Localizations

<u>Ex</u> $\mathbb{Q} = \{ \frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{Z} \setminus \{0\} \}$

$\phi : \mathbb{Z} \longrightarrow \mathbb{Q},\quad a \longmapsto \frac{a}{1}$

$\phi(a)$ is inv. $\Leftrightarrow$ $a \in \mathbb{Z} \setminus \{0\}$.

Let R : ring $S \subseteq R$ subset

<u>Q</u> When can we construct a ring $S^{-1}R$ of fractions $s^{-1}r$, $s \in S$, $r \in R$ with a morphism $R \longrightarrow S^{-1}R$, $r \longmapsto 1^{-1}r$

$s \longmapsto$ inv. elements.

<u>Def</u> We call $S \subseteq R$ a

1) multiplicative set if $(S, \cdot) \subseteq (R, \cdot)$ is a submonoid

2) regular if

$\forall a \in R$ $\quad Sa \ni 0 \Rightarrow a = 0$ (i.e $L_s : R \to R$ injective $\forall s \in S$)

$\forall a \in R$ $\quad aS \ni 0 \Rightarrow a = 0$ (i.e. $R_s : R \to R$ injective $\forall s \in S$)

<u>Ex</u> a) $R = \mathbb{Z}$, $S = \mathbb{Z} \setminus \{0\}$ is a regular mult. set

b) $R = \mathbb{Z}$, p : prime number $S = \{ m \in \mathbb{Z} \mid p \nmid m \}$ $\dashv\!\!/\!\!-$

<u>Motivation</u> 1) $1^{-1} = 1$, $s^{-1}t^{-1}$ should be $(ts)^{-1}$.

2) $sa = 0$ should imply $s^{-1}sa = a = 0$

Commutative rings

Assume R commutative, $S \subseteq R$ mult. set.

Define $S^{-1}R = S \times R / \sim$ where

$(s,a) \sim (t,b) \iff \exists u \in S : uta = usb$

Think $\frac{a}{us} = \frac{b}{ut}$

$\sim$ is an equivalence

Reflexive: use $1 \in S$

Symmetric: trivial

Transitive: use $SS \subseteq S$ (exercise)

Rmk 1) If S regular, then $(s,a) \sim (t,b) \iff ta = sb$

2) $(s,a) \sim (ts, ta)$ $\forall t \in S$ as $1 \cdot (ts) a = 1 \cdot s (ta)$

Write $[(s,a)] = s^{-1}a$ and define

$(s^{-1}a)(t^{-1}b) = (ts)^{-1}ab$

$s^{-1}a + t^{-1}b = (ts)^{-1}(ta + sb)$

Exercise $S^{-1}R$ is a well-defined ring with

$0 = 1^{-1}0$, $1 = 1^{-1}1$

Moreover $\forall s,t \in S$ $s^{-1}t$ is inv. with inverse $t^{-1}s$

Note $(s,a) \sim (1,0) \iff \exists u \in S : ua = 0$

So $s^{-1}a = 0 \iff Sa \ni 0$

Proposition $\iota : R \longrightarrow S^{-1}R$, $a \longmapsto 1^{-1}a$ is a ring morphism, which is injective iff S is regular.

proof
$\iota(a+b) = 1^{-1}a + 1^{-1}b = 1^{-1}(a+b) = \iota(a) + \iota(b)$.

$\iota(1) = 1^{-1}1 = 1$.

$\iota(ab) = 1^{-1}(ab) = 1^{-1}a\,1^{-1}b = \iota(a)\iota(b)$.

$1^{-1}a = 0 \Leftrightarrow Sa \ni 0$

so $\ker \iota = \{a \in R \mid Sa \ni 0\}$

so $\ker \iota = \{0\}$ iff S regular.

