi(s^{-1}a) = \underbrace{\psi(s)}_{=\phi(s)}$$

□

Ex 1)  $R = \mathbb{Z}$ ,  $S = \mathbb{Z} \setminus \{0\}$  reg.

$$S^{-1}R = \mathbb{Q}$$

2)  $R = \mathbb{Z}$   $p$  : prime,  $S = \{m \in \mathbb{Z} \mid p \nmid m\}$  reg.

$$S^{-1}R = \mathbb{Z}_{(p)} = \left\{ \frac{n}{m} \mid p \nmid m \right\} \subseteq \mathbb{Q}$$

3)  $R$  : com ring  $S = \{1, 0\}$   $\Rightarrow S^{-1}R = \{0\}$ .  
not reg. reg.

4)  $R = K[t]$ ,  $S = \{1, t, t^2, \dots\}$  reg.

$$S^{-1}R = K[t, t^{-1}]$$

5)  $R = K[x, y] / (xy)$   $S = \{1, x, x^2, \dots\}$  not reg.  
 $S_y \ni 0 \rightarrow 1^{-1}y = 0$   $S^{-1}R = K[x, x^{-1}]$

# Non-commutative rings

Now let  $R$  any ring.

Aim Generalize the above Thm.

Problems (1)  $(s^{-1}a)(t^{-1}b) = (s^{-1}t^{-1})ab$  does not make sense.

Idea  $s^{-1}at^{-1}b = s^{-1}t_1^{-1}a_1b = (t_1s)^{-1}a_1b$  where  
 $at^{-1} = t_1^{-1}a_1$  i.e.  $t_1a = a_1t$  This has a solution  $(t_1, a_1) \in S \times R$

if  $Sa \cap Rt \neq \emptyset$  (left Ore condition system Ore)

(2) We need  $aS = 0 \Rightarrow t^{-1}a = 0$

i.e.  $aS \ni 0 \Rightarrow Sa \ni 0$

Def Let  $R$ : ring. A multiplicative set  $S$

is left Ore if

i)  $\forall (t_1, a) \in S \times R \quad Sa \cap Rt \neq \emptyset$

$aS \cap tR \neq \emptyset$

$Sa \ni 0 \Rightarrow aS \ni 0$ .

ii)  $\forall a \in R \quad aS \ni 0 \Rightarrow Sa \ni 0$

Remark If  $S$ : regular then ii) is automatic.

Ex  $R = K \langle x, y \rangle \quad S = \{x^i \mid i \in \mathbb{N}\}$  is not Ore

$$Sy \cap Rx = \emptyset$$

Ex  $R = K \langle x, y \rangle / (xy + yx) \quad S = \{x^i \mid i \in \mathbb{N}\}$  is Ore

Indeed  $x^m y^n = (-1)^{mn} y^n x^m$  so  $x^m w = (-1)^{mn} w x^m$   
 $n = \text{number of } y\text{'s in } w$

and  $S \cap Rx^m \neq \emptyset$

Ex If  $R$ : comm. then any mult. set is left and right Ore.

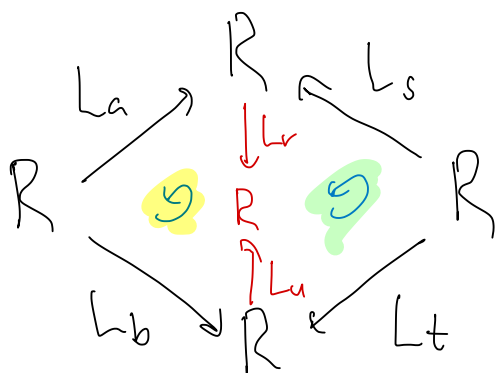
Definition

Assume  $S \subset R$  left Ore. Then

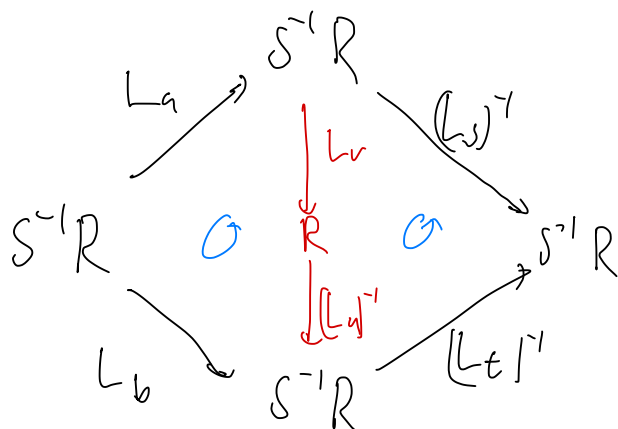
$$S^{-1}R = S \times R / \sim$$

where  $(s, a) \sim (t, b) \Leftrightarrow \exists (u, v) \in S \times R$   $ut = vs$  and  $ub = va$