Thm Let R commutative ring, $S \subseteq R$ multiplicative and $\iota : R \longrightarrow S^{-1}R$, $a \longmapsto 1^{-1}a$. Then $\iota(s)$ is inv. $\forall s \in S$ and if $\phi : R \longrightarrow R'$ st $\phi(s)$ is inv. $\forall s \in S$, then there is a unique $\psi : S^{-1}R \longrightarrow R'$ st $\psi \circ \iota = \phi$

$$\begin{tikzcd} R \arrow[r, "\iota"] \arrow[dr, "\forall \phi"'] & S^{-1}R \arrow[d, "\exists ! \psi"] \\ & R' \end{tikzcd}$$

$\phi(S) \subseteq (R')^{\times}$

In fact $\psi(s^{-1}a) = \phi(s)^{-1}\phi(a)$.

$$\begin{tikzcd} R \arrow[r, "\iota"] \arrow[dr, "\forall \phi"'] & S^{-1}R \arrow[d, "\exists ! \psi"] \\ & R' \end{tikzcd}$$

$\phi(S) \subseteq (R')^{\times}$

$$\psi(s^{-1}a) = \phi(s)^{-1}\phi(a).$$

proof ψ : well-defined $\left[uta = usb \right.$ $s,t,u \in S$ $a,b \in R$

mult by $\phi(ust)^{-1}$

$$\Rightarrow \phi(u)\phi(t)\phi(a) = \phi(u)\phi(s)\phi(b)$$

$$\Rightarrow \phi(s)^{-1}\phi(a) = \phi(t)^{-1}\phi(b) \left.\right]$$

$$(\psi \circ \iota)(a) = \psi(1^{-1}a) = \phi(1)^{-1}\phi(a) = \phi(a) \quad \text{ok}$$

$$\psi(s^{-1}a)\psi(t^{-1}b) = \phi(s)^{-1}\phi(a)\phi(t)^{-1}\phi(b) = \phi(s)^{-1}\phi(t)^{-1}\phi(a)\phi(b) \qquad \boxed{R \text{ comm.}}$$

$$= \phi(st)^{-1}\phi(ab) = \psi((st)^{-1}ab)$$

$$\psi(s^{-1}a + t^{-1}b) = \psi(s^{-1}a) + \psi(t^{-1}b) \quad \text{similar.}$$

ψ unique $\quad \underbrace{\psi(a)}_{=\phi(a)} = \psi(s s^{-1}a) = \underbrace{\psi(s)}_{=\phi(s)}\psi(s^{-1}a)$ $\square$

Ex 1) $R = \mathbb{Z}$, $S = \mathbb{Z} \setminus \{0\}$ reg.

$S^{-1}R = \mathbb{Q}$.

2) $R = \mathbb{Z}$ p : prime, $S = \{m \in \mathbb{Z} \mid p \nmid m\}$ reg

$S^{-1}R = \mathbb{Z}_{(p)} = \{ \frac{n}{m} \mid p \nmid m \} \subseteq \mathbb{Q}$

3) R : com ring $S = \{1, 0\}$ not reg. $\Rightarrow S^{-1}R = \{0\}$.

4) $R = K[t]$, $S = \{1, t, t^2, \ldots\}$ reg.

$S^{-1}R = K[t, t^{-1}]$

5) $R = K[x,y]/(xy)$ $S = \{1, x, x^2, \ldots\}$ not reg.

$S^{-1}R = K[x, x^{-1}]$

$Sy \ni 0 \to 1^{-1}y = 0$

Non-commutative rings

Now let R any ring.

Aim Generalize the above Thm.

Problems 1. $(s^{-1}a)(t^{-1}b) = (st)^{-1}ab$ does not make sense.

Idea $s^{-1}at^{-1}b = s^{-1}t_1^{-1}a_1b = (t_1s)^{-1}a_1b$ where $at^{-1} = t_1^{-1}a_1$ i.e. $t_1a = a_1t$ This has a solution $(t_1,a_1)\in S\times R$ iff $Sa \cap Rt \neq \emptyset$ (left Ore condition / System Ore)

2. We need $as = 0 \Rightarrow 1^{-1}a = 0$

i.e $aS \ni 0 \Rightarrow Sa \ni 0$

Def Let R: ring. A multiplicative set S is left Ore if
right

i) $\forall (t,a)\in S\times R$ $Sa\cap Rt \neq \emptyset$ — $aS \cap tR \neq \emptyset$

ii) $\forall a\in R$ $aS\ni 0 \Rightarrow Sa \ni 0$ — $Sa \ni 0 \Rightarrow aS \ni 0.$

Rmk If S: regular then ii) is automatic.

Ex $R = K\langle x,y\rangle$ $S = \{x^i \mid i\in\mathbb{N}\}$ is not Ore

$$Sy \cap Rx = \emptyset$$

Ex $R = K\langle x,y\rangle/(xy+yx)$ $S = \{x^i \mid i\in\mathbb{N}\}$ is Ore

Indeed $x^my^n = (-1)^{mn}y^nx^m$ so $x^m w = (-1)^{mn} w x^m$
n = number of y:s in w

and $Sf \cap Rx^m \neq \emptyset$

Ex If R: comm. then any mult. set is left and right Ore.

__Definition__ Assume $S \subset R$ left Ore. Then

$$S^{-1}R = S \times R/\sim$$

where $(s,a) \sim (t,b) \Leftrightarrow \exists (u,r) \in S \times R$ $ut = rs$ and $ub = ra$

Motivation:

$$\begin{tikzcd}
& R \arrow[d, "L_r"] & \\
R \arrow[ur, "L_a"] \arrow[dr, "L_b"'] & R & R \arrow[ul, "L_s"'] \arrow[dl, "L_t"] \\
& R \arrow[u, "L_u"'] &
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
& S^{-1}R \arrow[d, "L_r"] \arrow[dr, "(L_s)^{-1}"] & \\
S^{-1}R \arrow[ur, "L_a"] \arrow[dr, "L_b"'] & R \arrow[d, "{[L_u]^{-1}}"] & S^{-1}R \\
& S^{-1}R \arrow[ur, "{[L_t]^{-1}}"'] &
\end{tikzcd}$$

Note if R: commutative this the same equivalence relation as before

[If $ut = rs$, $ub = ra$ then $uta = rsa = sra = sub = usb$

If $uta = usb$ then $u' = su$, $r = ut$ Then $u't = sut = sr = rs$
$u'b = usb = uta = ra$

Write $[(s,a)] = s^{-1}a$ and define

$$s^{-1}at^{-1}b = (t_1 s)^{-1}a_1 b \quad \text{where} \quad t_1 a = a_1 t, \quad t_1 \in S,\ a_1 \in R$$

$$s^{-1}a + t^{-1}b = (t_1 s)^{-1}(t_1 a + s_1 b) \qquad t_1 s = s_1 t, \quad t_1 \in S,\ s_1 \in R$$

__Fact__ $\sim$: equivalence $S^{-1}R$ rings (quite tricky)

__Proposition__ $\iota : R \to S^{-1}R$, $a \mapsto 1^{-1}a$ is a ring morphism with $\ker \iota = \{a \in R \mid Sa \ni 0\}$.

__proof__ same proof works as $t = s = 1$ commute with everything $\square$.

Thm If $S \subseteq R$ left Ore, then $\iota : R \to S^{-1}R$ satisfies $\iota(s) \in (S^{-1}R)^\times$. If $\phi : R \to R'$ satisfies $\phi(s) \in (R')^\times$, then $\exists!\, \psi : S^{-1}R \to R'$ st $\psi \circ \iota = \phi$ i.e.

$$
\begin{tikzcd}
R \arrow[r, "\iota"] \arrow[d, "\forall \phi"'] & S^{-1}R \arrow[dl, "\exists!\, \psi"] \\
R' &
\end{tikzcd}
$$

$\phi(S) \subseteq (R')^\times$

Localization in categories

Idea replace

$$
\begin{tikzcd}
& R \arrow[d, "L_r"] & \\
R \arrow[ur, "L_a"] \arrow[dr, "L_b"'] & R & R \arrow[ul, "L_s"'] \arrow[dl, "L_t"] \\
& R \arrow[u, "L_u"] &
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
& S^{-1}R \arrow[d, "L_r"] \arrow[dr, "(L_s)^{-1}"] & \\
S^{-1}R \arrow[ur, "L_a"] \arrow[dr, "L_b"'] & R \arrow[d, "{[L_u]^{-1}}"] & S^{-1}R \\
& S^{-1}R \arrow[ur, "{[L_t]^{-1}}"'] &
\end{tikzcd}
$$

w/

$$X \xrightarrow{f} Y \xleftarrow{s} Y' \quad \rightsquigarrow \quad X \xrightarrow{f} Y \xrightarrow{s^{-1}} Y'$$

Need Ore conditions