Motivation:



$\rightsquigarrow$



Note if  $R$  : commutative this the same equivalence relation as before

$\Uparrow$  If  $ut = vs$  then  $uta = vsa = sva = sub = usb$   
 $ub = va$

If  $uta = usb$  then  $u' = su, v = ut$  Then  $u't = sut = sv = vs$   
 $u'b = ub = uta = va$

Write  $[(s, a)] = s^{-1}a$  and define

$$s^{-1}a t^{-1}b = (t, s)^{-1}a, b \quad \text{where} \quad \begin{array}{c} t, a = a, t \\ \uparrow \\ \downarrow \\ R \end{array}$$

$$s^{-1}a + t^{-1}b = (t, s)^{-1}(t, a + s, b) \quad \begin{array}{c} t, s = s, t \\ \uparrow \\ \downarrow \\ R \end{array}$$

Fact  $\sim$  : equivalence  $S^{-1}R$  r.h.s  
 (quite tricky)

Proposition  $\tau : R \rightarrow S^{-1}R, a \mapsto \tau^{-1}a$  is a ring morphism with  $\ker \tau = \{a \in R \mid Sa \ni 0\}$ .

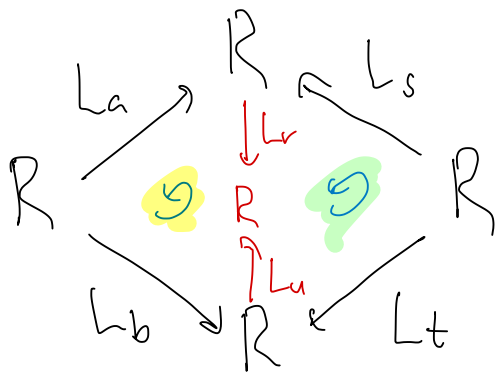
proof same proof works as  $t = s = 1$  commute with everything  $\square$ .

Thm If  $S \subseteq R$  left Ore, then  $\tau: R \rightarrow S^{-1}R$  satisfies  $\tau(s) \in (S^{-1}R)^\times$ . If  $\phi: R \rightarrow R'$  satisfies  $\phi(s) \in (R')^\times$ , then  $\exists! \psi: S^{-1}R \rightarrow R'$  s.t.  $\psi \circ \tau = \phi$  i.e.

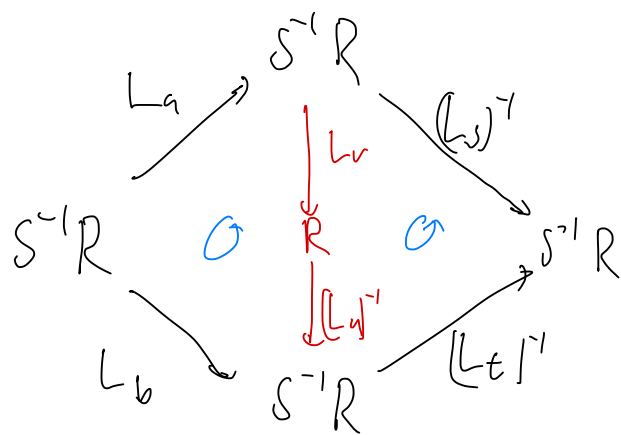
$$\begin{array}{ccc}
 R & \xrightarrow{\tau} & S^{-1}R \\
 \forall \phi \searrow & & \swarrow \exists! \psi \\
 \phi(s) \in (R')^\times & & R'
 \end{array}$$

## Localization in categories

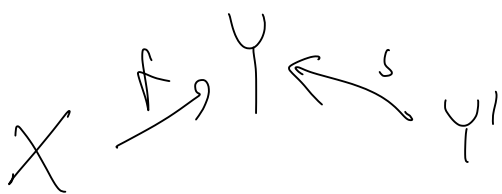
Idea replace



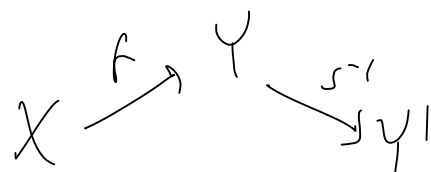
$\rightsquigarrow$



or



$\rightsquigarrow$



Need Ore conditions